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Localized curvature domination and rigidity of Harmonic maps

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Abstract: We establish a localized Bochner-type rigidity theorem for harmonic maps between Riemannian manifolds. Let $f : (M, g) \to (\hat{M}, \hat{g})$ be a harmonic map from a compact manifold. Instead of assuming global nonpositivity of the sectional curvature of the target manifold, we impose a curvature bound localized along the image $f(M)$, expressed in terms of the maximal sectional curvature encountered along this image. We prove that if the minimal Ricci curvature of the domain dominates this image–dependent curvature bound through a quantitative curvature pinching inequality involving the maximal energy density of f, then the map must be constant. In the critical case of equality, we obtain a homothetic classification: the differential of f is parallel and the image $f(M)$ is totally geodesic. Thus, the theorem replaces global curvature sign assumptions by an image–dependent curvature domination principle and provides a localized analogue of classical Yano–Ishihara–type rigidity results.

Keywords: Riemannian manifold, harmonic map, quantitative curvature pinching inequality, rigidity theory

MSC: 53C20, 53C43.

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1. Introduction

Harmonic maps between Riemannian manifolds play a fundamental role in geometric analysis, lying at the intersection of differential geometry, nonlinear partial differential equations, and mathematical physics (see, for example, [1]; [2] and [3]).

In their foundational work, Eells and Sampson showed in [4] that if the target manifold has nonpositive sectional curvature, harmonic maps possess strong existence and rigidity properties. Later results of Schoen and Yau (see [5]) established Liouville-type theorems under nonnegative Ricci curvature of the domain and nonpositive sectional curvature of the target. Yano and Ishihara obtained related rigidity results under curvature pinching conditions (see [6]).

A common feature of these results is the use of *global curvature sign conditions on the entire target manifold*. However, the Bochner identity actually involves curvature terms evaluated only along the differential of the map. Consequently, the geometry of regions of the target manifold that are never reached by the map should not influence the rigidity mechanism.

The purpose of the present paper is to formulate a *localized curvature domination principle* in which curvature control is required only along the image of the harmonic map. Instead of imposing a global curvature sign condition on the target manifold, we introduce the quantity $Sec_{max}(f(M))$, defined as the maximal sectional curvature of the target manifold taken over all two-planes based at points of the image $f(M)$.

We prove that if the minimal Ricci curvature of the compact domain dominates this image-dependent curvature bound in a suitable quantitative inequality involving the maximal energy density of the map, then the harmonic map must be constant. At the critical threshold we obtain a structural rigidity statement: the differential of the map is parallel and the image is totally geodesic.

Thus, the classical Bochner rigidity mechanism can be formulated in terms of curvature bounds along the image of the harmonic map, without imposing any global sign condition on the sectional curvature of the target manifold.

The result replaces global curvature sign assumptions with an image-dependent curvature domination principle and yields a localized analogue of classical Bochner-type rigidity results of Eells–Sampson, Schoen–Yau, and Yano–Ishihara (see [4–6]).

In the present paper we continue the study which we began in [7] where we proved some vanishing theorems for harmonic mappings into non-negatively curved manifolds and studied their applications.

The paper is organized as follows. In §2 we introduce the necessary notation and curvature quantities. §3 contains the main rigidity theorem and its proof. In the final section we present a geometric example illustrating the localization principle.

2. Preliminaries

Let (M, g) be a connected compact Riemannian manifold without boundary and let $(\overline{M},\, \overline{g})$ be a Riemannian manifold. Let $f : (M,\, g) \to (\overline{M},\, \overline{g})$ be a smooth map.

The *energy density* of f is defined by $e(f) = \frac{1}{2}|df|^2$. Denote $e_{max} = sup_M\, e(f)$. Since M is compact, e_{max} is finite and attained.

Let $Ric_{min}(M)$ denote the minimal Ricci curvature of the domain, defined as

$$Ric_{min}(M) = \min_{p\in M} \min_{\lfloor X\rfloor=1} (Ric(X, X)).$$

Since M is compact, this minimum is attained at some point of M.

On the other hand, because $f(M)$ is compact, the sectional curvature of the target manifold $(\overline{M},\, \overline{g})$ attains its maximum along the image. We define $\overline{Sec}_{max}(f(M))$ as the maximal sectional curvature of $(\overline{M},\, \overline{g})$ taken over all two-planes based at points of the image $f(M)$. Hence the quantity

$$\overline{Sec}_{max}(f(M)) := \max_{\substack{p\,\in\, f(M)\\ \sigma\,\subset\, T_p\overline{M}}} \overline{Sec}(\sigma),$$

is well defined and achieved for some two-plane $\sigma \subset T_p\overline{M}$ with $p \in f(M)$.

3. Main theorem

Theorem 1 (Rigidity under Localized Curvature Domination)**.** *Let $f : (M, g) \longrightarrow (\hat{M}, \hat{g})$ be a smooth harmonic map from a connected compact boundaryless n-dimensional Riemannian manifold into a connected Riemannian manifold. Denote by $Ric_{min}(M)$ the minimal Ricci curvature of M, and define $\overline{Sec}_{max}(f(M))$ as the maximal sectional curvature of the target manifold taken over all two-planes based at points of the image $f(M)$. Let $e(f)$ denote the energy density*

$$e\,(f) := \frac{1}{2}\;|\,df|^2,$$

and define $e_{max} := sup_M\, e(f)$. Assume that

$$Ric_{min}(M) > \frac{2(n-1)}{n}\overline{Sec}_{max}(f(M))e_{max}.$$

Then the harmonic map f is constant.

Proof. For a harmonic map f the Bochner–Weitzenböck identity reads (see [4–6])

$$\frac{1}{2}\Delta|df|^2 = |\nabla df|^2 + Q(f),$$

where the curvature term is

$$Q(f) = \sum_i Ric(e_i, e_i)|df(e_i)|^2 - \sum_{i,j}\langle \hat{R}(df(e_i), df(e_j))df(e_j), df(e_i)\rangle,$$

for a local orthonormal frame $\{e_1, \dots, e_n\}$ on M. Using the lower bound for the Ricci curvature of the domain and the definition of the maximal sectional curvature along $f(M)$ the image we can obtain the following inequality

$$Q(f) \geq |df|^2 \left(-Ric_{min}(M) \ \middle|\ \frac{n-1}{n} \overline{Sec}_{max}(f(M))|df|^2 \right).$$

Namely, let $\{e_1, \dots, e_n\}$ be a local orthonormal frame around a point $p \in M$ such that the pullback metric $g^* := f^*\hat{g}$ is diagonal at p, i.e. $\hat{g}(df(e_i), df(e_j)) = \lambda_i \delta_{ij}$ for $\lambda_i \geq 0$ and $\sum\limits_{i=1}^{n} \lambda_i = tr_g(g^*) = |df|^2$. Then by the definition of Ric_{inf}, we have

$$\sum_{i=1}^{n} Ric\,(e_i, e_i)\,|df\,(e_i)\,|^2 \geq Ric_{min} \sum_{i=1}^{n} \lambda_i = Ric_{min}|df|^2.$$

On the other hand, let $\bar{e}_i := df(e_i)$. Since $\hat{g}(\bar{e}_i, \bar{e}_j) = 0$ for $i \neq j$ and $|\bar{e}_i|^2 = \lambda_i$, we have for $i \neq j$

$$\left\langle \hat{R}\,(\bar{e}_i, \bar{e}_j)\,\bar{e}_j, \bar{e}_i \right\rangle = \overline{Sec}\ (\bar{e}_i, \bar{e}_j)\,|\bar{e}_i|^2 |\bar{e}_i|^2 = \overline{Sec}\,(\bar{e}_i, \bar{e}_j)\lambda_i \lambda_j.$$

Using the identity

$$\sum_{i<j} \lambda_i \lambda_j = \frac{1}{2} \left(\left(\sum_{i=1}^{n} \lambda_i \right)^2 - \sum_{i=1}^{n} \lambda_i^2 \right) \leq \frac{n-1}{2n} \left(\sum_{i=1}^{n} \lambda_i \right)^2,$$

where the inequality follows from Cauchy–Schwarz $\sum \lambda_i^2 \geq \frac{1}{n}(\sum \lambda_i)^2$, we obtain

$$\left\langle \hat{R}\,(\bar{e}_i, \bar{e}_j)\,\bar{e}_j, \bar{e}_i \right\rangle \leq 2\overline{Sec}_{max} \cdot \frac{n-1}{2n} \left(\sum_{i=1}^{n} \lambda_i \right)^2 = \frac{n-1}{n} \overline{Sec}_{max} |df|^4.$$

In turn, to estimate the curvature term coming from the target manifold, we use the definition of the maximal sectional curvature along the image $f(M)$ of the map. By definition, $\overline{Sec}_{max}\,(f\,(M))$ is the maximal sectional curvature of the target manifold taken over all two-dimensional planes based at points of the image $f(M)$. Therefore, for every two-plane σ based at a point of $f(M)$ one has

$$\overline{Sec}(\sigma) \ \leq\ \overline{Sec}_{max}(f(M)).$$

In the Bochner formula the curvature term of the target manifold appears with a minus sign. Consequently, multiplying the above inequality by -1 reverses the direction of the inequality and gives

$$-\ \overline{Sec}(\sigma) \geq -\ \overline{Sec}_{max}(f(M)).$$

Applying this estimate to each sectional curvature term occurring in the Bochner expression yields

$$Q(f) \geq |df|^2 \left(-Ric_{min}(M) \ \middle|\ \frac{n-1}{n} \overline{Sec}_{max}(f(M))|df|^2 \right).$$

This estimate is valid independently of the sign of the sectional curvature along the image.

If the sectional curvature along the image $f(M)$ is nonnegative, then $\overline{Sec}_{max}(f(M)) \geq 0$ and the second term in the parentheses represents a nonnegative curvature contribution.

If the sectional curvature along the image is negative, then $\overline{Sec}_{max}\,(f\,(M)) < 0$ and the second term becomes positive, which only strengthens the lower bound for $Q(f)$.

Thus the argument requires only the existence of the upper bound $\overline{Sec}_{max}\,(f\,(M))$ and does not require the sectional curvature along the image to be nonnegative.

Since $e(f) = \frac{1}{2}|df|^2$ this inequality can be written as

$$Q(f) \geq 2e(f) \left(Ric_{min}\,(M) - \frac{2\,(n-1)}{n} \overline{Sec}_{max}(f(M))\; e(f) \right).$$

Integrating the Bochner identity over the compact manifold M yields

$$\int_M (|\nabla df|^2 + Q(f))dV = 0.$$

If we suppose that

$$Ric_{min}(M) > \frac{2(n-1)}{n}\overline{Sec}_{max}(f(M))\ e_{max},$$

then the integrand is nonnegative everywhere. In this case $\nabla df \equiv 0 \ \ and \ \ Q(f) \equiv 0 \ \ \ on \ \ M$. In particular, $|df|^2$ is constant. Under the strict inequality, the identity $Q(f) \equiv 0$ forces $|df|^2 \equiv 0$, hence $df \equiv 0$ and f is constant. This completes the proof. □

Remark 1 (Threshold case)**.** Assume that equality holds

$$Ric_{min}(M) = \frac{2(n-1)}{n}\overline{Sec}_{max}(f(M))\ e_{max}.$$

Then the Bochner identity implies $\nabla df = 0$, so the differential of f is parallel and the energy density is constant. At the same time, the Cauchy–Schwarz inequality implies $\lambda_1 = \lambda_2 = ... = \lambda_n$. Thus, all singular values of df coincide and the differential acts as a homothety on the orthogonal complement of its kernel. Since $\nabla df = 0$, the second fundamental form of the image vanishes. Therefore, the image $f(M)$ is totally geodesic in the target manifold.

4. Geometric example illustrating the localization principle

We now present a geometric example illustrating the localized nature of the curvature domination condition. Consider the product manifold $\hat{M} = \mathbb{S}^2 \times \mathbb{H}^2$, where $\mathbb{S}^2$ denotes the standard sphere with constant sectional curvature equal to 1 and $\mathbb{H}^2$, denotes the hyperbolic plane with constant sectional curvature equal to -1.

Therefore, the sectional curvature of $\mathbb{S}^2 \times \mathbb{H}^2$ takes both positive and negative values.

Let (M, g) be a compact Riemannian manifold with positive Ricci curvature and let $\phi : (M, g) \to \mathbb{H}^2$ be a harmonic map Define a map $f : M \to \mathbb{S}^2 \times \mathbb{H}^2$ by $f(p) = (p_0 , \phi(p))$, where p_0 is a fixed point of the sphere $\mathbb{S}^2$.

In this case the image of f is contained in the submanifold $\{p_0\} \times \mathbb{H}^2$. Therefore, the maximal sectional curvature encountered along the image satisfies

$$\overline{Sec}_{max}(f(M)) \leq 0.$$

Hence the curvature domination condition of the main theorem is automatically satisfied if $Ric_{min}(M) > 0$. Consequently, the harmonic map f must be constant.

However, the ambient target manifold $\mathbb{S}^2 \times \mathbb{H}^2$ contains regions of positive sectional curvature coming from the sphere factor $\mathbb{S}^2$. These positively curved regions do not influence the rigidity result because they lie outside the image of the map.

Remark 2. The above example demonstrates that rigidity depends only on the curvature encountered along the image of the harmonic map rather than on the global curvature properties of the target manifold.

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