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$\mathcal{I}$-Statistical pre-Cauchy and frequent Cauchy criteria for compact sets in Hausdorff hyperspaces

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Abstract: In this paper, we study pairwise asymptotic criteria for sequences of compact sets in the Hausdorff hyperspace of a metric space. We introduce the notions of $\mathcal{I}$-statistically pre-Cauchy and frequent Cauchy compact-set sequences and examine their relation to convergence under suitable additional assumptions. After establishing basic permanence properties, we show that $\mathcal{I}$-statistical convergence implies the corresponding pre-Cauchy condition. For bounded sequences, we obtain two explicit characterizations, one in terms of double means of Hausdorff distances and the other in terms of bounded moduli. We also investigate the relation between frequent Cauchy behavior and frequent convergence under compactness assumptions on the hyperspace or on the closure of the range of the sequence. Finally, we present criteria under which pairwise asymptotic conditions can be upgraded to actual $\mathcal{I}$-statistical convergence, and we include examples that illustrate both the scope and the limitations of the pairwise approach in the hyperspace setting.

Keywords: Hausdorff hyperspace, compact sets, $\mathcal{I}$-statistical convergence, $\mathcal{I}$-statistically pre-Cauchy sequence, frequent convergence, frequent Cauchy sequence, bounded modulus

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1. Introduction

The Hausdorff hyperspace of nonempty compact subsets of a metric space provides a natural framework for the study of set-valued approximation, hyperspace dynamics, and compact-set iterations [1–3]. Once a metric space (X, d) is fixed, the family $\mathcal{K}(X)$ of all nonempty compact subsets of X, endowed with the Hausdorff distance, becomes a metric space in its own right. This makes it possible to formulate asymptotic questions for sequences of compact sets in a precise and flexible setting [4].

In many situations, ordinary convergence is too rigid to capture the long-term behavior of a compact-set sequence. Exceptional indices may occur sparsely yet persistently, and in such cases pairwise closeness can be more informative than a formulation based on a prescribed limit. For this reason, density-based and ideal-based methods have proved useful in asymptotic analysis. Various forms of ideal convergence, ideal Cauchy conditions, and localized statistical behavior have been investigated in different contexts, including metric and operator-theoretic settings [5–7].

For scalar sequences, the relation between statistical convergence, statistical Cauchy conditions, and statistical pre-Cauchy behavior has been examined in considerable detail [8–10]. Related questions involving ideal Cauchy sequences, localized statistical behavior, and operator-theoretic aspects of statistical convergence have also been studied from several perspectives [5–7]. In particular, every statistically convergent sequence is statistically pre-Cauchy, whereas the converse usually requires additional assumptions. For bounded sequences, one also has useful pairwise criteria expressed through double means and bounded moduli, both of which avoid prescribing the limit in advance [8,11]. These results indicate that pairwise formulations often retain the essential asymptotic information while remaining less rigid than direct limit tests.

A related viewpoint arises in the theory of frequent convergence. There the relevant object is not only the density of exceptional one-dimensional index sets, but also the frequency measure of subsets of $\mathbb{N} \times \mathbb{N}$ [12,13]. This leads naturally to the notion of a frequent Cauchy sequence, and in the scalar case frequent convergence is equivalent to the frequent Cauchy property [13,14]. More broadly, this line of thought fits well with subsequent developments on statistical and ideal convergence in metric, random n-normed, and fractal-type settings [5,7, 15–17]. Related probabilistic and approximation-theoretic aspects of deferred weighted statistical equivalence have also recently been studied in [18].

The purpose of the present paper is to develop pairwise asymptotic criteria for sequences of compact sets in the Hausdorff hyperspace. Our contribution is not the introduction of entirely new convergence notions, but the systematic transfer of pairwise criteria to compact-set sequences in the Hausdorff metric, together with bounded double-mean and bounded modulus characterizations and a collection of compact-set examples. In this way, the paper should be viewed as a hyperspace counterpart of earlier work on statistical and ideal asymptotic methods, rather than as an attractor-oriented study in the sense of fractal dynamics [6,15,16].

We begin by introducing $\mathcal{I}$-statistically pre-Cauchy sequences of compact sets and establishing the basic permanence properties needed later. We then derive bounded double-mean and bounded modulus criteria in terms of the Hausdorff distance. After that, we turn to frequent Cauchy sequences of compact sets and study their relation to frequent convergence under additional compactness assumptions. We also identify several situations in which pairwise asymptotic conditions can be upgraded to genuine convergence, and we conclude with examples illustrating both the scope and the limitations of the pairwise approach in the hyperspace setting.

Several arguments in §3, §4, and parts of §6 are metric in character and remain valid beyond the hyperspace context. What is specific to the present setting is their systematic formulation for compact-set sequences in the Hausdorff metric, together with the bounded pairwise criteria and the compact-set examples developed later in the paper. In this sense, the results of §3 and §4 are best understood as hyperspace adaptations of known pairwise criteria from the scalar and metric-space literature. By contrast, the frequent part in §5 requires a more delicate compactness-based argument, since the passage from pairwise frequency conditions to actual convergence is subtler in the hyperspace framework. The standing completeness assumption is therefore used only where such upgrade arguments genuinely require it.

The paper is organized as follows. §2 contains notation and preliminary facts. In §3, we introduce $\mathcal{I}$-statistically pre-Cauchy sequences of compact sets and prove their elementary properties. §4 is devoted to double-mean and bounded modulus criteria. §5 studies frequent Cauchy sequences of compact sets and their relation to frequent convergence. §6 discusses conditions under which pairwise criteria can be upgraded to actual convergence. The remaining sections contain examples, brief applications, and concluding remarks.

2. Preliminaries

Throughout the paper, $\mathcal{I}$ denotes a proper admissible ideal on $\mathbb{N}$ in the usual sense of ideal convergence theory [19]. Related ideal and lacunary ideal methods in nonlinear settings may be found, for instance, in [16]. Recall that a family $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if

$$A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}, \qquad B \subset A \in \mathcal{I} \implies B \in \mathcal{I}.$$

It is called admissible if every finite subset of $\mathbb{N}$ belongs to $\mathcal{I}$, and proper if $\mathbb{N} \notin \mathcal{I}$. The filter associated with $\mathcal{I}$ is given by

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \mathbb{N} \setminus M \in \mathcal{I}\}.$$

If $E \subset \mathbb{N}$, its natural density, whenever it exists, is defined by

$$d(E) = \lim_{n\to\infty} \frac{1}{n} \big|\{k \le n : k \in E\}\big|,$$

while the upper natural density is

$$\overline{d}(E) = \limsup_{n\to\infty} \frac{1}{n} \big|\{k \le n : k \in E\}\big|.$$

We shall also work with subsets of $\mathbb{N} \times \mathbb{N}$. For $\Omega \subset \mathbb{N} \times \mathbb{N}$, we write

$$\Omega(m,n) = \Omega \cap \big(\{1,\dots,m\} \times \{1,\dots,n\}\big).$$

Whenever the limit exists, the frequency measure of Ω is defined by

$$\mu(\Omega) = \lim_{m,n\to\infty} \frac{|\Omega(m,n)|}{mn},$$

and the corresponding upper frequency measure is

$$\overline{\mu}(\Omega) = \limsup_{m,n\to\infty} \frac{|\Omega(m,n)|}{mn}.$$

We begin with the scalar convergence notions that will later be transferred to the Hausdorff hyperspace.

Definition 1 ([19])**.** Let (Y,ρ) be a metric space. A sequence (y_n) in Y is said to be $\mathcal{I}$-convergent to $L \in Y$ if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \rho(y_n, L) \geq \varepsilon\} \in \mathcal{I}.$$

Definition 2 ([20])**.** Let (Y,ρ) be a metric space. A sequence (y_n) in Y is said to be $\mathcal{I}$-statistically convergent to $L \in Y$ if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\big|\{k \leq n : \rho(y_k, L) \geq \varepsilon\}\big| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}\text{-}st\text{-}\lim y_n = L$.

Definition 3 ([8])**.** Let (Y,ρ) be a metric space. A sequence (y_n) in Y is said to be $\mathcal{I}$-statistically pre-Cauchy if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\big|\{(j,k) : 1 \leq j,k \leq n,\ \rho(y_j, y_k) \geq \varepsilon\}\big| \geq \delta\right\} \in \mathcal{I}.$$

The pre-Cauchy formulation is pairwise and therefore does not require a candidate limit in advance. At the same time, it is generally weaker than convergence. For scalar sequences, statistical convergence implies statistical pre-Cauchy behavior, whereas converse implications usually require additional assumptions such as boundedness together with structural information on the limit-point set, or suitable mean-type conditions [8,11,14].

We now turn to the geometric setting of the paper. Let (X,d) be a metric space, and let $\mathcal{K}(X)$ denote the family of all nonempty compact subsets of X. For $x \in X$ and $A \in \mathcal{K}(X)$, set $d(x,A) = \inf_{a\in A} d(x,a)$. For $A, B \in \mathcal{K}(X)$, the Hausdorff distance is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}.$$

The metric space $(\mathcal{K}(X), d_H)$ will be referred to as the Hausdorff hyperspace of X [2,4].

When (X,d) is complete, it is well known that $(\mathcal{K}(X), d_H)$ is complete as well [4]. This completeness assumption is invoked only where completeness of the Hausdorff hyperspace is needed explicitly. In particular, the arguments in §3, §4, and parts of §6 are metric in nature and do not otherwise require completeness of X. For those parts, the metric structure of $(\mathcal{K}(X), d_H)$ is sufficient. This observation allows us to transfer scalar asymptotic notions to compact-set sequences simply by replacing the ambient metric ρ with the Hausdorff metric d_H.

Frequent convergence also plays an important role in the present framework. Motivated by the scalar theory, we use the pairwise frequency measure on $\mathbb{N} \times \mathbb{N}$ in order to formulate a Hausdorff version of the frequent Cauchy property [12–14]. Thus both $\mathcal{I}$-statistical pre-Cauchy behavior and frequent Cauchy behavior are expressed through the asymptotic smallness of sets of pairs of the form

$$\{(j,k) : d_H(A_j, A_k) \geq \varepsilon\}.$$

Finally, we record the bounded modulus notion used later. A function $f : [0,\infty) \to [0,\infty)$ is called a bounded modulus if it satisfies

$$f(t) = 0 \iff t = 0, \qquad f(s+t) \leq f(s) + f(t) \quad (s,t \geq 0),$$

$$f \text{ is increasing}, \qquad f \text{ is continuous from the right at } 0,$$

and $\sup_{t\geq 0} f(t) < \infty$. Such functions will be used to formulate bounded modulus criteria for pairwise Hausdorff differences.

The remainder of the paper is devoted to the interaction between these pairwise notions and actual convergence in the Hausdorff hyperspace.

3. $\mathcal{I}$-statistically Pre-Cauchy sequences of compact sets

We now transfer the scalar notions from the previous section to sequences of compact sets in the Hausdorff hyperspace. The key feature of the pre-Cauchy condition is its pairwise nature, which makes it particularly suitable for hyperspace problems, where mutual Hausdorff closeness is often more informative than a formulation based on a prescribed limit.

Definition 4. A sequence (A_n) in $(\mathcal{K}(X), d_H)$ is said to be $\mathcal{I}$-statistically convergent to $A \in \mathcal{K}(X)$ if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\big|\{k \leq n : d_H(A_k, A) \geq \varepsilon\}\big| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}\text{-}st\text{-}\lim A_n = A$.

Definition 5. A sequence (A_n) in $(\mathcal{K}(X), d_H)$ is said to be $\mathcal{I}$-statistically pre-Cauchy if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\big|\{(j,k) : 1 \leq j,k \leq n,\ d_H(A_j, A_k) \geq \varepsilon\}\big| \geq \delta\right\} \in \mathcal{I}.$$

The next proposition records two elementary permanence properties that will be used later.

Proposition 1. *Let (A_n) and (B_n) be sequences in $\mathcal{K}(X)$.*

(1) If $A_n = B_n$ for all sufficiently large n, then (A_n) is $\mathcal{I}$-statistically convergent if and only if (B_n) is $\mathcal{I}$-statistically convergent, and in that case they have the same $\mathcal{I}$-statistical limit.

(2) If $A_n = B_n$ for all sufficiently large n, then (A_n) is $\mathcal{I}$-statistically pre-Cauchy if and only if (B_n) is $\mathcal{I}$-statistically pre-Cauchy.

Proof. Assume that there exists $N \in \mathbb{N}$ such that $A_n = B_n$ for all $n \geq N$.

For part (1), let $A \in \mathcal{K}(X)$ and fix $\varepsilon > 0$ and $\delta > 0$. For each $n \in \mathbb{N}$, define

$$E_n(A,\varepsilon) = \{k \leq n : d_H(A_k, A) \geq \varepsilon\},$$

and

$$F_n(A,\varepsilon) = \{k \leq n : d_H(B_k, A) \geq \varepsilon\}.$$

Since the two sequences differ only in the first finitely many terms, we have

$$\big||E_n(A,\varepsilon)| - |F_n(A,\varepsilon)|\big| \leq N,$$

for all n. Hence
$$\left|\frac{1}{n}|E_n(A,\varepsilon)| - \frac{1}{n}|F_n(A,\varepsilon)|\right| \leq \frac{N}{n}.$$

It follows that the exceptional sets in Definition 4 differ by at most a finite set. Since $\mathcal{I}$ is admissible, the claim follows.

For part (2), fix $\varepsilon > 0$ and $\delta > 0$. Let

$$P_n(\varepsilon) = \{(j,k) : 1 \leq j,k \leq n,\ d_H(A_j, A_k) \geq \varepsilon\},$$

and

$$Q_n(\varepsilon) = \{(j,k) : 1 \leq j,k \leq n,\ d_H(B_j, B_k) \geq \varepsilon\}.$$

If both j and k are greater than N, then $A_j = B_j$ and $A_k = B_k$, so the pair conditions defining $P_n(\varepsilon)$ and $Q_n(\varepsilon)$ agree. Thus any discrepancy can occur only when at least one coordinate belongs to $\{1, \ldots, N\}$. Consequently,
$$\big||P_n(\varepsilon)| - |Q_n(\varepsilon)|\big| \leq 2Nn + N^2,$$
for all n. After division by n^2, we get

$$\left|\frac{1}{n^2}|P_n(\varepsilon)| - \frac{1}{n^2}|Q_n(\varepsilon)|\right| \leq \frac{2N}{n} + \frac{N^2}{n^2}.$$

Hence the exceptional sets in Definition 5 differ by at most a finite set, and the conclusion follows. □

The next theorem is the natural compact-set analogue of the familiar scalar fact that statistical convergence implies statistical pre-Cauchy behavior.

Theorem 1. *Let (A_n) be a sequence in $(\mathcal{K}(X), d_H)$. If $\mathcal{I}$-st-$\lim A_n = A$ for some $A \in \mathcal{K}(X)$, then (A_n) is $\mathcal{I}$-statistically pre-Cauchy.*

Proof. Fix $\varepsilon > 0$ and $\delta > 0$, and set $\eta = \varepsilon/2$. For each $n \in \mathbb{N}$, define

$$S_n(\eta) = \{k \leq n : d_H(A_k, A) \geq \eta\},$$

and
$$P_n(\varepsilon) = \{(j,k) : 1 \leq j,k \leq n,\ d_H(A_j, A_k) \geq \varepsilon\}.$$

If $(j,k) \in P_n(\varepsilon)$, then both inequalities $d_H(A_j, A) < \eta$ and $d_H(A_k, A) < \eta$ cannot hold simultaneously, because otherwise the triangle inequality would give

$$d_H(A_j, A_k) \leq d_H(A_j, A) + d_H(A_k, A) < 2\eta = \varepsilon,$$

which is impossible. Therefore every pair in $P_n(\varepsilon)$ has at least one coordinate in $S_n(\eta)$, and so

$$P_n(\varepsilon) \subset \big(S_n(\eta) \times \{1, \ldots, n\}\big) \cup \big(\{1, \ldots, n\} \times S_n(\eta)\big).$$

Consequently,
$$|P_n(\varepsilon)| \leq 2n\,|S_n(\eta)|.$$

Dividing by n^2, we obtain
$$\frac{1}{n^2}|P_n(\varepsilon)| \leq 2\,\frac{1}{n}|S_n(\eta)|.$$

Now suppose that
$$\frac{1}{n^2}|P_n(\varepsilon)| \geq \delta.$$

Then necessarily
$$\frac{1}{n}|S_n(\eta)| \geq \frac{\delta}{2}.$$

Hence

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}|P_n(\varepsilon)| \geq \delta\right\} \subset \left\{n \in \mathbb{N} : \frac{1}{n}|S_n(\eta)| \geq \frac{\delta}{2}\right\}.$$

Since $\mathcal{I}$-*st*- $\lim A_n = A$, the set on the right belongs to $\mathcal{I}$ by Definition 4. Hence the defining exceptional set for $\mathcal{I}$-statistical pre-Cauchy behavior also belongs to $\mathcal{I}$. Therefore (A_n) is $\mathcal{I}$-statistically pre-Cauchy. □

We also note, for later reference, that the pre-Cauchy property is preserved under Lipschitz self-mappings of the hyperspace.

Proposition 2. *Let $T : (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ be a Lipschitz mapping with constant $L \geq 0$. If (A_n) is $\mathcal{I}$-statistically pre-Cauchy, then $(T(A_n))$ is also $\mathcal{I}$-statistically pre-Cauchy.*

Proof. If $L = 0$, then T is constant and the conclusion is immediate. Assume that $L > 0$. Fix $\varepsilon > 0$ and $\delta > 0$. For each $n \in \mathbb{N}$, let

$$P_n(\varepsilon) = \{(j,k) : 1 \leq j,k \leq n,\ d_H(T(A_j), T(A_k)) \geq \varepsilon\},$$

and

$$Q_n(\varepsilon) = \left\{(j,k) : 1 \leq j,k \leq n,\ d_H(A_j, A_k) \geq \frac{\varepsilon}{L}\right\}.$$

Since

$$d_H(T(A_j), T(A_k)) \leq L\, d_H(A_j, A_k),$$

we have $P_n(\varepsilon) \subset Q_n(\varepsilon)$ for every n. Hence

$$\frac{1}{n^2}|P_n(\varepsilon)| \leq \frac{1}{n^2}|Q_n(\varepsilon)|.$$

Therefore

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}|P_n(\varepsilon)| \geq \delta\right\} \subset \left\{n \in \mathbb{N} : \frac{1}{n^2}|Q_n(\varepsilon)| \geq \delta\right\},$$

and the set on the right belongs to $\mathcal{I}$ because (A_n) is $\mathcal{I}$-statistically pre-Cauchy. This proves the result. □

Remark 1. At this stage we do not claim that $\mathcal{I}$-statistical pre-Cauchy behavior alone implies $\mathcal{I}$-statistical convergence. The pairwise condition is generally weaker, and additional assumptions are needed in order to recover a genuine limit. The next section is devoted to more explicit criteria of this kind for bounded sequences.

4. Double-mean and modulus criteria

The pre-Cauchy condition introduced in the previous section is expressed through the asymptotic smallness of exceptional pair sets in the Hausdorff hyperspace. For bounded compact-set sequences, this pairwise condition admits more explicit analytic formulations. In this section, we derive two such criteria, one based on double means of Hausdorff distances and the other on bounded moduli.

We begin by fixing the relevant boundedness notion.

Definition 6. A sequence (A_n) in $(\mathcal{K}(X), d_H)$ is said to be bounded in the Hausdorff hyperspace if there exists a constant $R > 0$ such that

$$d_H(A_j, A_k) \leq R \qquad \text{for all } j,k \in \mathbb{N}.$$

The first result shows that, under boundedness, the $\mathcal{I}$-statistically pre-Cauchy property is equivalent to the vanishing of the pairwise double mean of the Hausdorff distance.

Theorem 2. *Let (A_n) be a bounded sequence in $(\mathcal{K}(X), d_H)$. Then the following are equivalent.*

(1) *(A_n) is $\mathcal{I}$-statistically pre-Cauchy.*

(2)

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n^2} \sum_{j,k=1}^{n} d_H(A_j, A_k) = 0.$$

Proof. Assume first that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n^2}\sum_{j,k=1}^{n} d_H(A_j, A_k) = 0.$$

Fix $\varepsilon > 0$ and $\delta > 0$. For each $n \in \mathbb{N}$, set

$$P_n(\varepsilon) = \{(j,k) : 1 \le j,k \le n,\ d_H(A_j, A_k) \ge \varepsilon\},$$

and

$$M_n = \frac{1}{n^2}\sum_{j,k=1}^{n} d_H(A_j, A_k).$$

Since every pair in $P_n(\varepsilon)$ contributes at least ε to the double sum, we have

$$M_n \ge \frac{\varepsilon}{n^2}|P_n(\varepsilon)|.$$

Hence

$$\frac{1}{n^2}|P_n(\varepsilon)| \le \frac{M_n}{\varepsilon}.$$

Therefore

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}|P_n(\varepsilon)| \ge \delta\right\} \subset \{n \in \mathbb{N} : M_n \ge \varepsilon\delta\}.$$

The set on the right belongs to $\mathcal{I}$, so (A_n) is $\mathcal{I}$-statistically pre-Cauchy.

Conversely, assume that (A_n) is $\mathcal{I}$-statistically pre-Cauchy. Since the sequence is bounded in the Hausdorff hyperspace, there exists $R > 0$ such that $d_H(A_j, A_k) \le R$ for all $j,k \in \mathbb{N}$. Fix $\delta > 0$. If $\delta > R$, then

$$\{n \in \mathbb{N} : M_n \ge \delta\} = \varnothing,$$

so there is nothing to prove. We may therefore assume that $0 < \delta \le R$.

Put $\eta = \frac{\delta}{2}$. For each $n \in \mathbb{N}$, split the double sum according to whether $d_H(A_j, A_k) < \eta$ and $d_H(A_j, A_k) \ge \eta$. Then

$$M_n \le \frac{1}{n^2}\Big(\eta \cdot n^2 + R\,|P_n(\eta)|\Big) = \eta + R\,\frac{1}{n^2}|P_n(\eta)|.$$

If $M_n \ge \delta$, then

$$R\,\frac{1}{n^2}|P_n(\eta)| \ge \delta - \eta = \frac{\delta}{2},$$

and hence

$$\frac{1}{n^2}|P_n(\eta)| \ge \frac{\delta}{2R}.$$

Therefore

$$\{n \in \mathbb{N} : M_n \ge \delta\} \subset \left\{n \in \mathbb{N} : \frac{1}{n^2}|P_n(\eta)| \ge \frac{\delta}{2R}\right\}.$$

Since (A_n) is $\mathcal{I}$-statistically pre-Cauchy, the set on the right belongs to $\mathcal{I}$. Thus $\mathcal{I}\text{-}\lim_{n\to\infty} M_n = 0$. The proof is complete. □

As a direct consequence, one obtains a convenient reformulation of Theorem 1 for bounded sequences.

Corollary 1. *Let (A_n) be a bounded sequence in $(\mathcal{K}(X), d_H)$. If $\mathcal{I}$-st-$\lim A_n = A$ for some $A \in \mathcal{K}(X)$, then*

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n^2}\sum_{j,k=1}^{n} d_H(A_j, A_k) = 0.$$

Proof. By Theorem 1, the sequence (A_n) is $\mathcal{I}$-statistically pre-Cauchy. The conclusion now follows from Theorem 2. □

We next turn to a bounded modulus formulation. This criterion is often more flexible than the raw distance average and, at the same time, remains entirely pairwise.

Theorem 3. *Let (A_n) be a bounded sequence in $(\mathcal{K}(X), d_H)$, and let $f : [0,\infty) \to [0,\infty)$ be a bounded modulus. Then the following are equivalent.*

(1) *(A_n) is $\mathcal{I}$-statistically pre-Cauchy.*

(2)

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n^2} \sum_{j,k=1}^{n} f\big(d_H(A_j, A_k)\big) = 0.$$

Proof. Assume first that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n^2} \sum_{j,k=1}^{n} f\big(d_H(A_j, A_k)\big) = 0.$$

Fix $\varepsilon > 0$ and $\delta > 0$. Since f is increasing and vanishes only at 0, we have $f(\varepsilon) > 0$. For each $n \in \mathbb{N}$, let

$$P_n(\varepsilon) = \{(j,k) : 1 \le j,k \le n,\ d_H(A_j, A_k) \ge \varepsilon\},$$

and

$$N_n = \frac{1}{n^2} \sum_{j,k=1}^{n} f\big(d_H(A_j, A_k)\big).$$

Every pair in $P_n(\varepsilon)$ contributes at least $f(\varepsilon)$ to the sum defining N_n. Hence

$$N_n \ge \frac{f(\varepsilon)}{n^2} |P_n(\varepsilon)|.$$

Therefore

$$\frac{1}{n^2} |P_n(\varepsilon)| \le \frac{N_n}{f(\varepsilon)}.$$

It follows that

$$\left\{ n \in \mathbb{N} : \frac{1}{n^2} |P_n(\varepsilon)| \ge \delta \right\} \subset \{ n \in \mathbb{N} : N_n \ge \delta f(\varepsilon) \}.$$

The set on the right belongs to $\mathcal{I}$, so (A_n) is $\mathcal{I}$-statistically pre-Cauchy.

Conversely, assume that (A_n) is $\mathcal{I}$-statistically pre-Cauchy. Since f is bounded, there exists $K > 0$ such that $f(t) \le K$ for all $t \ge 0$. Fix $\delta > 0$. If $\delta > K$, then

$$\{ n \in \mathbb{N} : N_n \ge \delta \} = \varnothing,$$

so the claim is immediate. We may therefore assume that $0 < \delta \le K$.

By the right continuity of f at 0 and the fact that $f(0) = 0$, there exists $\eta > 0$ such that $f(\eta) < \frac{\delta}{2}$. For each $n \in \mathbb{N}$, split the double sum according to whether $d_H(A_j, A_k) < \eta$ or $d_H(A_j, A_k) \ge \eta$. Then

$$N_n \le \frac{1}{n^2} \Big(f(\eta) \cdot n^2 + K\, |P_n(\eta)| \Big) = f(\eta) + K \frac{1}{n^2} |P_n(\eta)|.$$

If $N_n \ge \delta$, then

$$K \frac{1}{n^2} |P_n(\eta)| \ge \delta - f(\eta) > \frac{\delta}{2},$$

and so $\frac{1}{n^2} |P_n(\eta)| \ge \frac{\delta}{2K}$. Consequently,

$$\{ n \in \mathbb{N} : N_n \ge \delta \} \subset \left\{ n \in \mathbb{N} : \frac{1}{n^2} |P_n(\eta)| \ge \frac{\delta}{2K} \right\}.$$

The set on the right belongs to $\mathcal{I}$ because (A_n) is $\mathcal{I}$-statistically pre-Cauchy. Hence $\mathcal{I}\text{-}\lim_{n\to\infty} N_n = 0$. This proves the result. □

Remark 2. The two criteria above show that, for bounded compact-set sequences, the $\mathcal{I}$-statistically pre-Cauchy condition admits two equivalent pairwise formulations. One is expressed through the average Hausdorff distance itself, while the other uses a bounded modulus of the Hausdorff distance. The modulus version is often more adaptable when one wants to compare different scales of pairwise closeness without changing the underlying asymptotic content.

5. Frequent Cauchy sequences of compact sets

The frequent side of the theory is formulated through frequency measures on $\mathbb{N} \times \mathbb{N}$. In contrast with the $\mathcal{I}$-statistical pre-Cauchy condition, which is governed by one-dimensional initial segments and an ideal on $\mathbb{N}$, the frequent Cauchy property measures how often two terms of the sequence remain close to each other in the Hausdorff metric.

Definition 7. A sequence (A_n) in $(\mathcal{K}(X), d_H)$ is said to be frequently convergent to $A \in \mathcal{K}(X)$ if for every $\varepsilon > 0$,

$$d\{n \in \mathbb{N} : d_H(A_n, A) \geq \varepsilon\} = 0.$$

Equivalently,

$$d\{n \in \mathbb{N} : d_H(A_n, A) < \varepsilon\} = 1,$$

for every $\varepsilon > 0$. In this case we write $flim\, A_n = A$.

Definition 8. A sequence (A_n) in $(\mathcal{K}(X), d_H)$ is said to be frequent Cauchy if for every $\varepsilon > 0$,

$$\mu\{(m,n) \in \mathbb{N} \times \mathbb{N} : d_H(A_m, A_n) \geq \varepsilon\} = 0.$$

Equivalently,

$$\mu\{(m,n) \in \mathbb{N} \times \mathbb{N} : d_H(A_m, A_n) < \varepsilon\} = 1,$$

for every $\varepsilon > 0$.

Lemma 1. *Let $E, F \subset \mathbb{N}$.*

(1) *If $d(E)$ and $d(F)$ exist, then*

$$\mu(E \times F) = d(E)\, d(F).$$

(2) *In general,*

$$\overline{\mu}(E \times F) \geq \overline{d}(E)\, \overline{d}(F).$$

Proof. For each $p, q \in \mathbb{N}$,

$$|(E \times F)(p,q)| = |E \cap \{1, \ldots, p\}| \cdot |F \cap \{1, \ldots, q\}|,$$

hence

$$\frac{|(E \times F)(p,q)|}{pq} = \frac{|E \cap \{1, \ldots, p\}|}{p} \frac{|F \cap \{1, \ldots, q\}|}{q}.$$

If $d(E)$ and $d(F)$ exist, passing to the limit as $p, q \to \infty$ yields

$$\mu(E \times F) = d(E)\, d(F).$$

For the upper version, choose sequences $p_j \to \infty$ and $q_j \to \infty$ such that

$$\frac{|E \cap \{1, \ldots, p_j\}|}{p_j} \to \overline{d}(E), \qquad \frac{|F \cap \{1, \ldots, q_j\}|}{q_j} \to \overline{d}(F).$$

Then

$$\frac{|(E \times F)(p_j, q_j)|}{p_j q_j} \to \overline{d}(E)\, \overline{d}(F),$$

and therefore

$$\overline{\mu}(E \times F) \geq \overline{d}(E)\, \overline{d}(F).$$

□

We begin with the direct implication from frequent convergence to the frequent Cauchy property.

Proposition 3. *If a sequence (A_n) in $(\mathcal{K}(X), d_H)$ is frequently convergent, then it is frequent Cauchy.*

Proof. Assume that $flim\, A_n = A$ for some $A \in \mathcal{K}(X)$. Fix $\varepsilon > 0$ and put

$$E(\varepsilon) = \{n \in \mathbb{N} : d_H(A_n, A) < \varepsilon/2\}.$$

By Definition 7, we have

$$d(E(\varepsilon)) = 1.$$

Hence, by Lemma 1,

$$\mu(E(\varepsilon) \times E(\varepsilon)) = 1.$$

If $(m, n) \in E(\varepsilon) \times E(\varepsilon)$, then

$$d_H(A_m, A_n) \leq d_H(A_m, A) + d_H(A_n, A) < \varepsilon.$$

Therefore

$$E(\varepsilon) \times E(\varepsilon) \subset \{(m, n) \in \mathbb{N} \times \mathbb{N} : d_H(A_m, A_n) < \varepsilon\}.$$

It follows that

$$\mu\{(m, n) \in \mathbb{N} \times \mathbb{N} : d_H(A_m, A_n) < \varepsilon\} = 1.$$

Thus (A_n) is frequent Cauchy. □

The next proposition shows that frequent Cauchy behavior is stronger than the $\mathcal{I}$-statistically pre-Cauchy condition introduced earlier.

Proposition 4. *If (A_n) is frequent Cauchy in $(\mathcal{K}(X), d_H)$, then it is $\mathcal{I}$-statistically pre-Cauchy for every proper admissible ideal $\mathcal{I}$ on $\mathbb{N}$.*

Proof. Fix $\varepsilon > 0$. Since (A_n) is frequent Cauchy,

$$\mu\{(m, n) \in \mathbb{N} \times \mathbb{N} : d_H(A_m, A_n) \geq \varepsilon\} = 0.$$

This means that

$$\lim_{p,q\to\infty} \frac{1}{pq} \big|\{(m, n) : 1 \leq m \leq p,\ 1 \leq n \leq q,\ d_H(A_m, A_n) \geq \varepsilon\}\big| = 0.$$

In particular, along the diagonal $p = q = n$,

$$\lim_{n\to\infty} \frac{1}{n^2} \big|\{(j, k) : 1 \leq j, k \leq n,\ d_H(A_j, A_k) \geq \varepsilon\}\big| = 0.$$

Hence for every $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} \big|\{(j, k) : 1 \leq j, k \leq n,\ d_H(A_j, A_k) \geq \varepsilon\}\big| \geq \delta \right\},$$

is finite. Since $\mathcal{I}$ is admissible, every finite set belongs to $\mathcal{I}$. Therefore (A_n) is $\mathcal{I}$-statistically pre-Cauchy. □

Lemma 2. *Let (A_n) be a sequence in a compact metric space (K, ρ). Then there exists a point $A \in K$ such that, for every $\varepsilon > 0$,*

$$\overline{d}\big(\{n \in \mathbb{N} : \rho(A_n, A) < \varepsilon\}\big) > 0.$$

Proof. We construct inductively a nested sequence of nonempty compact subsets of K whose diameters tend to 0.

Since K is compact, it can be covered by finitely many closed balls of radius 1:

$$K \subset \bigcup_{i=1}^{N_1} \overline{B}(C_i^{(1)}, 1).$$

For each i, define

$$E_i^{(1)} = \{n \in \mathbb{N} : A_n \in \overline{B}(C_i^{(1)}, 1)\}.$$

Because

$$\mathbb{N} = \bigcup_{i=1}^{N_1} E_i^{(1)},$$

and $\overline{d}(\mathbb{N}) = 1$, at least one of the sets $E_i^{(1)}$ has positive upper density. Choose such an index i_1, and set

$$K_1 = K \cap \overline{B}(C_{i_1}^{(1)}, 1).$$

Then K_1 is a nonempty compact subset of K,

$$\operatorname{diam}(K_1) \leq 2,$$

and

$$\overline{d}(\{n \in \mathbb{N} : A_n \in K_1\}) > 0.$$

Assume that for some $r \geq 1$ we have already chosen a nonempty compact set $K_r \subset K$ such that

$$\operatorname{diam}(K_r) \leq 2^{2-r},$$

and

$$\overline{d}(\{n \in \mathbb{N} : A_n \in K_r\}) > 0.$$

Since K_r is compact, it can be covered by finitely many closed balls of radius 2^{-r}:

$$K_r \subset \bigcup_{i=1}^{N_{r+1}} \overline{B}(C_i^{(r+1)}, 2^{-r}).$$

For each i, let

$$E_i^{(r+1)} = \{n \in \mathbb{N} : A_n \in K_r \cap \overline{B}(C_i^{(r+1)}, 2^{-r})\}.$$

Then

$$\{n \in \mathbb{N} : A_n \in K_r\} = \bigcup_{i=1}^{N_{r+1}} E_i^{(r+1)}.$$

Since the set on the left has positive upper density, at least one of the sets $E_i^{(r+1)}$ has positive upper density. Choose such an index i_{r+1}, and put

$$K_{r+1} = K_r \cap \overline{B}(C_{i_{r+1}}^{(r+1)}, 2^{-r}).$$

Then $K_{r+1} \subset K_r$, K_{r+1} is nonempty and compact,

$$\operatorname{diam}(K_{r+1}) \leq 2^{1-r},$$

and

$$\overline{d}(\{n \in \mathbb{N} : A_n \in K_{r+1}\}) > 0.$$

Thus we obtain a nested sequence

$$K_1 \supset K_2 \supset K_3 \supset \cdots,$$

of nonempty compact sets such that $\operatorname{diam}(K_r) \to 0$. Hence

$$\bigcap_{r=1}^{\infty} K_r = \{A\},$$

for some $A \in K$.

Now let $\varepsilon > 0$ be arbitrary. Choose r so large that $\operatorname{diam}(K_r) < \varepsilon$. Since $A \in K_r$, every point of K_r lies within distance $< \varepsilon$ of A. Therefore

$$\{n \in \mathbb{N} : A_n \in K_r\} \subset \{n \in \mathbb{N} : \rho(A_n, A) < \varepsilon\}.$$

Since the set on the left has positive upper density, so does the set on the right. This proves the lemma. □

The converse direction is more delicate in general metric spaces. A pairwise frequency condition need not automatically produce a limit unless some compactness is available. The next result gives a convenient form of the converse in the hyperspace setting.

Theorem 4. *Let (A_n) be a sequence in $(\mathcal{K}(X), d_H)$ such that the closure*

$$K := \overline{\{A_n : n \in \mathbb{N}\}},$$

is compact in $(\mathcal{K}(X), d_H)$. If (A_n) is frequent Cauchy, then it is frequently convergent to some point $A \in K$.

Proof. The set K is a compact metric subspace of $(\mathcal{K}(X), d_H)$. Applying Lemma 2 to the sequence (A_n) viewed in K, we obtain a point $A \in K$ such that for every $\varepsilon > 0$,

$$\overline{d}\big(\{n \in \mathbb{N} : d_H(A_n, A) < \varepsilon\}\big) > 0.$$

We claim that

$$flim\, A_n = A.$$

Assume to the contrary that (A_n) is not frequently convergent to A. Then there exists $\varepsilon_0 > 0$ such that, for the set

$$F = \{n \in \mathbb{N} : d_H(A_n, A) \geq \varepsilon_0\},$$

the density $d(F)$ either does not exist or is different from 0. In either case,

$$\overline{d}(F) > 0.$$

On the other hand, by the choice of A, the set

$$E = \{n \in \mathbb{N} : d_H(A_n, A) < \varepsilon_0/2\},$$

satisfies

$$\overline{d}(E) > 0.$$

Now let $m \in E$ and $n \in F$. Then

$$d_H(A_m, A_n) \geq d_H(A_n, A) - d_H(A_m, A) \geq \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}.$$

Hence

$$E \times F \subset \Big\{(m, n) \in \mathbb{N} \times \mathbb{N} : d_H(A_m, A_n) \geq \frac{\varepsilon_0}{2}\Big\}.$$

For each $p, q \in \mathbb{N}$,

$$|(E \times F)(p, q)| = |E \cap \{1, \ldots, p\}| \cdot |F \cap \{1, \ldots, q\}|,$$

and therefore

$$\frac{|(E\times F)(p,q)|}{pq}=\frac{|E\cap\{1,\ldots,p\}|}{p}\,\frac{|F\cap\{1,\ldots,q\}|}{q}.$$

Choose sequences $p_j\to\infty$ and $q_j\to\infty$ such that

$$\frac{|E\cap\{1,\ldots,p_j\}|}{p_j}\to\overline{d}(E),\qquad\frac{|F\cap\{1,\ldots,q_j\}|}{q_j}\to\overline{d}(F).$$

Then

$$\frac{|(E\times F)(p_j,q_j)|}{p_jq_j}\to\overline{d}(E)\,\overline{d}(F),$$

so

$$\overline{\mu}(E\times F)\geq\overline{d}(E)\,\overline{d}(F)>0.$$

Consequently,

$$\overline{\mu}\Big(\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:d_H(A_m,A_n)\geq\frac{\varepsilon_0}{2}\Big\}\Big)>0.$$

However, since (A_n) is frequent Cauchy, we have

$$\mu\Big(\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:d_H(A_m,A_n)\geq\varepsilon\Big\}\Big)=0\qquad\text{for every }\varepsilon>0.$$

In particular,

$$\mu\Big(\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:d_H(A_m,A_n)\geq\frac{\varepsilon_0}{2}\Big\}\Big)=0.$$

Hence its upper frequency is also zero:

$$\overline{\mu}\Big(\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:d_H(A_m,A_n)\geq\frac{\varepsilon_0}{2}\Big\}\Big)=0,$$

which is a contradiction.

Therefore, for every $\varepsilon>0$,

$$\overline{d}\big(\{n\in\mathbb{N}:d_H(A_n,A)\geq\varepsilon\}\big)=0.$$

It follows that the corresponding density exists and equals 0, that is,

$$d\big(\{n\in\mathbb{N}:d_H(A_n,A)\geq\varepsilon\}\big)=0\qquad\text{for every }\varepsilon>0.$$

Thus

$$flim\,A_n=A.$$

□

Corollary 2. *Assume that* $(\mathcal{K}(X),d_H)$ *is compact. Then a sequence* (A_n) *in* $(\mathcal{K}(X),d_H)$ *is frequently convergent if and only if it is frequent Cauchy.*

Proof. The forward implication follows from Proposition 3, and the reverse implication is a direct consequence of Theorem 4. □

The compactness of $(\mathcal{K}(X),d_H)$ is a strong assumption and is used here only to obtain a global equivalence between frequent convergence and the frequent Cauchy property. In most situations, it is enough to assume that the closure of the range of the sequence is compact, which is exactly the hypothesis used in Theorem 4.

Remark 3. Proposition 4 shows that frequent Cauchy behavior is stronger than $\mathcal{I}$-statistically pre-Cauchy behavior. The converse fails in general. Thus the frequent condition may be regarded as a genuinely sharper pairwise hypothesis, while the $\mathcal{I}$-statistical pre-Cauchy property remains the more flexible notion.

6. When pairwise criteria imply convergence

The results obtained so far show that pairwise conditions alone do not in general guarantee actual convergence. In this section, we record three useful upgrade criteria. The first two are one-point mean conditions, while the third shows that a pairwise hypothesis together with a sufficiently thick convergent subsequence already yields $\mathcal{I}$-statistical convergence.

Proposition 5. *Let (A_n) be a sequence in $(\mathcal{K}(X), d_H)$ and let $A \in \mathcal{K}(X)$. If*

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d_H(A_k, A) = 0,$$

then $\mathcal{I}$-st-$\lim A_n = A$.

Proof. Fix $\varepsilon > 0$ and $\delta > 0$. For each $n \in \mathbb{N}$, define

$$E_n(\varepsilon) = \{k \le n : d_H(A_k, A) \ge \varepsilon\},$$

and

$$M_n(A) = \frac{1}{n}\sum_{k=1}^{n} d_H(A_k, A).$$

Since every index in $E_n(\varepsilon)$ contributes at least ε to the sum, we have

$$M_n(A) \ge \frac{\varepsilon}{n}|E_n(\varepsilon)|.$$

Hence

$$\frac{1}{n}|E_n(\varepsilon)| \le \frac{M_n(A)}{\varepsilon}.$$

Therefore

$$\left\{n \in \mathbb{N} : \frac{1}{n}|E_n(\varepsilon)| \ge \delta\right\} \subset \{n \in \mathbb{N} : M_n(A) \ge \varepsilon\delta\}.$$

The set on the right belongs to $\mathcal{I}$ by hypothesis. Thus $\mathcal{I}$-st-$\lim A_n = A$. □

Proposition 6. *Let (A_n) be a sequence in $(\mathcal{K}(X), d_H)$, let $A \in \mathcal{K}(X)$, and let $f : [0, \infty) \to [0, \infty)$ be a bounded modulus. If*

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\big(d_H(A_k, A)\big) = 0,$$

then $\mathcal{I}$-st-$\lim A_n = A$.

Proof. Fix $\varepsilon > 0$ and $\delta > 0$. Since f is increasing and $f(t) = 0$ only for $t = 0$, we have $f(\varepsilon) > 0$. For each $n \in \mathbb{N}$, let

$$E_n(\varepsilon) = \{k \le n : d_H(A_k, A) \ge \varepsilon\},$$

and

$$N_n(A) = \frac{1}{n}\sum_{k=1}^{n} f\big(d_H(A_k, A)\big).$$

Each index in $E_n(\varepsilon)$ contributes at least $f(\varepsilon)$ to the average defining $N_n(A)$. Hence

$$N_n(A) \ge \frac{f(\varepsilon)}{n}|E_n(\varepsilon)|.$$

Therefore

$$\frac{1}{n}|E_n(\varepsilon)| \le \frac{N_n(A)}{f(\varepsilon)}.$$

It follows that
$$\left\{n \in \mathbb{N} : \frac{1}{n}|E_n(\varepsilon)| \geq \delta\right\} \subset \{n \in \mathbb{N} : N_n(A) \geq \delta f(\varepsilon)\}.$$

The set on the right belongs to $\mathcal{I}$ by assumption. Hence $\mathcal{I}$-st-$\lim A_n = A$. □

The next theorem shows that a pairwise hypothesis can also be upgraded to $\mathcal{I}$-statistical convergence if the sequence admits a convergent subsequence whose index set is not too thin.

Theorem 5. *Let (A_n) be a sequence in $(\mathcal{K}(X), d_H)$ which is $\mathcal{I}$-statistically pre-Cauchy. Assume that there exist a set $E \subset \mathbb{N}$ and a compact set $A \in \mathcal{K}(X)$ such that*

(1)
$$\liminf_{n\to\infty} \frac{|E \cap \{1, \ldots, n\}|}{n} > 0,$$

(2) *the subsequence $(A_n)_{n\in E}$ converges ordinarily to A in $(\mathcal{K}(X), d_H)$.*

Then $\mathcal{I}$-st-$\lim A_n = A$.

Proof. Let
$$\alpha = \liminf_{n\to\infty} \frac{|E \cap \{1, \ldots, n\}|}{n} > 0.$$

Fix $\varepsilon > 0$. Since $(A_n)_{n\in E} \to A$, there exists $N \in \mathbb{N}$ such that
$$n \in E,\ n \geq N \quad \Longrightarrow \quad d_H(A_n, A) < \frac{\varepsilon}{2}.$$

For each $n \in \mathbb{N}$, define
$$U_n = \{k \leq n : k \in E,\ k \geq N\},$$
and
$$V_n(\varepsilon) = \{k \leq n : d_H(A_k, A) \geq \varepsilon\}.$$

By the choice of N, every index in U_n satisfies
$$d_H(A_k, A) < \frac{\varepsilon}{2}.$$

Since the lower density of E is α, there exists $n_0 \in \mathbb{N}$ such that
$$|U_n| \geq \frac{\alpha}{2} n \qquad \text{for all } n \geq n_0.$$

We claim that $\mathcal{I}$-st-$\lim A_n = A$. Suppose not. Then there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that the set
$$S = \left\{n \in \mathbb{N} : \frac{1}{n}|V_n(\varepsilon_0)| \geq \delta_0\right\},$$
does not belong to $\mathcal{I}$.

Fix $n \in S$ with $n \geq n_0$. For every $j \in U_n$ and every $k \in V_n(\varepsilon_0)$, we have
$$d_H(A_j, A_k) \geq d_H(A_k, A) - d_H(A_j, A) \geq \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}.$$

Therefore
$$U_n \times V_n(\varepsilon_0) \subset \left\{(j,k) : 1 \leq j,k \leq n,\ d_H(A_j, A_k) \geq \frac{\varepsilon_0}{2}\right\}.$$

Hence
$$\frac{1}{n^2}\left|\{(j,k) : 1 \leq j,k \leq n,\ d_H(A_j, A_k) \geq \varepsilon_0/2\}\right| \geq \frac{|U_n|\,|V_n(\varepsilon_0)|}{n^2}.$$

Using
$$|U_n| \geq \frac{\alpha}{2} n \qquad \text{and} \qquad |V_n(\varepsilon_0)| \geq \delta_0 n,$$

we obtain

$$\frac{1}{n^2}\left|\{(j,k) : 1 \le j,k \le n,\ d_H(A_j, A_k) \ge \varepsilon_0/2\}\right| \ge \frac{\alpha\delta_0}{2},$$

for all $n \in S$ with $n \ge n_0$.

Consequently,

$$S \cap \{n \ge n_0\} \subset \left\{ n \in \mathbb{N} : \frac{1}{n^2}\left|\{(j,k) : 1 \le j,k \le n,\ d_H(A_j, A_k) \ge \varepsilon_0/2\}\right| \ge \frac{\alpha\delta_0}{2} \right\}.$$

Since (A_n) is $\mathcal{I}$-statistically pre-Cauchy, the set on the right belongs to $\mathcal{I}$. The finite set $\{1, 2, \ldots, n_0 - 1\}$ also belongs to $\mathcal{I}$, and hence $S \in \mathcal{I}$, a contradiction. Therefore $\mathcal{I}\text{-}st\text{-}\lim A_n = A$. □

Remark 4. The previous theorem shows that pairwise asymptotic closeness may force $\mathcal{I}$-statistical convergence once a sufficiently thick convergent subsequence is available. This is one of the ways in which pre-Cauchy information can be upgraded to an actual limit without assuming a full equivalence between the two notions.

7. Examples and applications

We close the main part of the paper with several examples illustrating the scope of the preceding results. The first one shows how sparse perturbations lead to $\mathcal{I}$-statistical convergence and hence to the pre-Cauchy property. The second example explains why a pre-Cauchy condition alone should not be identified with convergence. We then record a simple frequent Cauchy example and conclude with an application to a compact-set iteration generated by a contractive iterated function system.

Example 1. Let $X = [0, 1]$ with the usual metric, and consider the sequence (A_n) in $\mathcal{K}([0, 1])$ defined by

$$A_n = \begin{cases} \{1\}, & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Since the set of perfect squares has natural density zero, the sequence (A_n) is statistically convergent to $\{0\}$ in the Hausdorff hyperspace $(\mathcal{K}([0, 1]), d_H)$. Indeed,

$$d_H(A_n, \{0\}) = \begin{cases} 1, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise,} \end{cases}$$

so the exceptional indices form a density-zero set.

Consequently, for every proper admissible ideal $\mathcal{I}$, the sequence (A_n) is $\mathcal{I}$-statistically convergent to $\{0\}$, and Theorem 1 shows that (A_n) is $\mathcal{I}$-statistically pre-Cauchy. In this case the pairwise behavior is easy to see directly: the only pairs with Hausdorff distance 1 are those in which at least one coordinate is a square index, and such pairs are asymptotically negligible.

Example 2. The pre-Cauchy property does not by itself imply $\mathcal{Z}$-statistical convergence.

Let $\mathcal{Z}$ denote the density-zero ideal

$$\mathcal{Z} = \{E \subset \mathbb{N} : d(E) = 0\}.$$

Define $T_1 = 1$, and recursively set

$$L_{r+1} = T_r^2, \qquad T_{r+1} = T_r + L_{r+1} \qquad (r \ge 1).$$

Let

$$I_1 = \{1\}, \qquad I_r = (T_{r-1}, T_r] \quad (r \ge 2).$$

Now define a sequence (A_n) in $\mathcal{K}([0,1])$ by

$$A_n = \begin{cases} \{0\}, & \text{if } n \in I_r \text{ and } r \text{ is odd,} \\ \{1\}, & \text{if } n \in I_r \text{ and } r \text{ is even.} \end{cases}$$

We first show that (A_n) is $\mathcal{Z}$-statistically pre-Cauchy. Since

$$d_H(\{0\}, \{1\}) = 1,$$

it is enough to consider $0 < \varepsilon \le 1$. For $n \in I_r$, write

$$n = T_{r-1} + t, \qquad 1 \le t \le L_r.$$

Let z_n and o_n denote, respectively, the numbers of indices $k \le n$ for which $A_k = \{0\}$ and $A_k = \{1\}$. Then

$$z_n + o_n = n,$$

and the set of bad pairs

$$\{(j,k) : 1 \le j, k \le n,\ d_H(A_j, A_k) \ge \varepsilon\},$$

has cardinality exactly

$$2z_n o_n.$$

During the block I_r, one of the two values $\{0\}$ or $\{1\}$ is repeated throughout the whole block. Hence the minority count among the first n terms is at most T_{r-1}. Therefore

$$2z_n o_n \le 2T_{r-1}n,$$

and so

$$\frac{1}{n^2}\big|\{(j,k) : 1 \le j, k \le n,\ d_H(A_j, A_k) \ge \varepsilon\}\big| = \frac{2z_n o_n}{n^2} \le \frac{2T_{r-1}}{n}.$$

Fix $\delta > 0$. If

$$\frac{2z_n o_n}{n^2} \ge \delta,$$

then necessarily

$$n \le \frac{2}{\delta} T_{r-1}.$$

Thus, inside the block I_r, the exceptional indices can occur only among the first

$$\left\lceil \frac{2}{\delta} T_{r-1} \right\rceil,$$

positions. Since

$$|I_r| = L_r = T_{r-1}^2,$$

the number of exceptional indices in I_r is $O(T_{r-1})$, while the block length is T_{r-1}^2. Therefore the proportion of exceptional indices inside each block tends to 0 as $r \to \infty$. Because the block lengths grow quadratically relative to the previous endpoints, the union of all exceptional indices has natural density zero. Hence (A_n) is $\mathcal{Z}$-statistically pre-Cauchy.

We next show that (A_n) is not $\mathcal{Z}$-statistically convergent. At the end of an odd block,

$$n = T_{2m-1},$$

the last block consists entirely of the value $\{0\}$, and its length is

$$L_{2m-1} = T_{2m-2}^2.$$

Since
$$T_{2m-1} = T_{2m-2} + T_{2m-2}^2,$$
we obtain
$$\frac{L_{2m-1}}{T_{2m-1}} = \frac{T_{2m-2}^2}{T_{2m-2} + T_{2m-2}^2} \to 1.$$

Therefore, along the subsequence $n = T_{2m-1}$, the proportion of indices $k \le n$ with $A_k = \{0\}$ tends to 1. Similarly, along the subsequence $n = T_{2m}$, the proportion of indices $k \le n$ with $A_k = \{1\}$ tends to 1.

Thus the sequence cannot be $\mathcal{Z}$-statistically convergent to $\{0\}$, and it cannot be $\mathcal{Z}$-statistically convergent to $\{1\}$. Since the sequence takes only the two values $\{0\}$ and $\{1\}$, no other $\mathcal{Z}$-statistical limit is possible. Hence (A_n) is not $\mathcal{Z}$-statistically convergent.

This example shows that pairwise asymptotic closeness may still fail to determine a genuine statistical limit.

Example 3. Let $X = [0,1]$ and define
$$A_n = \left[0, \frac{1}{n}\right], \qquad n \in \mathbb{N}.$$

Then $A_n \to \{0\}$ ordinarily in $(\mathcal{K}([0,1]), d_H)$. In particular, $flim\, A_n = \{0\}$. By Proposition 3, the sequence is frequent Cauchy.

This elementary example is useful for interpretation. The pairwise Hausdorff distance satisfies
$$d_H(A_m, A_n) = \left|\frac{1}{m} - \frac{1}{n}\right|,$$
so pairwise closeness is eventually uniform on large rectangles in $\mathbb{N} \times \mathbb{N}$. Thus the frequent Cauchy property here is simply the two-dimensional reflection of the ordinary collapse of the intervals to the singleton $\{0\}$.

Example 4. We now indicate how the preceding criteria apply to a compact-set iteration generated by a contractive iterated function system.

Let $X = [0,1]$, and consider the two contractions
$$f_1(x) = \frac{x}{3}, \qquad f_2(x) = \frac{x+2}{3}.$$

The associated Hutchinson operator
$$F : \mathcal{K}([0,1]) \to \mathcal{K}([0,1]), \qquad F(A) = f_1(A) \cup f_2(A),$$
generates the classical Cantor set C as its unique attractor. Starting from
$$C_0 = [0,1],$$
define the exact orbit by
$$C_n = F^n(C_0), \qquad n \in \mathbb{N}.$$

It is well known that
$$C_n \to C,$$
in the Hausdorff metric.

We now perturb this orbit on the set of square indices. Define
$$A_n = \begin{cases} [0,1], & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\ C_n, & \text{otherwise.} \end{cases}$$

We claim that
$$\mathcal{I}\text{-}st\text{-}\lim A_n = C,$$

for every proper admissible ideal $\mathcal{I}$.

To verify this, consider the one-point mean

$$\frac{1}{n}\sum_{k=1}^{n} d_H(A_k, C).$$

Split the sum into square and non-square indices:

$$\frac{1}{n}\sum_{k=1}^{n} d_H(A_k, C) = \frac{1}{n}\sum_{\substack{1\le k\le n\\ k \text{ not square}}} d_H(C_k, C) + \frac{1}{n}\sum_{\substack{1\le k\le n\\ k \text{ square}}} d_H([0,1], C).$$

Since $C_k \to C$ in the Hausdorff metric, the first term tends to 0. For the second term, note that $d_H([0,1], C)$ is a fixed constant and the number of square indices not exceeding n is at most $\sqrt{n}$. Hence

$$0 \le \frac{1}{n}\sum_{\substack{1\le k\le n\\ k \text{ square}}} d_H([0,1], C) \le \frac{\sqrt{n}}{n} d_H([0,1], C) \to 0.$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n} d_H(A_k, C) \to 0,$$

in the ordinary sense. In particular,

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d_H(A_k, C) = 0,$$

for every admissible ideal $\mathcal{I}$. Proposition 5 therefore yields

$$\mathcal{I}\text{-}st\text{-}\lim A_n = C.$$

This example shows that the one-point mean criterion recovers the asymptotic compact-set limit even when the exact Hutchinson orbit is disturbed infinitely many times, provided that the perturbations remain sparse.

The examples above illustrate both the usefulness and the limitations of the pairwise approach. They show that pairwise Hausdorff control alone need not imply genuine convergence, but it becomes effective when combined with additional structural assumptions.

8. Conclusion

In this paper, we studied pairwise asymptotic criteria for sequences of compact sets in the Hausdorff hyperspace of a metric space. The main focus was on $\mathcal{I}$-statistically pre-Cauchy behavior and the frequent Cauchy property, both expressed through pairwise Hausdorff differences rather than through a prescribed limit.

We showed that $\mathcal{I}$-statistical convergence implies the corresponding pre-Cauchy condition, and for bounded sequences we obtained two explicit characterizations, one in terms of double means of Hausdorff distances and the other through bounded moduli. On the frequent side, we formulated a Hausdorff version of the frequent Cauchy property and examined how it interacts with frequent convergence under additional compactness assumptions. We also identified several hypotheses under which pairwise asymptotic conditions can be upgraded to actual $\mathcal{I}$-statistical convergence.

The examples illustrate both the scope and the limitations of the theory. In particular, they show that pairwise asymptotic smallness alone does not force genuine convergence, although it becomes effective when combined with additional structural assumptions. In this way, the results clarify what pairwise criteria can recover in the hyperspace setting and where stronger hypotheses are necessary.

Several directions remain open. It would be natural to investigate weighted, deferred, and lacunary versions of the present criteria, as well as corresponding forms in Wijsman and bornological hyperspace settings. Another interesting question is whether similar pairwise methods can be developed for broader classes of set-valued dynamics.

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