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Article

New integral formulas involving a finite series in the numerator

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Abstract: This paper presents several new integral formulas inspired by classical results and problems in the existing literature. We present a simpler alternative proof of a well-known technical result that avoids the use of the digamma function. Additionally, we derive a set of integral formulas involving finite series in the numerator, generalizing existing formulas.

Keywords: integral formula, zeta function, Euler gamma function, Nielsen beta function

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1. Introduction

Integral formulas play a fundamental role in mathematical analysis. They provide essential tools for evaluating functions, solving differential equations, and developing theoretical frameworks in applied mathematics and physics. Over the years, numerous classical integral formulas have been established, each contributing to the advancement of analytical methods and computational techniques. Nevertheless, the pursuit of new, more comprehensive integral formulas remains an active area of research, driven by the need to evaluate complex expressions efficiently and reveal deeper connections between special functions. For more information, we refer to the books [1–3] and the recent studies in [4–13].

This paper presents several new integral formulas involving finite series in the numerator of a ratio-type integrand. The first result is inspired by [7, Proposition 2.1], as stated formally below.

Theorem 1. [7, Proposition 2.1] *For any $n \in \mathbb{N} \setminus \{1\}$ and $x > 0$, we have*

$$\int_0^\infty \frac{\sum_{i=0}^{n-1} e^{-(x+i/n)t} - ne^{-nxt}}{1-e^{-t}} dt = n \ln n.$$

The original proof is rather technical, as it depends on the digamma function and its multiplication formula. Here, we present a simpler alternative that avoids the use of the digamma function. The idea is to work on the finite series first, and to apply manageable integral developments.

The second contribution of the paper can be presented as a collection of new integral formulas involving finite series in the numerator of a ratio-type integrand. It is inspired by [14, Problem 4. (a)], as stated formally below.

Theorem 2. [14, Problem 4. (a)] *Let $n \in \mathbb{N}$. Then, we have*

$$I_n = \int_0^1 \frac{\ln(1-x) + \sum_{i=1}^{n} \frac{x^i}{i}}{x^{n+1}} dx = -\frac{1}{n} H_n,$$

where

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

In particular, this problem has inspired us to a class of integral formulas defined as

$$I_n = \int_0^1 \frac{\ln(1-x) + \sum_{i=1}^{n} \frac{x^i}{i}}{f(x)} dx,$$

where f denotes a chosen weight function. We also consider other technical variations based on a transformation of the logarithmic term. Several new integral formulas are thus derived. These formulas involve a wide variety of special functions and rely on some existing results in the literature, notably those in [6]. In a sense, they can be viewed as generalizations. The formulas developed here are therefore expected to enrich the repository of known integral formulas and inspire further exploration of their properties and applications.

The remainder of the paper is as follows: §2 is devoted to the alternative proof of [7, Proposition 2.1]. §3 presents a collection of integral formulas derived from [14, Problem 4. (a)]. A conclusion is given in §4.

2. First integral result

The theorem below corresponds to [7, Proposition 2.1]. The main novelty is the proof, which relies on elementary tools and is considerably simpler than the original argument involving the digamma function and its properties.

Theorem 3. [7, Proposition 2.1] *For any $n \in \mathbb{N} \setminus \{1\}$ and $x > 0$, we have*

$$\int_0^{\infty} \frac{\sum_{i=0}^{n-1} e^{-(x+i/n)t} - ne^{-nxt}}{1-e^{-t}} dt = n \ln n.$$

New proof. We begin by the following finite geometric series result:

$$\sum_{i=0}^{n-1} e^{-(x+i/n)t} = e^{-xt} \sum_{i=0}^{n-1} e^{-(t/n)i} = e^{-xt} \frac{1-e^{-t}}{1-e^{-t/n}}.$$

Using this development, we set

$$I(x) = \int_0^{\infty} \frac{\sum_{i=0}^{n-1} e^{-(x+i/n)t} - ne^{-nxt}}{1-e^{-t}} dt = \int_0^{\infty} \left(\frac{e^{-xt}}{1-e^{-t/n}} - n\frac{e^{-nxt}}{1-e^{-t}} \right) dt.$$

Differentiating under the integral sign using the Leibniz rule gives

$$\begin{aligned} I'(x) &= \int_0^{\infty} \left(-t\frac{e^{-xt}}{1-e^{-t/n}} + n^2 t \frac{e^{-nxt}}{1-e^{-t}} \right) dt \\ &= -\int_0^{\infty} \frac{te^{-xt}}{1-e^{-t/n}} dt + \int_0^{\infty} \frac{n^2 te^{-nxt}}{1-e^{-t}} dt. \end{aligned}$$

In the second integral, we apply the change of variables $u = nt$, yielding

$$I'(x) = -\int_0^{\infty} \frac{te^{-xt}}{1-e^{-t/n}} dt + \int_0^{\infty} \frac{ue^{-xu}}{1-e^{-u/n}} du = 0.$$

Hence, $I(x)$ is constant for any $x > 0$. Let $I(x) = c$ for some $c \in \mathbb{R}$. To determine this constant, we consider the limit as $x \to \infty$, so that

$$c = \lim_{x\to\infty} I(x) = \lim_{x\to\infty} \int_0^\infty \left(\frac{e^{-xt}}{1-e^{-t/n}} - n\frac{e^{-nxt}}{1-e^{-t}} \right) dt.$$

Applying the change of variables $y = xt$, so $t = y/x$ and $dt = dy/x$, we obtain

$$c = \lim_{x\to\infty} \int_0^\infty \left(\frac{e^{-y}}{1-e^{-y/(nx)}} - n\frac{e^{-ny}}{1-e^{-y/x}} \right) \frac{dy}{x}.$$

Using the limit $\lim_{x\to\infty} x(1-e^{-y/(nx)}) = y/n$, the integrand tends to

$$\frac{n(e^{-y}-e^{-ny})}{y}.$$

Therefore, by the dominated convergence theorem, we have

$$c = n\int_0^\infty \frac{e^{-y}-e^{-ny}}{y} dy.$$

Finally, a standard identity (see [3, Entry 3.434(2), Page no. 363]) gives

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t} dt = \ln\left(\frac{b}{a}\right),$$

for any $a, b > 0$. We conclude that $c = n\ln n$. Therefore, we have

$$I(x) = n\ln n.$$

This completes the proof. □

Compared to the proof given in [7, Proposition 2.1], the approach proposed here is significantly simpler. This is because it directly uses the structure of the finite series in the numerator, avoiding the technical complexities associated with the digamma function.

3. Integral results of another type

We now focus on a new collection of integral formulas derived from [14, Problem 4. (a)].

3.1. Some preliminaries

Recall from [15, equation 25.5.3 and 25.5.4] that, the integral representations of the zeta function are given by

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x+1} dx, \quad s > 1, \tag{1}$$

and

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s+1)} \int_0^\infty \frac{e^x x^s}{(e^x+1)^2} dx, \quad s > 1,$$

where $\Gamma(s)$ is the classical Euler's gamma function. The Dirichlet eta function is defined by

$$\eta(x) = \sum_{i=1}^\infty \frac{(-1)^{i-1}}{i^x},$$

and the relation between the functions ζ and η is given in [1, Equation 23.2.19] by

$$\eta(x) = (1-2^{1-x})\zeta(x), \quad x > 0. \tag{2}$$

The Nielsen beta-function, first introduced in [16], can be represented through several equivalent forms, such as

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}dt, \quad x > 0,$$

and

$$\beta(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+x} = \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \quad x > 0,$$

where $\psi(x) = d\ln\Gamma(x)/dx$ is the psi or digamma function and $\Gamma(x)$ is the Euler gamma function. Recent developments on the Nielsen beta function can be found in [17] and [18].

3.2. Results with proofs

The theorem below is derived from [14, Problem 4. (a)], with the ratio-type integrand having a different denominator. This fundamentally changes the value of the integral.

Theorem 4. *Let $n \in \mathbb{N}$ and $\nu > 0$. Then, we have*

$$I_n = \int_0^1 \frac{\ln(1-x) + \sum_{i=1}^{n} \frac{x^i}{i}}{1+x^\nu}dx = -\frac{1}{\nu}\sum_{k=0}^{\infty} \frac{1}{n+k+1}\beta\left(\frac{n+k+2}{\nu}\right).$$

Furthermore, when $n = 1$ and $\nu = 2$, we have

$$\sum_{k=0}^{\infty} \frac{1}{k+2}\beta\left(\frac{k+3}{2}\right) = 2G - \left(\frac{4+\pi}{4}\right)\ln 2,$$

where G is the Catalan constant.

Proof. For any $x \in (0,1)$, we have

$$\ln(1-x) = -\sum_{i=1}^{\infty} \frac{x^i}{i}.$$

Then, exchanging the integral and sum signs by the monotone convergence theorem, we have

$$\begin{aligned} I_n &= -\int_0^1 \frac{\sum_{i=n+1}^{\infty} \frac{x^i}{i}}{1+x^\nu}dx = -\int_0^1 \sum_{i=n+1}^{\infty} \frac{x^i}{i(1+x^\nu)}dx \\ &= -\sum_{i=n+1}^{\infty} \frac{1}{i}\int_0^1 \frac{x^i}{1+x^\nu}dx. \end{aligned} \tag{3}$$

From [3, Entry 3.241(1), Page no. 324],

$$\int_0^1 \frac{x^{\mu-1}}{1+x^p}dx = \frac{1}{p}\beta\left(\frac{\mu}{p}\right), \Re(\mu) > 0, p > 0,$$

Eq. (3) becomes

$$\begin{aligned} I_n &= -\sum_{i=n+1}^{\infty} \frac{1}{i\nu}\beta\left(\frac{i+1}{\nu}\right) \\ &= -\frac{1}{\nu}\sum_{k=0}^{\infty} \frac{1}{n+k+1}\beta\left(\frac{n+k+2}{\nu}\right). \end{aligned}$$

This shows the first result. Next, we have

$$I = \int_0^1 \frac{\ln(1-x)+x}{1+x^2}dx = \int_0^1 \frac{\ln(1-x)}{1+x^2}dx + \int_0^1 \frac{x}{1+x^2}dx.$$

The second integral is elementary. Indeed, we have

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{1}{2}\ln 2.$$

For the first integral, set $x = \tan\theta$ with $\theta \in [0, \pi/4]$. Then $dx = \sec^2\theta d\theta$ and $1+x^2 = \sec^2\theta$, so

$$\int_0^1 \frac{\ln(1-x)}{1+x^2}dx = \int_0^{\pi/4} \ln(1-\tan\theta)d\theta.$$

Note that

$$1-\tan\theta = \frac{\cos\theta - \sin\theta}{\cos\theta} = \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\cos\theta}.$$

Hence, we have

$$\ln(1-\tan\theta) = \tfrac{1}{2}\ln 2 + \ln\cos\left(\theta + \frac{\pi}{4}\right) - \ln\cos\theta.$$

Integrating termwise and using the classical identities (see [3, Entry 4.224 (4 and 5), Page no. 534]), we obtain

$$\int_0^{\pi/2} \ln(\cos u)du = -\frac{\pi}{2}\ln 2, \qquad \int_0^{\pi/4} \ln(\cos u)du = -\frac{\pi}{4}\ln 2 + \frac{G}{2}.$$

Therefore, we have

$$\int_0^1 \frac{\ln(1-x)}{1+x^2}dx = \frac{\pi}{8}\ln 2 - G,$$

which implies that

$$I = \frac{\pi}{8}\ln 2 - G + \frac{1}{2}\ln 2 = \left(\frac{\pi+4}{8}\right)\ln 2 - G.$$

Thus, we derive

$$\sum_{k=0}^{\infty} \frac{1}{k+2}\beta\left(\frac{k+3}{2}\right) = 2G - \left(\frac{4+\pi}{4}\right)\ln 2.$$

This completes the proof. □

Another variation is proposed in the theorem below.

Theorem 5. *Let $n \in \mathbb{N}$. Then, we have*

$$I_n = \int_0^1 \frac{\ln(1-x) + \sum_{i=1}^{n} \frac{x^i}{i}}{\sqrt{1-x}}dx = -2\sum_{k=0}^{\infty} \frac{1}{n+k+1} \cdot \frac{(2n+2k+2)!!}{(2n+2k+3)!!},$$

where

$$n!! = \begin{cases} n \cdot (n-2) \cdot (n-4) \cdots 2, & \text{if } n \text{ is even,} \\ n \cdot (n-2) \cdot (n-4) \cdots 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, when $n = 1$, we have

$$I_1 = -2\sum_{k=0}^{\infty} \frac{1}{k+2} \cdot \frac{(2k+4)!!}{(2k+5)!!} = -\frac{8}{3}.$$

Proof. For any $x \in (0,1)$, we have

$$\ln(1-x) = -\sum_{i=1}^{\infty} \frac{x^i}{i}.$$

Then, exchanging the integral and sum signs by the monotone convergence theorem, we have

$$I_n = -\int_0^1 \frac{\sum_{i=n+1}^{\infty} \frac{x^i}{i}}{\sqrt{1-x}} dx = -\sum_{i=n+1}^{\infty} \frac{1}{i} \int_0^1 \frac{x^i}{\sqrt{1-x}} dx. \tag{4}$$

The standard identity [3, Entry 3.226(1), Page no. 322], gives

$$\int_0^1 \frac{x^n\, dx}{\sqrt{1-x}} = 2\frac{(2n)!!}{(2n+1)!!},$$

then the Eq. (4) becomes

$$I_n = -2\sum_{i=n+1}^{\infty} \frac{1}{i} \frac{(2i)!!}{(2i+1)!!} = -2\sum_{k=0}^{\infty} \frac{1}{n+k+1} \frac{(2n+2k+2)!!}{(2n+2k+3)!!}.$$

The first result is proved. For the second result, i.e., when $n = 1$, we evaluate

$$I_1 = \int_0^1 \frac{\ln(1-x) + x}{\sqrt{1-x}} dx.$$

Let us make the change of variables $1 - x = t^2$, so that $x = 1 - t^2$ and $dx = -2tdt$. When $x = 0$, we have $t = 1$, and, when $x = 1$, we have $t = 0$. Substituting gives

$$I_1 = \int_1^0 \frac{\ln(t^2) + (1-t^2)}{t}(-2tdt) = 2\int_0^1 \left[\ln(t^2) + 1 - t^2\right] dt.$$

Simplifying, we obtain

$$I_1 = 2\int_0^1 (2\ln t + 1 - t^2)dt = 4\int_0^1 \ln t dt + 2\int_0^1 (1-t^2)dt.$$

Now, using the known integrals

$$\int_0^1 \ln t dt = -1, \qquad \int_0^1 (1-t^2)dt = 1 - \frac{1}{3} = \frac{2}{3},$$

we get

$$I_1 = 4(-1) + 2\left(\frac{2}{3}\right) = -4 + \frac{4}{3} = -\frac{8}{3}.$$

Therefore, we have

$$I_1 = -2\sum_{k=0}^{\infty} \frac{1}{k+2} \frac{(2k+3)!!}{(2k+5)!!} = -\frac{8}{3}.$$

This ends the proof. □

The theorem below considers another numerator term than that in [14, Problem 4. (a)].

Theorem 6. *Let $n \in \mathbb{N} \setminus \{1\}$. Then, we have*

$$I_n = \int_0^1 \frac{\{\ln(1-x)\}^2 - \sum_{i=2}^{n} \frac{2H_{i-1}}{i} x^i}{x^{n+1}} dx = \sum_{k=0}^{\infty} \frac{2H_{k+n}}{(n+k+1)(k+1)}.$$

Proof. Utilizing the series representation of $\{\ln(1-x)\}^2$ (see [3, Entry 1.516(1), Page. no 54]), we have

$$\{\ln(1-x)\}^2 = \sum_{i=2}^{\infty} 2\frac{H_{i-1}}{i} x^i.$$

Then, exchanging the integral and sum signs by the monotone convergence theorem, we have

$$I_n = \int_0^1 2\frac{\sum_{i=n+1}^{\infty} \frac{H_{i-1}}{i} x^i}{x^{n+1}} dx = 2\sum_{i=n+1}^{\infty} \frac{H_{i-1}}{i} \int_0^1 x^{i-n-1} dx = 2\sum_{i=n+1}^{\infty} \frac{H_{i-1}}{i(i-n)}$$
$$= 2\sum_{k=0}^{\infty} \frac{H_{k+n}}{(n+k+1)(k+1)}.$$

The desired result is obtained, ending the proof. □

Remark 1. For $n = 2$, one can show that

$$\sum_{k=0}^{\infty} \frac{H_{k+2}}{(k+3)(k+1)} = \frac{3}{4} + \frac{\pi^2}{12}.$$

The theorem below considers another domain of integration and functions than those in [14, Problem 4. (a)].

Theorem 7. *Let $n \in \mathbb{N}$. Then, we have*

$$I_n = \int_0^{\infty} \frac{e^{-x} - \sum_{i=0}^{n} \frac{(-1)^i x^i}{i!}}{1+e^x} dx = \sum_{k=0}^{\infty} (-1)^{n+k+1} \eta(n+k+2),$$

where η is the Dirichlet eta function.

Proof. For any $x > 0$, we have

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}.$$

Then, exchanging the integral and sum signs by the dominated convergence theorem, we have

$$I_n = \int_0^{\infty} \frac{\sum_{i=n+1}^{\infty} \frac{(-1)^i x^i}{i!}}{1+e^x} dx = \sum_{i=n+1}^{\infty} \frac{(-1)^i}{i!} \int_0^{\infty} \frac{x^i}{1+e^x} dx. \tag{5}$$

By Eq. (1), Eq. (5) becomes

$$I_n = \sum_{i=n+1}^{\infty} \frac{(-1)^i}{i!} \zeta(i+1)\Gamma(i+1)\left(1 - 2^{1-(i+1)}\right).$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{n+k+1}}{(n+k+1)!} \zeta(n+k+2)\Gamma(n+k+2)\left(1 - 2^{1-(n+k+2)}\right).$$

Moreover, by Eq. (2), we have

$$I_n = \sum_{k=0}^{\infty} (-1)^{n+k+1} \eta(n+k+2).$$

This completes the proof. □

Remark 2. For $n = 0$ and $n = 1$, we have

$$\sum_{k=0}^{\infty} (-1)^k \eta(k+2) = 2\ln(2) - 1, \tag{6}$$

and

$$\sum_{k=0}^{\infty}(-1)^k\eta(k+3) = -2\ln(2) + \frac{\pi^2}{12} + 1. \tag{7}$$

Adding Eqs. (6) and (7), we obtain

$$\sum_{k=0}^{\infty}(-1)^k\left(\eta(k+2)+\eta(k+3)\right) = \frac{\pi^2}{12}.$$

A sophisticated variation of [14, Problem 4. (a)] is introduced in the theorem below.

Theorem 8. *Let $n \in \mathbb{N}$ and $\alpha \in (0,1)$. Then, we have*

$$\begin{aligned} I_n &= \int_0^1 \frac{\ln(1+\alpha x) - \sum_{i=1}^{n} \frac{(-1)^{i+1}\alpha^i}{i}x^i}{1+\alpha x^2}dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{n+k}\alpha^{n+k+1}}{(n+k+1)(n+k+2)}{}_2F_1\left(1, \frac{n+k+2}{2}; \frac{n+k+4}{2}; -\alpha\right). \end{aligned}$$

Proof. For $|\alpha x| < 1$, it is well known that the power series expansion of $\ln(1+\alpha x)$ is given by

$$\ln(1+\alpha x) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}\alpha^i x^i}{i}.$$

Hence, exchanging the integral and sum signs by the dominated convergence theorem, we obtain

$$I_n = \int_0^1 \sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}\alpha^i x^i}{i(1+\alpha x^2)}dx = \sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}\alpha^i}{i}\int_0^1 \frac{x^i}{1+\alpha x^2}dx. \tag{8}$$

The standard identity [3, Entry 3.194] gives

$$\int_0^1 \frac{x^{a-1}}{(1+bx)^{\nu}}dx = \frac{1}{a}{}_2F_1(\nu, a; 1+a; -b), \quad |\arg(1+b)| < \pi, \Re(a) > 0,$$

the integral in Eq. (8) can be evaluated so that

$$\begin{aligned} I_n &= \sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}\alpha^i}{i(i+1)}{}_2F_1\left(1, \frac{i+1}{2}; \frac{i+3}{2}; -\alpha\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{n+k}\alpha^{n+k+1}}{(n+k+1)(n+k+2)}{}_2F_1\left(1, \frac{n+k+2}{2}; \frac{n+k+4}{2}; -\alpha\right). \end{aligned}$$

The proof is complete. □

Remark 3. When $n = 1$, it follows from [6, Proposition 2.1] that

$$\int_0^1 \frac{\ln(1+\alpha x)}{1+\alpha x^2}dx = \frac{1}{2\sqrt{\alpha}}\arctan(\sqrt{\alpha})\ln(1+\alpha).$$

Consequently, the corresponding series sum is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\alpha^{k+2}}{(k+2)(k+3)}{}_2F_1\left(1, \frac{k+3}{2}; \frac{k+5}{2}; -\alpha\right) = \frac{1}{2\alpha}\left(\sqrt{\alpha}\arctan(\sqrt{\alpha}) - 1\right)\ln(1+\alpha).$$

Another variation is presented in the theorem below.

Theorem 9. *Let $n \in \mathbb{N}$ and $\alpha \in (0,1)$. Then, we have*

$$I_n = \int_0^1 \frac{\ln(1+\alpha x) - \sum_{i=1}^{n} \frac{(-1)^{i+1}\alpha^i}{i} x^i}{(1+\alpha x^2)^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}\alpha^{n+k+1}}{(n+k+1)(n+k+2)} {}_2F_1\left(2, \frac{n+k+2}{2}; \frac{n+k+4}{2}; -\alpha\right).$$

Proof. For $|\alpha x| < 1$, it is well known that the power series expansion of $\ln(1+\alpha x)$ is given by

$$\ln(1+\alpha x) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}\alpha^i x^i}{i}.$$

Hence, exchanging the integral and sum signs by the dominated convergence theorem, we obtain

$$I_n = \int_0^1 \sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}\alpha^i x^i}{i(1+\alpha x^2)^2} dx = \sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}\alpha^i}{i} \int_0^1 \frac{x^i}{(1+\alpha x^2)^2} dx. \tag{9}$$

From [3, Entry 3.194], the integral in (9) can be evaluated to yield

$$I_n = \sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}\alpha^i}{i(i+1)} {}_2F_1\left(2, \frac{i+1}{2}; \frac{i+3}{2}; -\alpha\right) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}\alpha^{n+k+1}}{(n+k+1)(n+k+2)} {}_2F_1\left(2, \frac{n+k+2}{2}; \frac{n+k+4}{2}; -\alpha\right).$$

□

Remark 4. When $n = 1$, it follows from [6, Proposition 3.4] that

$$\int_0^1 \frac{\ln(1+\alpha x)}{(1+\alpha x^2)^2} dx = \frac{4}{\sqrt{\alpha}} \arctan(\sqrt{\alpha}) \ln(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha}\arctan(\sqrt{\alpha}) - \frac{3}{4}\ln(1+\alpha)\right\}.$$

Consequently, the corresponding series sum is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\alpha^{k+2}}{(k+2)(k+3)} {}_2F_1\left(2, \frac{k+3}{2}; \frac{k+5}{2}; -\alpha\right) = \frac{4}{\sqrt{\alpha}} \arctan(\sqrt{\alpha}) \ln(1+\alpha) - \frac{1}{1+\alpha}\left\{\frac{1}{2}\sqrt{\alpha}\arctan(\sqrt{\alpha}) - \frac{3}{4}\ln(1+\alpha)\right\} + \frac{1}{2(1+\alpha)}.$$

4. Conclusion

In conclusion, this paper draws inspiration from [7] and [14] to advance the study of integral formulas involving a finite series in the numerator. It provides new results and alternative proofs that extend several known results in the literature, including those in [6]. The paper presents one alternative proof of a known theorem, and several new theorems have been rigorously established , each offering potential avenues for further research. These findings enrich the current repository of integral formulas. They also suggest directions for future investigations, such as exploring generalizations, applications to special functions, and connections with analytic and computational techniques in integral evaluation and special function identities.

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