Open Journal of Mathematical Sciences

PSR Press

Article

Integral representations for specific Maclaurin series and their applications to normalized remainder of exponential function

Chun-Ying He[1] and Feng Qi[2,*]

[1] School of Mathematics and Physics, Hulunbuir University, Hulunbuir, Inner Mongolia, 021008, China
[2] 17709 Sabal Court, University Village, Dallas, TX 75252-8024, USA
* Correspondence: qifeng618@gmail.com

Received: 28 Nov 2025; Revised: 30 Dec 2025; Accepted: 15 Jan 2026; Published: 29 Mar 2026

Abstract: In the work, by establishing integral representations for a class of specific Maclaurin power series, the authors restate recently-published results related to the normalized remainder of the Maclaurin power series of the exponential function, alternatively prove some of these results, and pose some new problems in terms of the majorizing relations.

Keywords: Maclaurin power series, integral representation, normalized remainder, absolute monotonicity, inequality, problem

MSC: Primary 41A58; Secondary 26A48, 30E20, 33B10.

1. Motivations

We recall two definitions and related conclusions.

Definition 1 ([1, p. 19]). If a function $f(u)$ has derivatives of all orders throughout a neighborhood of a point ξ, then the Taylor series of the function $f(u)$ is

$$f(\xi)+\frac{u-\xi}{1!}f'(\xi)+\frac{(u-\xi)^2}{2!}f''(\xi)+\frac{(u-\xi)^3}{3!}f'''(\xi)+\cdots, \tag{1}$$

and its remainder is

$$R_n[f(u);\xi]=f(u)-f(\xi)-\sum_{k=1}^{n}\frac{(u-\xi)^k}{k!}f^{(k)}(\xi). \tag{2}$$

Theorem 1 ([1, p. 19]). *The Taylor series* (1) *converges to the function* $f(u)$ *if the remainder* (2) *approaches zero as* $n\to\infty$.

When $\xi=0$, we call the series (1) the Maclaurin series and we denote the remainder (2) by $R_n[f(u)]$.

Definition 2 ([2, Section 1], [3, Section 1], and [4, Definition 1]). Let f be an infinitely differentiable real-valued function defined on an interval $I\subseteq\mathbb{R}$ with $\xi\in I$ being an inner point. The informal series expansion of f around the point ξ is

$$\sum_{n=0}^{\infty}f^{(n)}(\xi)\frac{(u-\xi)^n}{n!}. \tag{3}$$

Its partial sum of the first $n+1$ terms is

$$S_n(u,\xi)=\sum_{k=0}^{n}f^{(k)}(\xi)\frac{(u-\xi)^k}{k!},\quad n\in\mathbb{N}_0.$$

1. If $f(\xi) \neq 0$, then we define

$$T_{-1}[f(u), \xi] = \frac{f(u)}{f(\xi)}, \quad u \in I.$$

2. If $f^{(n+1)}(\xi) \neq 0$ for $n \in \mathbb{N}_0$, then we define

$$T_n[f(u), \xi] = \begin{cases} \frac{1}{f^{(n+1)}(\xi)} \frac{(n+1)!}{(u-\xi)^{n+1}}[f(u) - S_n(u, \xi)], & u \in I \setminus \{\xi\}; \\ 1, & u = \xi. \end{cases} \tag{4}$$

We call the quantity $T_n[f(u), \xi]$ the nth normalized remainder of the informal series expansion (3) of f around ξ.

When $\xi = 0$, we simply write $T_n[f(u), 0]$ as $T_n[f(u)]$.

When applying $f(u) = \mathrm{e}^u$ and $\xi = 0$ in (2), we have the following integral representation for the remainder $R_n[\mathrm{e}^u]$.

Theorem 2 ([5, p. 502], [6, Lemma 4], and [2, Lemma 2]). *For $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, we have the integral representation*

$$R_n[\mathrm{e}^u] = \frac{u^{n+1}}{n!} \int_0^1 v^n \mathrm{e}^{u(1-v)} \, \mathrm{d}v. \tag{5}$$

When applying $f(u) = \mathrm{e}^u$ in (4) and employing the integral representation (5), we obtain the following integral representation for the normalized remainder $T_n[\mathrm{e}^u]$.

Theorem 3 ([7, Lemma 1]). *For $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, the normalized remainder $T_n[\mathrm{e}^u]$ has the integral representation*

$$T_n[\mathrm{e}^u] = (n+1) \int_0^1 v^n \mathrm{e}^{u(1-v)} \, \mathrm{d}v. \tag{6}$$

In the paper [4], we derived the following integral representation for a specific Maclaurin series.

Theorem 4 ([4, Lemma 1]). *For $n \in \mathbb{N}$ and $u \in \mathbb{R}$, we have the integral representation*

$$\sum_{j=0}^{\infty} \frac{j+1}{(j+n+1)!} u^j = \frac{1}{(n-1)!} \int_0^1 v^{n-1}(1-v) \mathrm{e}^{u(1-v)} \, \mathrm{d}v. \tag{7}$$

We can reformulate the integral representation (7) as

$$\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n}} \frac{u^j}{j!} = n \int_0^1 v^{n-1}(1-v) \mathrm{e}^{u(1-v)} \, \mathrm{d}v, \tag{8}$$

for $n \in \mathbb{N}$ and $u \in \mathbb{R}$, as well as we can rewrite the integral representation (6) as

$$\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n+1}} \frac{u^j}{j!} = (n+1) \int_0^1 v^n \mathrm{e}^{u(1-v)} \, \mathrm{d}v, \tag{9}$$

for $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$. Differentiating with respect to u on both sides of (9) and replacing n by $n-1$ leads to (8). Accordingly, the integral representations from (6) to (9) are equivalent to each other.

In the papers [4,6], we established the following inequalities:

1. For $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, the inequality

$$\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+3}{n+1}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n+1}} \frac{u^j}{j!} \geq \left[\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+2}{n+1}} \frac{u^j}{j!} \right]^2, \tag{10}$$

is sound; see [6, Corollary 2].

2. For $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, the inequality

$$\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+2}{n+1}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n}{n}} \frac{u^j}{j!} < \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n+1}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n}} \frac{u^j}{j!}, \tag{11}$$

is valid; see [4, Theorem 1].

3. For $n \in \mathbb{N}$ and $u \in \mathbb{R}$, the inequality

$$\sum_{j=0}^{\infty} \frac{j+1}{(j+n+2)!} u^j \sum_{j=0}^{\infty} \frac{j+1}{(j+n)!} u^j < \left[\sum_{j=0}^{\infty} \frac{j+1}{(j+n+1)!} u^j\right]^2, \tag{12}$$

is sound; see [4, Lemma 3].

We notice that the inequality (12) can be rearranged as

$$\frac{n+1}{n+2} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+3}{n+2}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n}} \frac{u^j}{j!} < \left[\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+2}{n+1}} \frac{u^j}{j!}\right]^2, \tag{13}$$

for $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$.

As the inequalities from (11) to (13) are, as guessed in [7, Remark 1], the inequality (10) should be strict too. We will verify this strictness later in this paper.

In [2, Remark 7] and [4, Remark 3], influenced by the inequalities (10) and (11), Qi conjectured that, for given $n \in \mathbb{N}_0$, the functions

$$G_n(u) = \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n+1}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n}} \frac{u^j}{j!} - \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+2}{n+1}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n}{n}} \frac{u^j}{j!}, \tag{14}$$

and

$$H_n(u) = \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+3}{n+1}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n+1}} \frac{u^j}{j!} - \left[\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+2}{n+1}} \frac{u^j}{j!}\right]^2, \tag{15}$$

are absolutely monotonic in $u \in \mathbb{R}$, that is, $G_n^{(k)}(u) \ge 0$ and $H_n^{(k)}(u) \ge 0$ for $k, n \in \mathbb{N}_0$ and $u \in \mathbb{R}$. Similarly, we guess that the function

$$J_n(u) = \left[\sum_{j=0}^{\infty} \frac{1}{\binom{j+n+2}{n+1}} \frac{u^j}{j!}\right]^2 - \frac{n+1}{n+2} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+3}{n+2}} \frac{u^j}{j!} \sum_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n}} \frac{u^j}{j!}, \tag{16}$$

for $n \in \mathbb{N}_0$ is absolutely monotonic in $u \in \mathbb{R}$, that is, $J_n^{(k)}(u) \ge 0$ for $k, n \in \mathbb{N}_0$ and $u \in \mathbb{R}$.

These guesses are stronger than the inequalities from (10) to (13). For detailed information on the notion of absolutely monotonic functions and its generalizations such as the logarithmically absolutely monotonic functions, the logarithmically completely monotonic functions, and the completely monotonic degrees, please refer to the chapters [8, Chapter XIII] and [9, Chapter IV], the monograph [10], the papers [11–15], and closely-related references therein.

We observe that all of the Maclaurin series in the above-mentioned equations from (7) to (15) are special cases of the Maclaurin series

$$S_{m,n}(u) = \sum_{j=0}^{\infty} \frac{1}{\binom{j+m+n}{n}} \frac{u^j}{j!}, \tag{17}$$

for $m, n \in \mathbb{N}_0$ and $u \in \mathbb{R}$. It is easy to see that $S_{m,0}(u) = \mathrm{e}^u$ for $m \in \mathbb{N}_0$.

In this paper, we concentrate our attention on discussion and applications of the Maclaurin series $S_{m,n}(u)$.

2. Integral representations of a kind of Maclaurin series

In this section, we derive integral representations of the Maclaurin series $S_{m,n}(u)$ defined in (17) and its generalization.

Theorem 5. *For $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $u \in \mathbb{R}$, we have the integral representation*

$$S_{m,n}(u) = \sum_{j=0}^{\infty} \frac{1}{\binom{j+m+n}{n}} \frac{u^j}{j!} = n \int_0^1 v^{n-1}(1-v)^m \mathrm{e}^{u(1-v)} \,\mathrm{d}v. \tag{18}$$

In particular, for $u \in \mathbb{R}$ and $m, n \in \mathbb{N}_0$, we have

$$S_{m,n+1}(u) = T_n^{(m)}[\mathrm{e}^u]. \tag{19}$$

Consequently, the Maclaurin series $S_{m,n}(u)$ for $m, n \in \mathbb{N}_0$ and the normalized remainder $T_n[\mathrm{e}^u]$ for $n \in \mathbb{N}_0$ are absolutely monotonic functions of $u \in \mathbb{R}$.

Proof. Straightforward operation yields

$$\begin{aligned}
\int_0^1 v^{n-1}(1-v)^m \mathrm{e}^{u(1-v)} \,\mathrm{d}v &= \int_0^1 v^{n-1}(1-v)^m \left[\sum_{j=0}^{\infty} \frac{u^j(1-v)^j}{j!}\right] \mathrm{d}v \\
&= \sum_{j=0}^{\infty} \left[\int_0^1 v^{n-1}(1-v)^{j+m} \,\mathrm{d}v\right] \frac{u^j}{j!} \\
&= \sum_{j=0}^{\infty} B(j+m+1, n) \frac{u^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{\Gamma(j+m+1)\Gamma(n)}{\Gamma(j+m+n+1)} \frac{u^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{(j+m)!(n-1)!}{(j+m+n)!} \frac{u^j}{j!} \\
&= \frac{1}{n} \sum_{j=0}^{\infty} \frac{1}{\binom{j+m+n}{n}} \frac{u^j}{j!} \\
&= \frac{1}{n} S_{m,n}(u),
\end{aligned}$$

for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $u \in \mathbb{R}$, where the classical beta function

$$B(a, b) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \,\mathrm{d}t, \quad \Re(\alpha), \Re(\beta) > 0,$$

the classical Euler gamma function

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

and their relations can be found in [16, Chapter 3]. The integral representation (18) is thus proved.

The relation (19) follows differentiating $m \in \mathbb{N}_0$ times on both sides of the integral representation (6) in Theorem 3 and then comparing with the integral representation (18).

Differentiating $k \in \mathbb{N}_0$ times with respect to u at the very ends of the integral representation (18) gives

$$S_{m,n}^{(k)}(u) = n \int_0^1 v^{n-1}(1-v)^{m+k} \mathrm{e}^{u(1-v)} \,\mathrm{d}v > 0,$$

for $u \in \mathbb{R}$. So, the function $S_{m,n}(u)$ for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$ is absolutely monotonic in $u \in \mathbb{R}$. Moreover, it is easy to see that $S_{m,0}(u) = \mathrm{e}^u$ for $m \in \mathbb{N}_0$, which is an absolutely monotonic function in $u \in \mathbb{R}$. The proof of Theorem 5 is thus complete. □

Remark 1. The integral representation (6) in Theorem 3 and its equivalence (9) is the special case $S_{0,n}(u)$ in the integral representation (18) for $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

The integral representation (7) in Theorem 4 and its equivalence (8) is the special case $S_{1,n}(u)$ in the integral representation (18) for $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

Remark 2. Taking $m = 0, 1, 2$ and $n = 1, 2, 3$ in (18) gives the integral representations

$$\begin{aligned}
\frac{\mathrm{e}^u - 1}{\mathrm{e}^u u} &= \int_0^1 \mathrm{e}^{-uv} \,\mathrm{d}v, \\
\frac{\mathrm{e}^u - 1 - u}{\mathrm{e}^u u^2} &= \int_0^1 v \,\mathrm{e}^{-uv} \,\mathrm{d}v, \\
\frac{\mathrm{e}^u - 1 - u - \frac{u^2}{2!}}{\mathrm{e}^u u^3} &= \frac{1}{2} \int_0^1 v^2 \,\mathrm{e}^{-uv} \,\mathrm{d}v, \\
\frac{\mathrm{e}^u (u-1) + 1}{\mathrm{e}^u u^2} &= \int_0^1 (1-v) \,\mathrm{e}^{-uv} \,\mathrm{d}v, \\
\frac{\mathrm{e}^u (u-2) + u + 2}{\mathrm{e}^u u^3} &= \int_0^1 v(1-v) \,\mathrm{e}^{-uv} \,\mathrm{d}v, \\
\frac{2\,\mathrm{e}^u (u-3) + u^2 + 4u + 6}{\mathrm{e}^u u^4} &= \int_0^1 v^2(1-v) \,\mathrm{e}^{-uv} \,\mathrm{d}v, \\
\frac{\mathrm{e}^u \left(u^2 - 2 + 2\right) - 2}{\mathrm{e}^u u^3} &= \int_0^1 (1-v)^2 \,\mathrm{e}^{-uv} \,\mathrm{d}v, \\
\frac{\mathrm{e}^u \left(u^2 - 4u + 6\right) - 2u - 6}{\mathrm{e}^u u^4} &= \int_0^1 v(1-v)^2 \,\mathrm{e}^{-uv} \,\mathrm{d}v,
\end{aligned}$$

and

$$\frac{\mathrm{e}^u \left(u^2 - 6u + 12\right) - u^2 - 6u - 12}{\mathrm{e}^u u^5} = \frac{1}{2} \int_0^1 v^2(1-v)^2 \,\mathrm{e}^{-uv} \,\mathrm{d}v,$$

for $u \neq 0$. These functions are completely monotonic in $u \in \mathbb{R}$. A function $f(u)$ is said to be completely monotonic on an interval I if $(-1)^k f^{(k)}(u) \geq 0$ for all $k \in \mathbb{N}_0$ and $u \in I$. For detailed information, please refer to the chapters [8, Chapter XIII] and [9, Chapter IV], the monograph [10], the papers [11,13,14], and closely-related references therein.

In [16, p. 73], the classical binomial numbers $\binom{n}{k}$ for $k, n \in \mathbb{N}_0$ are generalized to

$$\binom{z}{w} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, \quad z \neq -1, -2, \dots,$$

for general complex values of w and z. See also [17, p. 711] and [18, Section 1].

We now generalize the integral representation (18) in Theorem 5 as follows.

Theorem 6. *For $\Re(\alpha) > -1$, $\Re(\beta) > 0$, and $u \in \mathbb{R}$, we have the integral representation*

$$S_{\alpha,\beta}(u) = \sum_{j=0}^{\infty} \frac{1}{\binom{j+\alpha+\beta}{\beta}} \frac{u^j}{j!} = \beta \int_0^1 v^{\beta-1} (1-v)^\alpha \,\mathrm{e}^{u(1-v)} \,\mathrm{d}v. \tag{20}$$

Consequently, for $\alpha \in (-1, \infty)$ and $\beta \in (0, \infty)$, the function $S_{\alpha,\beta}(u)$ is absolutely monotonic in $u \in \mathbb{R}$.

Proof. Direct computation gives

$$\begin{aligned}
\int_0^1 v^{\beta-1} (1-v)^\alpha \,\mathrm{e}^{u(1-v)} \,\mathrm{d}v &= \int_0^1 v^{\beta-1} (1-v)^\alpha \left[\sum_{j=0}^{\infty} \frac{u^j (1-v)^j}{j!} \right] \mathrm{d}v \\
&= \sum_{j=0}^{\infty} \left[\int_0^1 v^{\beta-1} (1-v)^{j+\alpha} \,\mathrm{d}v \right] \frac{u^j}{j!} \\
&= \sum_{j=0}^{\infty} B(j+\alpha+1, \beta) \frac{u^j}{j!}
\end{aligned}$$

$$= \sum_{j=0}^{\infty} \frac{\Gamma(j+\alpha+1)\Gamma(\beta)}{\Gamma(j+\alpha+\beta+1)} \frac{u^j}{j!}$$
$$= \frac{1}{\beta} \sum_{j=0}^{\infty} \frac{1}{\binom{j+\alpha+\beta}{\beta}} \frac{u^j}{j!}$$
$$= \frac{1}{\beta} S_{\alpha,\beta}(u),$$

for $\Re(\alpha) > -1$, $\Re(\beta) > 0$, and $u \in \mathbb{R}$.

Differentiating $k \in \mathbb{N}_0$ times with respect to u at the very ends of the integral representation (20) gives

$$S_{\alpha,\beta}^{(k)}(u) = \beta \int_0^1 v^{\beta-1}(1-v)^{\alpha+k} e^{u(1-v)} \, dv > 0,$$

for $u \in \mathbb{R}$. Consequently, the function $S_{\alpha,\beta}(u)$ for $\alpha \in (-1, \infty)$ and $\beta \in (0, \infty)$ is absolutely monotonic in $u \in \mathbb{R}$. The proof of Theorem 6 is thus complete. □

Remark 3. The Dawson integral is defined [19, p. 298] by

$$F(u) = \frac{1}{e^{u^2}} \int_0^u e^{t^2} \, dt, \quad x \in \mathbb{R}.$$

Taking $\alpha = \pm\frac{1}{2}$ and $\beta = 1$ in the integral representation (20) results in

$$\frac{\sqrt{u} - F(\sqrt{u})}{u^{3/2}} = \int_0^1 \sqrt{1-v} \, e^{-uv} \, dv \quad \text{and} \quad \frac{F(\sqrt{u})}{\sqrt{u}} = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{1-v}} e^{-uv} \, dv,$$

respectively. These two functions are completely monotonic in $u \in \mathbb{R}$.

3. Integral representations of inequalities for series

In light of the integral representation (18) in Theorem 5, we can reformulate the inequality (10) for $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, the inequality (11) for $n \in \mathbb{N}$ and $u \in \mathbb{R}$, the inequality (13) for $n \in \mathbb{N}$ and $u \in \mathbb{R}$, and their equivalences as follows.

1. For $u \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$\int_0^1 v^n (1-v)^2 e^{-uv} \, dv \int_0^1 v^n e^{-uv} \, dv \geq \left[\int_0^1 v^n (1-v) e^{-uv} \, dv \right]^2. \tag{21}$$

2. For $u \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\int_0^1 v^n (1-v) e^{-uv} \, dv \int_0^1 v^{n-1} e^{-uv} \, dv < \int_0^1 v^n e^{-uv} \, dv \int_0^1 v^{n-1} (1-v) e^{-uv} \, dv. \tag{22}$$

3. For $u \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\frac{n}{n+1} \int_0^1 v^{n+1} (1-v) e^{-uv} \, dv \int_0^1 v^{n-1} (1-v) e^{-uv} \, dv < \left[\int_0^1 v^n (1-v) e^{-uv} \, dv \right]^2. \tag{23}$$

The Čebyšev integral inequality [8, p. 239, Chapter IX] reads that,

1. if $f, h : I \to \mathbb{R}$ are two integrable functions, either both increasing or both decreasing, then

$$\int_I p(v) \, dv \int_I p(v) f(v) h(v) \, dv \geq \int_I p(v) f(v) \, dv \int_I p(v) h(v) \, dv, \tag{24}$$

where $I \subseteq \mathbb{R}$ is an interval and $p : I \to [0, \infty)$ is a non-negative and integral function;

2. if one of the functions f and h is non-increasing and the other non-decreasing, then the inequality in (24) is reversed;

3. the equality in (24) validates if and only if one of the functions f and h reduces to a scalar.

Taking $p(v) = v^n \mathrm{e}^{-uv}$ for $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, $f(v) = h(v) = 1 - v$, and $I = [0, 1]$ in (24) results in the strict version of the integral inequality (21). Letting $p(v) = v^{n-1} \mathrm{e}^{-uv}$ for $n \in \mathbb{N}$ and $u \in \mathbb{R}$, $f(v) = v$, $h(v) = 1 - v$, and $I = [0, 1]$ in (24) leads to the integral inequality (22). Accordingly, we alternatively proved the inequalities (21) and (22), that is, the inequality (10) for $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$ and the inequality (11) for $n \in \mathbb{N}$ and $u \in \mathbb{R}$, by a different method from those in the papers [4,6,7].

Setting $p(v) = v^{n-1}(1 - v) \mathrm{e}^{-uv}$ for $n \in \mathbb{N}$ and $u \in \mathbb{R}$, $f(v) = h(v) = v$, and $I = [0, 1]$ in (24) arrives at the inequality

$$\int_0^1 v^{n+1}(1-v)\,\mathrm{e}^{-uv}\,\mathrm{d}v \int_0^1 v^{n-1}(1-v)\,\mathrm{e}^{-uv}\,\mathrm{d}v > \left[\int_0^1 v^{n}(1-v)\,\mathrm{e}^{-uv}\,\mathrm{d}v\right]^2, \tag{25}$$

for $n \in \mathbb{N}$ and $u \in \mathbb{R}$, which is the reversed version of the inequality (23).

In terms of the notation $S_{m,n}(u)$, we sum up and restate the above inequalities between (10) and (25) as follows.

Theorem 7. *For $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, we have*

$$S_{2,n+1}(u)S_{0,n+1}(u) > S_{1,n+1}^2(u) \quad \textit{and} \quad S_{1,n+1}(u)S_{0,n}(u) < S_{0,n+1}(u)S_{1,n}(u).$$

For $n \in \mathbb{N}$ and $u \in \mathbb{R}$, we have

$$\frac{n+1}{n+2} S_{1,n+2}(u)S_{1,n}(u) < S_{1,n+1}^2(u) < \frac{n+1}{n}\frac{n+1}{n+2} S_{1,n+2}(u)S_{1,n}(u). \tag{26}$$

The left inequality in (26) is also valid for $n = 0$.

4. Several limits of ratios

At the site https://math.stackexchange.com/q/4956563 and in [4, Remark 1], Qi posed the following two problems on limits:

1. For given $u \in \mathbb{R}$, what is the limit

$$\lim_{n\to\infty} \frac{\sum\limits_{j=0}^{\infty} \frac{1}{\binom{j+n+1}{n}} \frac{u^j}{j!}}{\sum\limits_{j=0}^{\infty} \frac{1}{\binom{j+n}{n}} \frac{u^j}{j!}}?$$

2. For given $n \in \mathbb{N}_0$ and $u \in \mathbb{R}$, what is the limit

$$\lim_{m\to\infty} \frac{\sum\limits_{j=0}^{\infty} \frac{1}{\binom{j+n+m+1}{n+1}} \frac{u^j}{j!}}{\sum\limits_{j=0}^{\infty} \frac{1}{\binom{j+n+m}{n+1}} \frac{u^j}{j!}}?$$

See also [2, Section 2].

In terms of the notation $S_{m,n}(u)$ defined by (17), we can reformulate these two limits as

$$\lim_{n\to\infty} \frac{S_{1,n}(u)}{S_{0,n}(u)} \quad \text{and} \quad \lim_{m\to\infty} \frac{S_{m,n}(u)}{S_{m-1,n}(u)},$$

for $m, n \in \mathbb{N}$ and $u \in \mathbb{R}$. Further utilizing the integral representation (18) in Theorem 5, we conclude the following problem.

Problem 1. For $m, n \in \mathbb{N}$, calculate the limits

$$\lim_{n\to\infty} \frac{S_{m,n}(u)}{S_{m-1,n}(u)} = \lim_{n\to\infty} \frac{\int_0^1 v^{n-1}(1-v)^m\,\mathrm{e}^{-uv}\,\mathrm{d}v}{\int_0^1 v^{n-1}(1-v)^{m-1}\,\mathrm{e}^{-uv}\,\mathrm{d}v},$$

and
$$\lim_{m\to\infty}\frac{S_{m,n}(u)}{S_{m-1,n}(u)}=\lim_{m\to\infty}\frac{\int_0^1 v^{n-1}(1-v)^m\,\mathrm{e}^{-uv}\,\mathrm{d}v}{\int_0^1 v^{n-1}(1-v)^{m-1}\,\mathrm{e}^{-uv}\,\mathrm{d}v}.$$

In terms of the integral representation (20) in Theorem 6, we can generalize Problem 1 as follows.

Problem 2. For $\Re(\alpha)>0$, $\Re(\beta)>0$, and $u\in\mathbb{R}$, compute the limits
$$\lim_{\beta\to\infty}\frac{S_{\alpha,\beta}(u)}{S_{\alpha-1,\beta}(u)}=\lim_{\beta\to\infty}\frac{\int_0^1 v^{\beta-1}(1-v)^{\alpha}\,\mathrm{e}^{-uv}\,\mathrm{d}v}{\int_0^1 v^{\beta-1}(1-v)^{\alpha-1}\,\mathrm{e}^{-uv}\,\mathrm{d}v},$$
and
$$\lim_{\alpha\to\infty}\frac{S_{\alpha,\beta}(u)}{S_{\alpha-1,\beta}(u)}=\lim_{\alpha\to\infty}\frac{\int_0^1 v^{\beta-1}(1-v)^{\alpha}\,\mathrm{e}^{-uv}\,\mathrm{d}v}{\int_0^1 v^{\beta-1}(1-v)^{\alpha-1}\,\mathrm{e}^{-uv}\,\mathrm{d}v}.$$

5. Absolute monotonicity

In terms of the notation $S_{m,n}(u)$, we can reformulate the functions from (14) to (16) as
$$\begin{aligned}G_n(u)&=S_{0,n+1}(u)S_{1,n}(u)-S_{1,n+1}(u)S_{0,n}(u)\\&=-\begin{vmatrix}S_{0,n}(u)&S_{0,n+1}(u)\\S_{1,n}(u)&S_{1,n+1}(u)\end{vmatrix},\\H_n(u)&=S_{2,n+1}(u)S_{0,n+1}(u)-S_{1,n+1}^2(u),\end{aligned}$$
and
$$J_n(u)=S_{1,n+1}^2(u)-\frac{n+1}{n+2}S_{1,n+2}(u)S_{1,n}(u),$$
for $n\in\mathbb{N}_0$ and $u\in\mathbb{R}$. Based on the right inequality in (26), we introduce another function
$$K_n(u)=\frac{n+1}{n+2}S_{1,n+2}(u)S_{1,n}(u)-\frac{n}{n+1}S_{1,n+1}^2(u),$$
for $n\in\mathbb{N}_0$ and $u\in\mathbb{R}$.

The function $G_n(u)$ can be generalized as
$$G_{m,n;k,\ell}(u)=-\begin{vmatrix}S_{m,n}(u)&S_{m,n+\ell}(u)\\S_{m+k,n}(u)&S_{m+k,n+\ell}(u)\end{vmatrix},\tag{27}$$
for $m,n\in\mathbb{N}_0$ and $k,\ell\in\mathbb{N}$.

Let $\boldsymbol{s}_n=(s_1,s_2,\dots,s_n)$ and $\boldsymbol{t}_n=(t_1,t_2,\dots,t_n)\in\mathbb{R}^n$. A n-tuple $\boldsymbol{s}_n$ is said to strictly majorize $\boldsymbol{t}_n$, denoted by $\boldsymbol{t}_n\prec\boldsymbol{s}_n$ or $\boldsymbol{s}_n\succ\boldsymbol{t}_n$, if
$$(s_{[1]},s_{[2]},\dots,s_{[n]})\ne(t_{[1]},t_{[2]},\dots,t_{[n]}),\qquad\sum_{i=1}^n s_i=\sum_{i=1}^n t_i,$$
and
$$\sum_{i=1}^k s_{[i]}>\sum_{i=1}^k t_{[i]},\quad 1\le k\le n-1,$$
are valid, where
$$s_{[1]}\ge s_{[2]}\ge\dots\ge s_{[n]}\quad\text{and}\quad t_{[1]}\ge t_{[2]}\ge\dots\ge t_{[n]},$$
are rearrangements of $\boldsymbol{s}_n$ and $\boldsymbol{t}_n$ in a descending order. See [20, p. 8, Defnition A.1].

Making use of the notion of majorization, we generalize the functions $H_n(u)$, $J_n(u)$, and $K_n(u)$ as follows:

1. For $n \in \mathbb{N}_0$ and $(s_1, s_2), (t_1, t_2) \in \mathbb{N}_0^2$ such that $(s_1, s_2) \succ (t_1, t_2)$, define

$$H_{n;s_2,t_2}(u) = S_{s_1,n+1}(u)S_{s_2,n+1}(u) - S_{t_1,n+1}(u)S_{t_2,n+1}(u). \tag{28}$$

2. For $m \in \mathbb{N}_0$ and $(s_1, s_2), (t_1, t_2) \in \mathbb{N}_0^2$ such that $(s_1, s_2) \succ (t_1, t_2)$, define

$$J_{m;s_2,t_2}(u) = S_{m,t_1}(u)S_{m,t_2}(u) - A(m; s_1, s_2; t_1, t_2)S_{m,s_1}(u)S_{m,s_2}(u). \tag{29}$$

3. For $m \in \mathbb{N}_0$ and $(s_1, s_2), (t_1, t_2) \in \mathbb{N}_0^2$ such that $(s_1, s_2) \succ (t_1, t_2)$, define

$$K_{m;s_2,t_2}(u) = S_{m,s_1}(u)S_{m,s_2}(u) - B(m; s_1, s_2; t_1, t_2)S_{m,t_1}(u)S_{m,t_2}(u). \tag{30}$$

Problem 3. For $m, n \in \mathbb{N}_0$ and $k, \ell \in \mathbb{N}$, the function $G_{m,n;k,\ell}(u)$ defined by (27) is absolutely monotonic in $u \in \mathbb{R}$.

For $n \in \mathbb{N}_0$ and $(s_1, s_2), (t_1, t_2) \in \mathbb{N}_0^2$ such that $(s_1, s_2) \succ (t_1, t_2)$, the function $H_{n;s_2,t_2}(u)$ defined by (28) is absolutely monotonic in $u \in \mathbb{R}$.

For $m \in \mathbb{N}_0$ and $(s_1, s_2), (t_1, t_2) \in \mathbb{N}_0^2$ such that $(s_1, s_2) \succ (t_1, t_2)$, what are the best constants $0 < A(m; s_1, s_2; t_1, t_2) < 1$ and $0 < B(m; s_1, s_2; t_1, t_2) < 1$ such that the functions $J_{m;s_2,t_2}(u)$ and $K_{m;s_2,t_2}(u)$ defined by (29) and (30) respectively are absolutely monotonic in $u \in \mathbb{R}$?

Author Contributions: All authors contributed equally to the manuscript and read and approved the final manuscript.

Funding Information: The authors were partially supported by the Youth Project of Hulunbuir City for Basic Research and Applied Basic Research (Grant No. GH2024020) and by the Natural Science Foundation of Inner Mongolia Autonomous Region (Grant No. 2025QN01041).

References

[1] Gradshteyn, I. S., & Ryzhik, I. M. (2014). *Table of Integrals, Series, and Products.* Academic press.
[2] Qi, F. (2024). Absolute monotonicity of normalized tail of power series expansion of exponential function. *Mathematics, 12*(18), 2859.
[3] Zhang, G. Z., & Qi, F. E. N. G. (2024). On convexity and power series expansion for logarithm of normalized tail of power series expansion for square of tangent. *Journal of Mathematical Inequalities, 18*(3), 937-952.
[4] Zhang, T., & Qi, F. (2025). Decreasing ratio between two normalized remainders of Maclaurin series expansion of exponential function. *AIMS Mathematics, 10*(6), 14739-14756.
[5] Alzer, H. O. R. S. T., Brenner, J. L., & Ruehr, O. G. (1993). Inequalities for the tails of some elementary series. *Journal of Mathematical Analysis and Applications, 179*(2), 500-506.
[6] Bao, Z. H., Agarwal, R. P., Qi, F., & Du, W. S. (2024). Some properties on normalized tails of Maclaurin power series expansion of exponential function. *Symmetry, 16*(8), 989.
[7] Li, Y. W., & Qi, F. (2025). Elegant proofs for properties of normalized remainders of Maclaurin power series expansion of exponential function. *Mathematica Slovaca, 75*(5), 1035-1044.
[8] Mitrinovic, D. S., Pecaric, J., & Fink, A. M. (2013). *Classical and New Inequalities in Analysis* (Vol. 61). Springer Science & Business Media.
[9] Bellman, R., & Roth, R. S. (1984). *The Laplace Transform* (Vol. 3). World Scientific.
[10] Schilling, R. L., Song, R., & Vondracek, Z. (2012). *Bernstein Functions: Theory and Applications* (Vol. 37). Walter de Gruyter.
[11] Berg, C. (2004). Integral representation of some functions related to the gamma function. *arXiv preprint math/0411550.*
[12] Guo, B. N., & Qi, F. (2010). A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, 72*(2), 21-30.
[13] Mahmoud, M. M. M., & Qi, F. (2022). Bounds for completely monotonic degrees of remainders in asymptotic expansions of the digamma function. *Mathematical Inequalities & Applications, 25*(1), 291-308.
[14] Qi, F. (2021). Bounds for completely monotonic degree of a remainder for an asymptotic expansion of the trigamma function. *Arab Journal of Basic and Applied Sciences, 28*(1), 314-318.
[15] Qi, F., & Chen, C. P. (2004). A complete monotonicity property of the gamma function. *Journal of Mathematical Analysis and Applications, 296*(2), 603-607.

[16] Temme, N. M. (1996). *Special Functions: An Introduction to the Classical Functions of Mathematical Physics.* John Wiley & Sons.

[17] Qi, F. (2022). Taylor's series expansions for real powers of two functions containing squares of inverse cosine function, closed-form formula for specific partial Bell polynomials, and series representations for real powers of Pi. *Demonstratio Mathematica, 55*(1), 710-736.

[18] Wei, C. F. (2022). Integral representations, monotonic properties, and inequalities on extended central binomial coefficient. *Mathematical Methods in the Applied Sciences, 45*(9), 5412-5422.

[19] Abramowitz, M., & Stegun, I. A. (Eds.). (1948). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Vol. 55). US Government printing office.

[20] Marshall, A. W., Olkin, I., & Arnold, B. C. (1979). *Inequalities: Theory of Majorization and Its Applications*. Part of the book series: Springer Series in Statistics.