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New maximal monotonicity results and existence of solution for evolution variational inequality problem for multi-valued perturbation of p -Laplace operator

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Abstract: In this paper, we shall prove an existence of solution for the constrained evolution variational inequality problem of finding $\zeta \in \mathcal{K} \subseteq Y = L^p(0, T; W_0^{1,p}(\Omega))$ (where \mathcal{K} is a nonempty, closed, convex and symmetric subset of Y) with $p \geq 2$, $T > 0$, $\zeta' \in Y^*$ and $\zeta(0) = \zeta(T)$, such that

$$\langle \mathcal{A}\zeta - \Delta_p \zeta, v - \zeta \rangle + \langle Q\zeta, v - \zeta \rangle + \Phi(v) - \Phi(\zeta) \geq \langle f^*, v - \zeta \rangle, \quad (*)$$

for all $v \in \mathcal{K}$, where $f^* \in Y^*$,

$$\begin{aligned} \langle Q\zeta, w \rangle &= \sum_{i=1}^N \int_{\Gamma} q_i(x, t, \zeta(x, t), \nabla \zeta(x, t)) \frac{\partial w(x, t)}{\partial x_i} dx dt \\ &+ \int_{\Gamma} q_0(x, t, \zeta(x, t), \nabla \zeta(x, t)) w(x, t) dx dt, \quad w \in Y, \zeta \in Y, \end{aligned} \quad (**)$$

$\Gamma = [0, T] \times \Omega$, Ω is a nonempty, bounded and open subset of \mathbf{R}^N with $N \in \mathbf{Z}_+$, Δ_p is the p -Laplacian operator, $q_i : \overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}$ satisfies mild conditions, $\mathcal{A}\zeta = \zeta'$ (where ζ' is the derivative of ζ in the sense of distributions) and $\Phi : Y \supseteq D(\Phi) \rightarrow \mathbf{R} \cup \{\infty\}$ is a proper, convex and lower-semi-continuous function. In order to address problems like (*), we shall establish new maximal monotonicity results for the sum of two maximal monotone operators \mathcal{N} and \mathcal{M} defined from reflexive Banach space into its dual, provided that $\text{int}(D(\psi_{\mathcal{M}}) - D(\psi_{\mathcal{N}})) \neq \emptyset$, where $\psi_{\mathcal{N}}$ and $\psi_{\mathcal{M}}$ are corresponding convex functions introduced by Simons. The question of maximal monotonicity of two maximal monotone operators is one of the outstanding problems in monotone operator theory. The significant contribution of Rockafellar gave a foundation in the study of nonlinear problems. In this paper, we give new maximality results, which present generalizations of the existing criteria, and provide a positive solution for Simons problem [1, Problem 21.5, p.84]. With the help of these results, existence of solution for (*) is proved possibly allowing $D(\mathcal{A}) \cap \text{int}\mathcal{K} = \emptyset$.

Keywords: Simons problems, maximality theorems, variational inequality problems, Parabolic inequalities, evolution problems, convex functions

MSC: 47H07, 47H11, 47H14.

1. Introduction

The main goal of this paper is to prove existence of solution (s) for problem (*). The essential strategy is to prove existence of solution for corresponding abstract variational inequality problem. Indeed, the theory of variational inequality problems has been an essential approach to deal with nonlinear problems in differential equations and optimization problems corresponding to various research fields (cf., [2–7] and the references therein). The theory of maximal monotone operators provides formulations for various types of variational inequality problems. For further references on monotone operator theory, the reader is referred to the books by D. Motreanu, V. Motreanu and Papageorgiou [5], Zeidler [7], Barbu [8], Showalter [9], Pascali-Sburlan [10], Cioranescu [11], and references therein.

In particular, the fundamental properties of maximal monotone operators have been played a great role in the treatment of nonlinear inclusion problems. In most cases, variational inequality problems are formulated in the form of operator inclusion problems involving the subdifferentials of convex functions (in the sense of convex analysis, cf., [7]) and subdifferentials of locally Lipschitz functions (in the sense of Clarke, [6]) perturbed by maximal monotone and /or pseudomonotone operators. Because of these, it is fundamental to derive abstract existence results corresponding to a given variational inequality problem. In order to successfully apply abstract variational inequality results corresponding to (*), it is essential to impose mild suitable conditions on the functions q_i and Φ as given below:

- (c₁) $T > 0$, Ω is a nonempty, bounded and open subset of \mathbf{R}^N (with $N \in \mathbf{Z}_+$) and $\Gamma = (0, T) \times \Omega$, $Y = L^p(0, T; W_0^{1,p}(\Omega))$ ($p \geq 2$), q is the conjugate exponent of p (i.e., $p^{-1} + q^{-1} = 1$), \mathcal{K} is a nonempty, closed and convex subset of Y and $f^* \in Y^*$, where Y^* is the dual space of Y .
- (c₂) $\Phi : Y \supseteq D(\Phi) \rightarrow \mathbf{R} \cup \{\infty\}$ is a proper, convex and lower semi-continuous function.
- (c₃) $Q : Y \rightarrow Y^*$ is generated by the differential operator

$$(\mathcal{D}u)(x, t) = - \sum_{i=0}^N \frac{\partial}{\partial x_i} q_i(x, t, u(x, t), \nabla u(x, t)), \quad u \in Y, \quad (x, t) \in \overline{\Omega} \times [0, T],$$

where $\frac{\partial}{\partial x_0} q_0(x, t, u(x, t), \nabla u(x, t))$ denotes $q_0(x, t, u(x, t), \nabla u(x, t))$, provided that the following conditions hold.

- (c₃¹) For each $i = 0, 1, 2, \dots, N$, $q_i : \overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}$ is Carathéodory function, i.e., $(x, t) \mapsto q_i(x, t, \zeta, \eta)$ is measurable for almost all $(\zeta, \eta) \in \mathbf{R} \times \mathbf{R}^N$ and $(\zeta, \eta) \mapsto q_i(x, t, \zeta, \eta)$ is continuous for almost all $(x, t) \in \overline{\Omega} \times [0, T]$.
- (c₃²) For each $\eta = (\eta_i) \in \mathbf{R}^N$ and $\eta^* = (\eta_i^*) \in \mathbf{R}^N$ with $\eta \neq \eta^*$, we have

$$\sum_{i=1}^N (q_i(x, t, \zeta, \eta) - q_i(x, t, \zeta, \eta^*)) (\eta_i - \eta_i^*) > 0,$$

- for all $(x, t) \in \overline{\Omega} \times [0, T]$ and $\zeta \in \mathbf{R}$.
- (c₃³) For some $c \in (0, pq(Np + 2q)^{-1})$, we assume

$$|q_i(x, t, \zeta, \eta)| \leq c(|\zeta|^{p-1} + |\eta|^{p-1}),$$

for all $(x, t) \in \overline{\Omega} \times [0, T]$, $\zeta \in \mathbf{R}$, $\eta \in \mathbf{R}^N$ and $i = 0, 1, 2, \dots, N$.

- (c₄) There exist $\theta \geq 1$ and $\pi \in \mathbf{R}$ such that

$$q_0(x, t, \zeta, \eta) \zeta \geq \theta |\zeta|^p - \pi,$$

for all $(x, t, \zeta, \eta) \in \overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N$.

Example 1. For each $i = 1, 2, \dots, N$, let $p \geq 2$ is an even integer, $c \in [1, pq(Np + 2q)^{-1})$ and

$$q_i(x, t, s, \zeta) = c|\zeta|^{p-2} \zeta_i \text{ and } q_0(x, t, s, \zeta) = c|s|^{p-2},$$

for all $(x, t, s, \zeta) \in \overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N$. Then each q_i ($i = 0, 1, 2, \dots, N$) satisfies conditions (c₃) and (c₄). Indeed, each q_i ($i = 0, 1, 2, \dots, N$) is a continuous function defined from $\overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N$ into \mathbf{R} , i.e., (c₃¹) is satisfied. For $p \geq 2$, it is a well-known fact (cf., [5]) that there exists $c > 0$ such that

$$\left(|\eta|^{p-2} \eta - |\eta^*|^{p-2} \eta^* \right) (\eta - \eta^*) \geq c |\eta - \eta^*|^p,$$

for all $\eta = (\eta_i) \in \mathbf{R}^N$ and $\eta^* = (\eta_i^*) \in \mathbf{R}^N$. Consequently, we obtain

$$\sum_{i=1}^N (q_i(x, t, \zeta, \eta) - q_i(x, t, \zeta, \eta^*)) (\eta_i - \eta_i^*) = \sum_{i=1}^N (|\eta|^{p-2} \eta_i - |\eta^*|^{p-2} \eta_i^*) (\eta_i - \eta_i^*)$$

$$\begin{aligned}
 &= \left(|\eta|^{p-2}\eta - |\eta^*|^{p-2}\eta^* \right) (\eta - \eta^*) \\
 &\geq c|\eta - \eta^*|^p > 0,
 \end{aligned}
 \tag{1}$$

for all $(x, t) \in \overline{\Omega} \times [0, T]$, $\eta = (\eta_i) \in \mathbf{R}^N$ and $\eta^* = (\eta_i^*) \in \mathbf{R}^N$ with $\eta \neq \eta^*$. This shows that (c_3^2) holds. In addition (c_3^3) holds because $|\eta_i| \leq |\eta|$ for all $i = 0, 1, \dots, N$. On the other hand, q_0 fulfills condition (c_4) for all $\pi \geq 0$, because one can take $\theta = c \geq 1$ and $q_0(x, t, \zeta, \eta)\zeta = c|\zeta|^{p-2}\zeta^2 \geq c|\zeta|^p - \pi$ for all $(x, t, s, \zeta) \in \overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N$.

Example 2. Let Ω be a nonempty, bounded and open subset of \mathbf{R}^N ($N \in \mathbf{Z}_+$), $p \geq 2$ and $\phi : \mathbf{R} \rightarrow [0, \infty]$ (with $\phi(0) = 0$) be even, convex, proper and lower-semicontinuous function. Define

$$\begin{cases} \Phi(u) = \int_{\Gamma} \phi(u(x, t)) dx dt, & u \in D(\Phi), \\ \infty, & \text{otherwise,} \end{cases}
 \tag{2}$$

where

$$D(\Phi) = \{u \in Y : \phi(u) \in L^1(\Gamma)\}.$$

Obviously, Φ is proper because Φ is not identically ∞ on Y . and convex. It is convex because ϕ is convex. It remain to verify that Φ is lower semicontinuous, i.e.,

$$\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n),$$

whenever $u_n \rightarrow u$ in Y , as $n \rightarrow \infty$. It is known that this condition is equivalent to say: For each $\lambda \in \mathbf{R} \cup \{\infty\}$, the level set $E_\lambda = \{u \in Y : \Phi(u) \leq \lambda\}$ is closed (cf., Barbu [8, pp. 4-8]). Thus it is sufficient to show that E_λ is a closed subset of Y . To this end, let $\{u_n\} \subseteq E_\lambda$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$. Clearly, $\lambda = \infty$ implies $E_\lambda = Y$, which is closed. Assume that $\lambda \in \mathbf{R}$. We assume that there exists a subsequence, denoted again by $\{u_n\}$, such that $u_n(x, t) \rightarrow u(x, t)$ a.e., in $[0, T] \times \overline{\Omega}$. The lower semi-continuity of ϕ on \mathbf{R} shows that

$$\phi(u(x, t)) \leq \liminf_{n \rightarrow \infty} \phi(u_n(x, t)) \text{ a.e., in } \Gamma. \tag{3}$$

Consequently, (3) along with the generalized Fatou’s Lemma confirms that

$$\begin{aligned}
 \Phi(u) &= \int_{\Gamma} \phi(u(x, t)) dx dt \\
 &\leq \int_{\Gamma} \liminf_{n \rightarrow \infty} \phi(u_n(x, t)) dx dt \\
 &\leq \liminf_{n \rightarrow \infty} \int_{\Gamma} \phi(u_n(x, t)) dx dt \leq \lambda,
 \end{aligned}
 \tag{4}$$

i.e., $u \in E_\lambda$. Thus E_λ is closed. Thus Φ is a proper, convex and lower semi-continuous function, i.e., Φ satisfies (c_2) .

Example 3. Let

$$\mathcal{K} = \{v \in Y : |\nabla v(x, t)| \leq \omega(x, t), \text{ almost everywhere in } \Omega \times [0, T]\},$$

where $\omega : [0, T] \times \overline{\Omega} \rightarrow \mathbf{R}$ is a non negative function. Then \mathcal{K} is a nonempty, closed, convex and symmetric subset of Y . This constraint set is known as gradient obstacle constraint. The function ω is called an obstacle function. For more, constraint sets corresponding to elliptic as well as parabolic partial differential equations and variational inequality problems, the reader is referred to the book by Showalter [9, pp. 70-75], Kenmochi [4] and references therein.

In Examples 1–3, it is demonstrated that \mathcal{K} , Φ and q_i ($i = 0, 1, \dots, N$) satisfy the corresponding conditions in (c_1) through (c_4) .

For the purpose of clarity, we shall provide brief definitions and properties of relevant operators which correspond with our main goal. Throughout the paper, we shall denote a reflexive Banach space by E and, its

dual space by E^* . Whenever necessary, we shall use a different symbol to denote a Banach space. The symbol $\|\cdot\|$ denotes norms in E and E^* . For any subset D of E , we shall use $\text{int}D$ and ∂D to denote the interior and boundary of D , respectively. For any subsets C and D of E , the symbol $C - D$, is understood to be the algebraic difference between C and D , i.e., $C - D = \{c - d : c \in C, d \in D\}$. In addition, for $f^* \in E^*$ and $x \in E$, the symbol $\langle f^*, x \rangle$ is used to denote the value $f^*(x)$. An operator $\mathcal{A} : E \supseteq D(\mathcal{A}) \rightarrow 2^{E^*}$ ($D(\mathcal{A})$ denotes its domain) is called monotone if $\langle g^* - h^*, x - y \rangle \geq 0$ for all $(x, g^*) \in G(\mathcal{A})$ and $(y, h^*) \in G(\mathcal{A})$, where $G(\mathcal{A})$ denotes the graph of \mathcal{A} . It is well-known that a monotone operator \mathcal{A} is maximal monotone if $f^* \in \mathcal{A}x$ ($x \in D(\mathcal{A})$) whenever $\langle f^* - g^*, x - y \rangle \geq 0$ for all $(y, g^*) \in G(\mathcal{A})$. In the case of reflexive Banach spaces, Rockafellar [12] (cf., Browder [13]) proved that, the monotone operator \mathcal{A} is maximal monotone if and only if $R(\mathcal{A} + \lambda J) = E^*$ for all (equivalently, some) $\lambda > 0$. A fundamental result due to Rockafellar [12, Theorem 1], states that the sum of two maximal monotone operators $\mathcal{A} : E \supseteq D(\mathcal{A}) \rightarrow 2^{E^*}$ and $\mathcal{B} : E \supseteq D(\mathcal{B}) \rightarrow 2^{E^*}$, is again maximal monotone if

$$\text{int}(D(\mathcal{A})) \cap D(\mathcal{B}) \neq \emptyset. \tag{5}$$

We notice here that under suitable domain translation, one can assume that $0 \in \text{int}(D(\mathcal{A})) \cap D(\mathcal{B})$. This sufficient condition, is commonly known as Rockafellar’s condition. Furthermore, Brèzis, Crandall and Pazy [14, Theorem 2.3] gave a maximality theorem provided that $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and

$$|\mathcal{B}x| \leq k(\|x\|)|\mathcal{A}x| + C(\|x\|),$$

where k and C are non-decreasing functions of r with $k(r) < 1$ for all r . In addition, Brèzis, Crandall and Pazy [14, Theorem 2.2] gave an alternative proof of Rockafellar’s characterization of maximal monotone operators. The question of maximality of the sum of two maximal monotone operators remained to be one of the problems in monotone operator theory. In the case of reflexivity of E , this problem is an active area of research and various results have been contributed to weaken the Rockafellar’s criterion. When $E = H$ is a Hilbert space, Attouch [15, Theorem 1] proved maximality of $\mathcal{A} + \mathcal{B}$ if

$$0 \in \text{int}(D(\mathcal{A}) - D(\mathcal{B})). \tag{6}$$

When $\mathcal{B} = \partial\Theta$ (where Θ is a proper, convex and lower semi-continuous function from H into $(-\infty, \infty]$), Attouch [15, Theorem 2] gave a maximality result for $\mathcal{A} + \partial\Theta$ under relatively general criterion that $0 \in \text{int}(D(\mathcal{A}) - D(\Theta))$. In addition, if $\mathcal{A} = \partial\Pi$ (sub-differential of Π) and $\mathcal{B} = \partial\psi$ (sub-differential of ψ), where Π and ψ are proper, convex and lower semi-continuous function from H into $(-\infty, \infty]$, then maximality of $\mathcal{A} + \mathcal{B}$ is achieved (cf., [15, Theorem 3]) if $0 \in \text{int}(D(\Pi) - D(\psi))$. It is clear to see that $0 \in \text{int}(D(\mathcal{A})) \cap D(\mathcal{B})$ implies $0 \in \text{int}(D(\mathcal{A})) \subseteq D(\mathcal{A}) - D(\mathcal{B})$, and hence $0 \in \text{int}(D(\mathcal{A}) - D(\mathcal{B}))$, i.e. Attouch’s condition is weaker than Rockafellar’s criterion. Hassan [16] used the condition

$$\bigcup_{\lambda > 0} \lambda(D(\mathcal{A}) - D(\mathcal{B})) = C \text{ is a closed subspace of } H,$$

to prove the maximality of $\mathcal{A} + \mathcal{B}$. Later, this condition is proved to be sufficient for maximality (when $C = E$, E is a reflexive Banach space) by Attouch, Riahi and Thera [17, Theorem 4]. Analogous maximality result is proved by Chu [18] assuming $\text{int}(coD(\mathcal{A}) - coD(\mathcal{B})) \neq \emptyset$, where the symbols $coD(\mathcal{A})$ and $coD(\mathcal{B})$, denote the convex hull of $D(\mathcal{A})$ and $D(\mathcal{B})$, respectively. For further properties of maximality qualification conditions and equivalencies between various criteria, the reader is advised to refer the books by Simons [1,19] (cf., Theorems 23.1 and 23.2 in [1]) and references therein. On the other hand, a variant of Brèzis, Crandall and Pazy [14, Theorem 2.3] type criterion was used by Webb and Zhao [20]. More specifically, Webb and Zhao [20] established maximality results for $\mathcal{A} + \mathcal{B}$ (where \mathcal{B} is single-valued and $D(\mathcal{A}) \subseteq D(\mathcal{B})$) under various sufficient conditions involving estimates on $\|\mathcal{B}x\|$, $\|x^*\|$ ($x^* \in \mathcal{A}x$) and $c(\|x\|)$, where c is a nonnegative and nondecreasing function. Among these sufficient conditions, the authors [20, Theorem 1] assumed that \mathcal{A} and \mathcal{B} to satisfy the estimate

$$\|\mathcal{B}x\| \leq k(\|x\|)|\mathcal{A}x| + d|(\mathcal{A} + \mathcal{B})x| + c(\|x\|), \quad x \in D(\mathcal{A}),$$

where $0 \leq k(r) < 1$, $c(r) \geq 0$, $0 \leq d < 1$ and, k and c are nondecreasing functions and $|\mathcal{A}x| = \inf\{\|w^*\| : x \in D(\mathcal{A}), w^* \in \mathcal{A}x\}$ and $|(\mathcal{A} + \mathcal{B})(x)| = \inf\{z^* : x \in D(\mathcal{A}) \cap D(\mathcal{B}), z^* \in (\mathcal{A} + \mathcal{B})x\}$. For additional results, we refer the reader to read Propositions 1 and 2 in [20]. Recently, Asfaw [21, Theorem 5] gave a maximality result if conditions (i) and (ii) below are satisfied.

- (i) There exists $R > 0$ and $x_0 \in E$ such that for each bounded subset B of $D(\mathcal{A})$ and each $y \in B(x_0, R)$, there exists $N(B, y) \in \mathbf{R}$ such that

$$\langle g^*, x - y \rangle \geq N(B, y),$$

for all $x \in B$ and $g^* \in \mathcal{A}x$.

- (ii) For each bounded subset D of $D(\mathcal{B})$, there exists $N(D, x_0) \in \mathbf{R}$ such that

$$\langle w^*, x - x_0 \rangle \geq N(D, x_0),$$

for all $x \in D$ and $w^* \in \mathcal{B}x$.

It is shown (cf., Corollary 8 in [21]) that the Rockafellar’s condition is stronger than conditions (i) and (ii) in Theorem 5 there. For further maximality results, the reader is referred to [21, Corollaries 9, 11, 12, Theorems 10, 13 and 16].

We like to point out that various outstanding contributions have been known in the direction of giving a positive answer as to whether or not Rockafellar’s criterion is sufficient for maximality of sum of maximal monotone operators if E is non-reflexive. Among these, we refer the reader to the papers due to Borwein [22,23], Bauschke, Wang and Yao [24], Laszlo and Mosoni [25], Simons [1,19,26], Burachik and Svalter [27], Verona and Verona [28], and the references therein.

Motivated by the above contributions and the importance of maximality of the sum of maximal monotone operators to address operator inclusion problems (such as nonlinear partial differential equations and variational inequality problems), we shall provide new maximality results for $\mathcal{A} + \mathcal{B}$ (in the setting of reflexive Banach spaces) under a weaker condition

$$\text{int}(D(\psi_{\mathcal{A}}) - D(\psi_{\mathcal{B}})) \neq \emptyset, \tag{7}$$

where $\psi_{\mathcal{A}} : E \rightarrow \mathbf{R} \cup \{\infty\}$ and $\psi_{\mathcal{B}} : E \rightarrow \mathbf{R} \cup \{\infty\}$ are defined by

$$\psi_{\mathcal{A}}(x) = \sup_{(w, g^*) \in G(\mathcal{A})} \frac{\langle g^*, x - w \rangle}{1 + \|w\|} \quad \text{and} \quad \psi_{\mathcal{B}}(x) = \sup_{(w, h^*) \in G(\mathcal{B})} \frac{\langle h^*, x - w \rangle}{1 + \|w\|}, \quad x \in E.$$

The functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{B}}$ are first introduced by Simons [1]. Clearly, these functions are convex and lower-semi-continuous such that $D(\mathcal{A}) \subseteq D(\psi_{\mathcal{A}})$ and $D(\mathcal{B}) \subseteq D(\psi_{\mathcal{B}})$. For further definitions, properties, important results and conjectures, the reader is advised to the books by Simons [1,19] and the references therein.

Our results in the current paper, provide new maximality theorems as well as positive solution(s) for Simons problem [1, Problem 21.5] that states as to whether or not, the sum $\mathcal{A} + \mathcal{B}$ is maximal monotone if (7) holds. It is worth mentioning that (7) is weaker than the sufficient conditions due to Rockafellar [12], Attouch [15] (where E is a Hilbert space) and, Attouch, Riahi and Thera [17]. It holds that $D(\mathcal{A}) \subseteq D(\psi_{\mathcal{A}})$, $D(\mathcal{B}) \subseteq D(\psi_{\mathcal{B}})$, $coD(\mathcal{A}) \subseteq D(\psi_{\mathcal{A}})$, $coD(\mathcal{B}) \subseteq D(\psi_{\mathcal{B}})$ and $0 \in \text{int}(D(\mathcal{A})) \cap D(\mathcal{B})$ because $D(\psi_{\mathcal{A}})$ and $D(\psi_{\mathcal{B}})$ are convex. Thus the inclusion relation

$$0 \in \text{int}(D(\mathcal{A}) - D(\mathcal{B})) \subseteq \text{int}(coD(\mathcal{A}) - coD(\mathcal{B})) \subseteq \text{int}(D(\psi_{\mathcal{A}}) - D(\psi_{\mathcal{B}})), \tag{8}$$

holds. In addition, our results are applied to establish an existence result which guarantee the solvability of the variational inequality problem (*). To this end, we shall recall essential contents related to (*). For a proper, convex and lower-semicontinuous function Φ , the operator $\partial\Phi : E \supseteq D(\partial\Phi) \rightarrow 2^{E^*}$, defined by

$$\partial\Phi(x) = \{x^* \in E^* : \langle x^*, x - y \rangle \geq \Phi(x) - \Phi(y) \text{ for all } y \in E\},$$

is known as the subdifferential operator corresponding to Φ . It is known that $\partial\Phi$ is a maximal monotone operator. When $I_{\mathcal{K}}$ is the indicator function on a closed and convex subset \mathcal{K} of E (i.e., $I_{\mathcal{K}}(x) = \{0\}$ for $x \in \mathcal{K}$, and $I_{\mathcal{K}}(x) = \infty$ for all $x \notin \mathcal{K}$), it follows that $I_{\mathcal{K}} : E \rightarrow \mathbf{R} \cup \{\infty\}$ is a proper, convex and lower semicontinuous function, and $v^* \in \partial I_{\mathcal{K}}(x)$ if and only if $\langle v^*, x - y \rangle \geq 0$ for all $y \in \mathcal{K}$. Browder and Hess [29] introduced the definition of multi-valued pseudomonotone operator given in Definition 1 below:

Definition 1. An operator $Q : E \rightarrow 2^{E^*}$ is pseudomonotone if conditions (p_1) - (p_3) are satisfied.

- (p_1) For each $z \in E$, Qz is a nonempty, closed, convex and bounded subset of E^* .
- (p_2) Q is upper semicontinuous from each finite dimensional subspace F of E to the weak topology on E^* , i.e., to a given element $z_0 \in F$ and a weak neighborhood V of Qz_0 in E^* , there exists a neighborhood \mathcal{U} of z_0 in F such that $Qu \subseteq V$ for all $u \in \mathcal{U}$.
- (p_3) For each $\{z_n\}$ and $w_n^* \in Qz_n$ such that $z_n \rightarrow z$ and $\limsup_{n \rightarrow \infty} \langle w_n^*, z_n - z \rangle \leq 0$, we have that, for each $y \in E$, there exists $w(y) \in Qz$ such that

$$\langle w(y), z - y \rangle \leq \liminf_{n \rightarrow \infty} \langle w_n^*, z_n - y \rangle.$$

Concerning problem $(*)$, one can easily see that $\zeta \in D(\mathcal{A}) \cap D(\partial\Phi) \cap \mathcal{K}$ is a solution of $(*)$ if and only if ζ is a solution of the inclusion problem

$$f^* \in \mathcal{A}\zeta - \Delta_p \zeta + Q\zeta + \partial\Phi(\zeta) + \partial I_{\mathcal{K}}(\zeta), \tag{9}$$

where $Q : Y \rightarrow Y^*$ is as defined by $(**)$ and $\mathcal{A} : Y \supseteq D(\mathcal{A}) \rightarrow Y^*$ defined by $\mathcal{A}\zeta = \zeta'$, where $D(\mathcal{A}) = \{\zeta \in Y : \zeta' \in Y^*, \zeta(0) = \zeta(T)\}$ and ζ' stands for the weak derivative of ζ . It is well-known that \mathcal{A} is linear maximal monotone (cf., the result due to Brèzis [7, Theorem 32.L]). Here, we see that

$$\Delta_p \zeta = -\operatorname{div} \left(|\nabla \zeta|^{p-2} \nabla \zeta \right), \zeta \in Y, p \geq 2,$$

and $-\Delta_p : Y \rightarrow Y^*$ is defined by

$$\langle \Delta_p \zeta, v \rangle = \int_{\Gamma} |\nabla \zeta|^{p-2} \nabla \zeta \nabla v \, dxdt, v \in Y.$$

We notice here that $-\Delta_p$ is a continuous and monotone operator with effective domain Y , i.e., it is a maximal monotone operator, and hence pseudomonotone (cf., Propositions 8 and 9 [29]). For further properties of maximal monotone operators when the base space is reflexive, we shall refer to the paper by Browder and Hess [29], and the books by Barbu [8], Pascali and Sburlan [10], Zeidler [7], Cioranescu [11] and the references therein. Moreover, the operator $-\Delta_p + Q$ is pseudomonotone because the class of pseudomonotone operators is closed under addition. Various results on single pseudomonotone operator (cf., Browder and Hess [29, Proposition 9]) as well as maximal monotone perturbations of pseudomonotone operators, can be found in the paper by Browder and Hess [29].

The significance of the current paper is based on the following main reasons:

- (R_1) The constraint set \mathcal{K} (which is assumed to be nonempty, closed and convex) is not required to satisfy $\operatorname{int}\mathcal{K} \neq \emptyset$. In many of the previous important existence results concerning variational inequality problems, the condition $\operatorname{int}\mathcal{K} \neq \emptyset$, has been used effectively. In most cases, the reason for this condition on \mathcal{K} , is to guarantee the fulfillment of Rockafellar’s maximality criterion. In this paper, we shall allow $\operatorname{int}\mathcal{K}$ to be empty, and instead, we assume \mathcal{K} to be symmetric subset of Y .
- (R_2) The differential operator \mathcal{D} is not required to generate coercive operator for the higher order derivatives. Instead, we effectively used the coercivity property of $-\Delta_p$ and q_0 , along with suitable choices of the constants c, p, q and N satisfying conditions under (c_3) .
- (R_3) One of the difficulties in the study of inclusion problems (i.e., operator equations and variational inequality problems) in monotone operator theory, is to determine reasonable condition(s) which guarantee (s) maximality of the sum of maximal monotone operators. With respect to the goal of the current paper, it is essential to prove the maximality of the monotone operator $\mathcal{A} + \partial\Phi + \partial I_{\mathcal{K}}$ because the

solvability of (*) is equivalent to the solvability of (9). On the other hand, there are no existence results concerning (9) without requiring $\mathcal{A} + \partial\Phi + \partial I_{\mathcal{K}}$ to be maximal monotone. In reference to solvability of operator inclusion problems, the only available result is for single maximal monotone perturbation of pseudomonotone operators due to Le [30, Theorem 2.1] and, Asfaw and Kartsatos [31, Theorem 2.12]. In fact, we shall require maximality of the operator $\mathcal{A} + \partial\Phi + \partial I_{\mathcal{K}}$ under a general condition as compared with the Rockafellar's type criterion. In the first step, we shall prove maximality of the sum of two maximal monotone operators in Theorems 1, 2 and 3. As a consequence of Theorem 4, the maximality of $\mathcal{A} + \partial\Phi + \partial I_{\mathcal{K}}$ is established under reasonably general criteria. In line with this, the solvability of (*) is established in §4.

Based on the above reasons, we plan to prove the solvability of (9) (and hence solvability of (*) by establishing new maximal monotonicity results for the sum of maximal monotone operators. Once this objective is achieved in §3, we shall apply the existence result due to Le [30, Theorem 2.1] and the same result independently proved by Asfaw and Kartsatos [31, Theorem 2.12], which guarantees the solvability of operator inclusion problem of the type $Tw + Sw \ni f^*$ (where T and S are maximal monotone and pseudomonotone operators, respectively) provided that $T + S$ is coercive.

The paper is organized as follows. In §2, we shall present essential lemmas. §3 provides a new result (Theorem 1) to confirm the maximality of $\mathcal{A} + \mathcal{B}$ provided that $D(\psi_{\mathcal{A}}) \subseteq D(\psi_{\mathcal{B}})$ and (7) is replaced by

$$\text{int}(D(\psi_{\mathcal{A}}) - D(\psi_{\mathcal{B}})) \neq \emptyset. \quad (10)$$

Theorem 2 gives another maximality result if (7) holds, and $D(\psi_{\mathcal{A}})$ and $D(\psi_{\mathcal{B}})$ are symmetric. When \mathcal{M} is a maximal monotone operator from E into E^* , Theorem 3 gives a new sufficient condition for the maximality of $\mathcal{M} + \partial\Phi$, where Φ is a proper, convex and lower semi-continuous function on E . Consequently, $\mathcal{A} + \partial\Phi + \partial I_{\mathcal{K}}$ is proved to be maximal monotone (cf., Theorem 4) under mild conditions. The arguments of the proofs are based on the results by Browder [13, Theorem 2], Rockafellar [12], Brèzis, Crandall and Pazy [14], Eberlein-Smulian Theorem, uniform boundedness principle, surjectivity result for maximal monotone perturbations of pseudomonotone operators due to Le [30, Theorem 2.2] or Asfaw-Kartsatos [31, Theorem 12] and essential lemmas to be proved in §2 below. These new results gave solution(s) to Simons problem [1, Problem 21.5] and gave improvements and / generalizations of the known results. In §4, we applied the abstract theorems to derive existence results on the solvability of problem (*) provided that suitable mild conditions are fulfilled.

2. Background lemmas

In this section, we shall provide known as well as new lemmas, which give ingredients of the proofs of the abstract theorems in §3 and an application in §4. Lemma 1 is commonly known as the Young's inequality.

Lemma 1. *Let a and b be complex numbers, and $p > 1$ and q be its conjugate exponent (i.e., $1/p + 1/q = 1$). Then*

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

The content of Lemma 2 is the result due to Brèzis, Crandall and Pazy [14, Lemma 1.2].

Lemma 2. *Let $\mathcal{A} : E \supseteq D(\mathcal{A}) \rightarrow 2^{E^*}$ be a maximal monotone operator. Suppose $(u_n, u_n^*) \in G(\mathcal{A})$ for all n , such that $u_n \rightharpoonup u_0$, $u_n^* \rightharpoonup u_0^*$ as $n \rightarrow \infty$, and*

$$\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u_0 \rangle \leq 0.$$

Then $(u_0, u_0^) \in G(\mathcal{A})$ and $\langle u_n^*, u_n \rangle \rightarrow \langle u_0^*, u_0 \rangle$ as $n \rightarrow \infty$.*

Lemma 3 is essential to complete the proofs of the results in §3. Its proof is part of the arguments used to prove Theorem 2.7 due to Asfaw [32]. We shall give the proof here for convenience of readers.

Lemma 3. *Let $\mathcal{A} : E \supseteq D(\mathcal{A}) \rightarrow 2^{E^*}$ be a maximal monotone operator, $\{\lambda_n\} \subseteq (0, \infty)$ and $\{x_n\} \subseteq E$ be bounded. Let \mathcal{A}_{λ_n} and \mathcal{R}_{λ_n} be the Yosida approximant and resolvent of \mathcal{A} , respectively. Then $\{\mathcal{R}_{\lambda_n} x_n\}$ is bounded.*

Proof. Suppose $\{w_n\}$ is bounded. It is enough to prove the boundedness of $\{w_n - \mathcal{R}_{\lambda_n} w_n\}$. We recall that the duality mapping J satisfies $\langle Jx, x \rangle = \|x\|^2$ for all $x \in E$. Choose $w_0 \in D(\mathcal{M})$ and $w_0^* \in \mathcal{M}w_0$. Since \mathcal{M} is monotone, $(w_0, w_0^*) \in G(\mathcal{M})$, $(\mathcal{R}_{\lambda_n} w_n, \mathcal{M}_{\lambda_n} w_n) \in G(\mathcal{M})$, $\lambda_n \mathcal{M}_{\lambda_n} w_n = J(w_n - \mathcal{R}_{\lambda_n} w_n)$, we obtain that $\langle \mathcal{M}_{\lambda_n} w_n - w_0^*, \mathcal{R}_{\lambda_n} w_n - w_0 \rangle \geq 0$ for all n . Furthermore, we arrive at

$$\begin{aligned} \|w_n - \mathcal{R}_{\lambda_n} w_n\|^2 &= \langle J(w_n - \mathcal{R}_{\lambda_n} w_n), w_n - \mathcal{R}_{\lambda_n} w_n \rangle \\ &= \langle J(w_n - \mathcal{R}_{\lambda_n} w_n), w_n - w_0 \rangle - \lambda_n \langle \mathcal{M}_{\lambda_n} w_n - w_0^*, \mathcal{R}_{\lambda_n} w_n - w_0 \rangle - \lambda_n \langle w_0^*, \mathcal{R}_{\lambda_n} w_n - w_0 \rangle \\ &\leq \|w_n - \mathcal{R}_{\lambda_n} w_n\| \|w_n - w_0\| + \lambda_n \|w_0^*\| \|w_n - \mathcal{R}_{\lambda_n} w_n\| + \lambda_n \|w_0^*\| \|w_n - w_0\| \\ &\leq \|w_n - \mathcal{R}_{\lambda_n} w_n\| \mathcal{D} + \lambda_n \|w_0^*\| \|w_n - \mathcal{R}_{\lambda_n} w_n\| + \mathcal{Q}, \end{aligned} \tag{11}$$

for all n , where \mathcal{D} and \mathcal{Q} are upper bounds for $\{\|w_n - w_0\|\}$ and $\{\lambda_n \|w_0^*\| \|w_n - w_0\|\}$, respectively. We claim that $\{w_n - \mathcal{R}_{\lambda_n} w_n\}$ is bounded. Otherwise, we extract a subsequence of $\{w_n - \mathcal{R}_{\lambda_n} w_n\}$, denoted again by the same symbol $\{w_n - \mathcal{R}_{\lambda_n} w_n\}$, such that $\|w_n - \mathcal{R}_{\lambda_n} w_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Dividing (11) by $\|w_n - \mathcal{R}_{\lambda_n} w_n\|$ for all large n , we get

$$\|w_n - \mathcal{R}_{\lambda_n} w_n\| \leq \mathcal{D} + \lambda_n \|w_0^*\| + \frac{\mathcal{Q}}{\|w_n - \mathcal{R}_{\lambda_n} w_n\|}, \tag{12}$$

for all n . Letting $n \rightarrow \infty$ on both sides of (12) gives $\infty \leq \mathcal{D} + \mathcal{P}$, where \mathcal{P} is upper bounded for $\{\lambda_n \|w_0^*\|\}$. However, this is impossible. Thus the claim holds. The boundedness of $\{w_n\}$ proves the boundedness of $\{\mathcal{R}_{\lambda_n} w_n\}$. The proof is complete. \square

In §4, we use Lemma 4 below. In fact, Lemma 4 is due to Le [30, Theorem 2.2] as well as due to Asfaw-Kartsatos [31, Theorem 12].

Lemma 4. Let $\mathcal{T} : E \supseteq D(\mathcal{T}) \rightarrow 2^{E^*}$ be maximal monotone, $\mathcal{S} : E \rightarrow 2^{E^*}$ be bounded pseudomonotone and $f^* \in E^*$. Suppose there exists $R > 0$ and $u_0 \in B_R(0)$ such that $D(\mathcal{T}) \cap B_R(0) \neq \emptyset$ and

$$\langle a^* + b^*, u - u_0 \rangle > 0,$$

for all $u \in D(\mathcal{T}) \cap \partial B_R(0)$, $a^* \in \mathcal{T}u$ and $b^* \in \mathcal{S}u$. Then $f^* \in R(\mathcal{T} + \mathcal{S})$. In addition, $\mathcal{T} + \mathcal{S}$ is surjective if $\mathcal{T} + \mathcal{S}$ is coercive.

The proofs of Theorems 1, 2 and 3, are based on Lemma 5 below:

Lemma 5. Let $\mathcal{A} : E \supseteq D(\mathcal{A}) \rightarrow 2^{E^*}$ and $\mathcal{B} : E \supseteq D(\mathcal{B}) \rightarrow 2^{E^*}$ be maximal monotone operators. Let $\{\lambda_n\} \subseteq (0, \infty)$ such that $\lambda_n \downarrow 0^+$ as $n \rightarrow \infty$, and \mathcal{A}_{λ_n} be the Yosida approximant of \mathcal{A} and $f^* \in E^*$. Then the following conclusions hold.

- (i) Let $h_n^* \in E^*$ for all n . If for each $y \in B(0, \varepsilon)$ ($\varepsilon > 0$), the sequence $\{\langle h_n^*, y \rangle\}$ is bounded, then $\{h_n^*\}$ is bounded.
- (ii) Suppose there exists $w_n \in D(\mathcal{B})$ and $g_n^* \in \mathcal{B}w_n$ such that

$$g_n^* + \mathcal{A}_{\lambda_n} w_n + Jw_n = f^*, \tag{13}$$

for all n . If $\{w_n\}$, $\{\mathcal{A}_{\lambda_n} w_n\}$ and $\{g_n^*\}$ are bounded, then $f^* \in R(\mathcal{A} + \mathcal{B} + J)$.

Proof. Proof of (i). Suppose hypotheses hold. We observe here that for each $y \in E$, there exists $d(y)$ such that

$$\left| \langle h_n^*, \frac{\varepsilon y}{1 + \|y\|} \rangle \right| \leq d(y),$$

or

$$|\langle h_n^*, y \rangle| \leq d(y)(1 + \|y\|)\varepsilon^{-1},$$

i.e., for each $y \in E$, the sequence $\{\langle h_n^*, y \rangle\}$ is bounded. The well-known uniform boundedness principle yields the boundedness of $\{h_n^*\}$.

Proof of (ii). For each n , let \mathcal{R}_{λ_n} be the Yosida resolvent of \mathcal{A} . Because of the reflexivity of E and E^* , there exist subsequences denoted again by the same symbols, such that $w_n \rightharpoonup w_0$, $g_n^* \rightharpoonup g_0^*$ and $\mathcal{A}_{\lambda_n} w_n \rightharpoonup a_0^*$ as

$n \rightarrow \infty$. It is known due to Brèzis, Crandall and Pazy [14, Lemma 1.3] that $\mathcal{R}_{\lambda_n} w_n \in D(\mathcal{A})$, $(\mathcal{R}_{\lambda_n} w_n, \mathcal{A}_{\lambda_n} w_n) \in G(\mathcal{A})$,

$$\mathcal{A}_{\lambda_n} w_n = \lambda_n^{-1} J(w_n - \mathcal{R}_{\lambda_n} w_n),$$

and

$$\langle \mathcal{A}_{\lambda_n} w_n, w_n - \mathcal{R}_{\lambda_n} w_n \rangle = \lambda_n \langle \mathcal{A}_{\lambda_n} w_n, J^{-1}(\mathcal{A}_{\lambda_n} w_n) \rangle = \lambda_n \|\mathcal{A}_{\lambda_n} w_n\|^2, \tag{14}$$

for all n . Clearly, we have $\mathcal{R}_{\lambda_n} w_n \rightarrow w_0$ as $n \rightarrow \infty$ because $w_n - \mathcal{R}_{\lambda_n} w_n = \lambda_n J^{-1}(\mathcal{A}_{\lambda_n} w_n)$ for all n , and $\lambda_n J^{-1}(\mathcal{A}_{\lambda_n} w_n) \rightarrow 0$ as $n \rightarrow \infty$. An application of the monotonicity of \mathcal{A} and J , along with (13) yields

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \mathcal{A}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n - w_0 \rangle &= \limsup_{n \rightarrow \infty} (\langle \mathcal{A}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n - w_n \rangle + \langle \mathcal{A}_{\lambda_n} w_n, w_n - w_0 \rangle) \\ &= \limsup_{n \rightarrow \infty} \left(-\lambda_n \|\mathcal{A}_{\lambda_n} w_n\|^2 + \langle \mathcal{A}_{\lambda_n} w_n, w_n - w_0 \rangle \right) \\ &\leq \limsup_{n \rightarrow \infty} \langle \mathcal{A}_{\lambda_n} w_n, w_n - w_0 \rangle \\ &= \limsup_{n \rightarrow \infty} (-\langle g_n^*, w_n - w_0 \rangle - \langle Jw_n - Jw_0, w_n - w_0 \rangle - \langle Jw_0 - f^*, w_n - w_0 \rangle) \\ &\leq -\liminf_{n \rightarrow \infty} \langle g_n^*, w_n - w_0 \rangle. \end{aligned}$$

We shall verify that

$$d = \liminf_{n \rightarrow \infty} \langle g_n^*, w_n - w_0 \rangle \geq 0.$$

Otherwise (i.e., $d < 0$), assume there exists a subsequence of $\{F_n = \langle g_n^*, w_n - w_0 \rangle\}$ denoted again by $\{F_n\}$, such that $F_n \rightarrow d$ as $n \rightarrow \infty$, i.e.,

$$\lim_{n \rightarrow \infty} \langle g_n^*, w_n - w_0 \rangle = d < 0.$$

Lemma 2 shows that $w_0 \in D(\mathcal{B})$, $g_0^* \in \mathcal{B}w_0$ and $\langle g_n^*, w_n \rangle \rightarrow \langle g_0^*, w_0 \rangle$ as $n \rightarrow \infty$, i.e., $d = 0$, which is a contradiction. Thus, we get

$$\limsup_{n \rightarrow \infty} \langle \mathcal{A}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n - w_0 \rangle \leq 0.$$

It is known that $(\mathcal{R}_{\lambda_n} w_n, \mathcal{A}_{\lambda_n} w_n) \in G(\mathcal{A})$ for all n . Again, we apply Lemma 2 to conclude that $w_0 \in D(\mathcal{A})$, $a_0^* \in \mathcal{A}w_0$ and

$$\langle \mathcal{A}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n \rangle \rightarrow \langle a_0^*, w_0 \rangle \text{ as } n \rightarrow \infty,$$

i.e., we have

$$\begin{aligned} \langle \mathcal{A}_{\lambda_n} w_n, w_n \rangle &= \langle \mathcal{A}_{\lambda_n} w_n, w_n - \mathcal{R}_{\lambda_n} w_n \rangle + \langle \mathcal{A}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n \rangle \\ &= \lambda_n \|\mathcal{A}_{\lambda_n} w_n\|^2 + \langle \mathcal{A}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n \rangle \rightarrow \langle a_0^*, w_0 \rangle, \end{aligned} \tag{15}$$

because $\lambda_n \downarrow 0^+$ and $\{\mathcal{A}_{\lambda_n} w_n\}$ is bounded. In addition, we apply (13), monotonicity of J (i.e., $\liminf_{n \rightarrow \infty} \langle Jw_n - Jw_0, w_n - w_0 \rangle \geq 0$) and $\langle \mathcal{A}_{\lambda_n} w_n, w_n \rangle \rightarrow \langle a_0^*, w_0 \rangle$ as $n \rightarrow \infty$, to obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle g_n^*, w_n - w_0 \rangle &= \limsup_{n \rightarrow \infty} (-\langle \mathcal{A}_{\lambda_n} w_n, w_n - w_0 \rangle - \langle Jw_n - f^*, w_n - w_0 \rangle) \\ &\leq \limsup_{n \rightarrow \infty} (-\langle \mathcal{A}_{\lambda_n} w_n, w_n - w_0 \rangle) - \liminf_{n \rightarrow \infty} \langle Jw_n - Jw_0 - f^*, w_n - w_0 \rangle - \lim_{n \rightarrow \infty} \langle Jw_0, w_n - w_0 \rangle \\ &\leq \lim_{n \rightarrow \infty} -\langle \mathcal{A}_{\lambda_n} w_n, w_n - w_0 \rangle = 0. \end{aligned} \tag{16}$$

Again, we use Lemma 2 (cf., Brèzis, Crandall and Pazy [14, Lemma 1.2]) to conclude that $w_0 \in D(\mathcal{B}) \cap D(\mathcal{A})$, $g_0^* \in \mathcal{B}w_0$ and $\langle g_n^*, w_n \rangle \rightarrow \langle g_0^*, w_0 \rangle$ as $n \rightarrow \infty$. At last, applying (13), $\langle g_n^*, w_n \rangle \rightarrow \langle g_0^*, w_0 \rangle$, $\langle \mathcal{A}_{\lambda_n} w_n, w_n \rangle \rightarrow \langle a_0^*, w_0 \rangle$ and properties of J , we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Jw_n, w_n - w_0 \rangle &= -\lim_{n \rightarrow \infty} \langle g_n^* + \mathcal{A}_{\lambda_n} w_n - f^*, w_n - w_0 \rangle \\ &= -\lim_{n \rightarrow \infty} \langle g_n^*, w_n - w_0 \rangle - \lim_{n \rightarrow \infty} \langle \mathcal{A}_{\lambda_n} w_n, w_n - w_0 \rangle + \lim_{n \rightarrow \infty} \langle f^*, w_n - w_0 \rangle = 0, \end{aligned}$$

i.e., $w_n \rightarrow w_0$ and $Jw_n \rightarrow Jw_0$ (as $n \rightarrow \infty$) because J is continuous and of type (S_+) . Finally, letting $n \rightarrow \infty$ in (13) shows that $f^* \in R(\mathcal{A} + \mathcal{B} + J)$. The proof is complete \square

3. New maximal monotonicity results

Theorem 1 is one of the main results. We like to indicate that Theorem 1 is a new result and gives a positive solution for Simons problem [1, Problem 28.15] if the inclusion $D(\psi_{\mathcal{N}}) \subseteq D(\psi_{\mathcal{M}})$ holds.

Theorem 1. *Let E and E^* be strictly convex reflexive Banach spaces. Suppose $\mathcal{M} : E \supseteq D(\mathcal{M}) \rightarrow 2^{E^*}$ and $\mathcal{N} : E \supseteq D(\mathcal{N}) \rightarrow 2^{E^*}$ are maximal monotone operators such that $D(\mathcal{N}) \cap D(\mathcal{M}) \neq \emptyset$, $D(\psi_{\mathcal{N}}) \subseteq D(\psi_{\mathcal{M}})$ and*

$$\text{int}(D(\psi_{\mathcal{N}}) - D(\psi_{\mathcal{M}})) \neq \emptyset.$$

Then $\mathcal{N} + \mathcal{M}$ is maximal monotone.

Proof. For each $\lambda > 0$, let $\mathcal{M}_\lambda : E \rightarrow E^*$ and $\mathcal{R}_\lambda : E \rightarrow D(\mathcal{M})$ be the Yosida approximant and resolvent of \mathcal{M} , respectively. It is well-known that \mathcal{M}_λ is bounded, monotone and demicontinuous (c.f., [14]) (i.e., \mathcal{M}_λ is maximal monotone) and \mathcal{R}_λ is bounded and continuous. In addition, it is known due to Browder [13] that $\mathcal{N} + \mathcal{M}_\lambda$ is maximal monotone. By applying the Rockafellar [12] characterization of maximal monotone operators, we conclude that (for each $\lambda > 0$), the operator $\mathcal{N} + \mathcal{M}_\lambda + J$ is surjective, i.e., for each $f^* \in E^*$ and $\lambda_n \downarrow 0^+$, there exist $w_n \in D(\mathcal{N})$ and $g_n^* \in \mathcal{N}w_n$ such that

$$g_n^* + \mathcal{M}_{\lambda_n}w_n + Jw_n = f^*, \tag{17}$$

for all $n \in \mathbb{Z}_+$. It is known that $\mathcal{R}_{\lambda_n}w_n \in D(\mathcal{M})$, $\mathcal{M}_{\lambda_n}w_n \in \mathcal{M}(\mathcal{R}_{\lambda_n}w_n)$ and

$$\mathcal{M}_{\lambda_n}w_n = \lambda_n^{-1}J(w_n - \mathcal{R}_{\lambda_n}w_n),$$

and

$$\langle \mathcal{M}_{\lambda_n}w_n, w_n - \mathcal{R}_{\lambda_n}w_n \rangle = \langle \mathcal{M}_{\lambda_n}w_n, \lambda_n J^{-1}(\mathcal{M}_{\lambda_n}w_n) \rangle = \lambda_n \|\mathcal{M}_{\lambda_n}w_n\|^2,$$

for all n . The details about \mathcal{M}_{λ_n} and \mathcal{R}_{λ_n} can be found in [14, Lemma 1.3]. For each $w_0 \in D(\mathcal{M}) \cap D(\mathcal{N})$ and $g_0^* \in \mathcal{N}w_0$, one can use the monotonicity of \mathcal{M}_{λ_n} , \mathcal{N} and properties of J , to get the estimate

$$\begin{aligned} \|w_n\|^2 - \|w_n\|\|w_0\| &\leq \langle Jw_n, w_n - w_0 \rangle \\ &= -\langle \mathcal{M}_{\lambda_n}w_n - \mathcal{M}_{\lambda_n}w_0, w_n - w_0 \rangle - \langle \mathcal{M}_{\lambda_n}w_0, w_n - w_0 \rangle \\ &\quad - \langle g_n^* - g_0^*, w_n - w_0 \rangle - \langle g_0^*, w_n - w_0 \rangle + \langle f^*, w_n - w_0 \rangle \\ &\leq (\|\mathcal{M}_{\lambda_n}w_0\| + \|f^*\| + \|g_0^*\|) \|w_n - w_0\| \\ &\leq (|\mathcal{M}w_0| + \|f^*\| + \|g_0^*\|) \|w_n - w_0\|, \end{aligned} \tag{18}$$

for all n , where $|\mathcal{M}w_0| = \inf\{\|h^*\| : h^* \in \mathcal{M}w_0\}$, i.e., we have

$$\|w_n\|^2 \leq \|w_n\|\|w_0\| + (|\mathcal{M}w_0| + \|f^*\| + \|g_0^*\|) \|w_n - w_0\|, \tag{19}$$

for all $n \in \mathbb{Z}_+$. We claim that $\{w_n\}$ is bounded. Suppose not, i.e., there exists a subsequence of $\{w_n\}$, denoted again by the same symbol $\{w_n\}$, such that $\|w_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Dividing both sides of (19) by $\|w_n\|$ for all large n (i.e., $\|w_n\| > 0$ for sufficiently large n), we obtain that

$$\|w_n\| \leq \|w_0\| + (|\mathcal{M}w_0| + \|f^*\| + \|g_0^*\|) \left(1 + \frac{\|w_0\|}{\|w_n\|}\right),$$

for all sufficiently large n . Letting $n \rightarrow \infty$ on both sides of this inequality yields

$$\infty \leq \|w_0\| + (|\mathcal{M}w_0| + \|f^*\| + \|g_0^*\|),$$

which is impossible. Thus our claim holds, i.e., $\{w_n\}$ is bounded. The boundedness of $\{Jw_n\}$ holds because J is bounded. Thus Lemma 3 confirms the boundedness of $\{\mathcal{R}_{\lambda_n}w_n\}$. Suppose

$$x_0 \in \text{int}(D(\psi_{\mathcal{N}}) - D(\psi_{\mathcal{M}})),$$

and $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq D(\psi_{\mathcal{N}}) - D(\psi_{\mathcal{M}})$, where $x_0 = c - d$ with $c \in D(\psi_{\mathcal{N}})$ and $d \in D(\psi_{\mathcal{M}})$. For each $y \in B(0, \varepsilon)$, it follows that $y + x_0 \in B(x_0, \varepsilon)$, i.e., $y + x_0 \in D(\psi_{\mathcal{N}}) - D(\psi_{\mathcal{M}})$. Set $y + x_0 = A - B$ for some $A \in D(\psi_{\mathcal{N}}) \subseteq D(\psi_{\mathcal{M}})$ and $B \in D(\psi_{\mathcal{N}})$. It is known due to the definitions and properties of \mathcal{M}_{λ_n} and \mathcal{R}_{λ_n} , that

$$\mathcal{R}_{\lambda_n}w_n \in D(\mathcal{M}), \mathcal{M}_{\lambda_n}w_n \in \mathcal{M}(\mathcal{R}_{\lambda_n}w_n), (\mathcal{R}_{\lambda_n}w_n, \mathcal{M}_{\lambda_n}w_n) \in G(\mathcal{M}), w_n - \mathcal{R}_{\lambda_n}w_n = \lambda_n J^{-1}(\mathcal{M}_{\lambda_n}w_n), \tag{20}$$

and

$$\langle \mathcal{M}_{\lambda_n}w_n, \lambda_n J^{-1}(\mathcal{M}_{\lambda_n}w_n) \rangle = \lambda_n \|\mathcal{M}_{\lambda_n}w_n\|^2, \tag{21}$$

for all n (cf., [14]). Thus the definitions of $\psi_{\mathcal{N}}$ and $\psi_{\mathcal{M}}$ imply

$$\frac{\langle \mathcal{M}_{\lambda_n}w_n, A - \mathcal{R}_{\lambda_n}w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n}w_n\|} \leq \psi_{\mathcal{M}}(A) \quad \text{and} \quad \frac{\langle g_n^*, B - w_n \rangle}{1 + \|w_n\|} \leq \psi_{\mathcal{N}}(B),$$

for all n . Furthermore, (17) yields the estimates

$$\langle \mathcal{M}_{\lambda_n}w_n, w_n \rangle = -\langle g_n^*, w_n \rangle - \langle Jw_n - f^*, w_n \rangle, \tag{22}$$

and

$$\langle \mathcal{M}_{\lambda_n}w_n, B - w_n \rangle = -\langle g_n^*, B - w_n \rangle - \langle Jw_n - f^*, B - w_n \rangle, \tag{23}$$

for all n . Then for each $y \in B(0, \varepsilon)$, (17), (20), (21), (22) and (23) show that

$$\begin{aligned} \langle g_n^*, y + x_0 \rangle &= -\langle g_n^*, w_n - (A - B) \rangle + \langle g_n^*, w_n \rangle \\ &= \langle \mathcal{M}_{\lambda_n}w_n, w_n - (A - B) \rangle + \langle Jw_n - f^*, w_n - (A - B) \rangle + \langle g_n^*, w_n \rangle \\ &= \langle \mathcal{M}_{\lambda_n}w_n, w_n - \mathcal{R}_{\lambda_n}w_n \rangle - \langle \mathcal{M}_{\lambda_n}w_n, A - \mathcal{R}_{\lambda_n}w_n \rangle + \langle g_n^*, w_n \rangle + \langle \mathcal{M}_{\lambda_n}w_n, B - w_n \rangle + \langle \mathcal{M}_{\lambda_n}w_n, w_n \rangle \\ &= \lambda_n \|\mathcal{M}_{\lambda_n}w_n\|^2 - \langle \mathcal{M}_{\lambda_n}w_n, A - \mathcal{R}_{\lambda_n}w_n \rangle + \langle g_n^*, w_n \rangle - \langle g_n^*, B - w_n \rangle - \langle Jw_n - f^*, B - w_n \rangle \\ &\quad - \langle g_n^*, w_n \rangle - \langle Jw_n - f^*, w_n \rangle \\ &\geq -\langle \mathcal{M}_{\lambda_n}w_n, A - \mathcal{R}_{\lambda_n}w_n \rangle - \langle g_n^*, B - w_n \rangle - \langle Jw_n - f^*, B - w_n \rangle - \langle Jw_n - f^*, w_n \rangle \\ &= -(1 + \|\mathcal{R}_{\lambda_n}w_n\|) \frac{\langle \mathcal{M}_{\lambda_n}w_n, A - \mathcal{R}_{\lambda_n}w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n}w_n\|} - (1 + \|w_n\|) \frac{\langle g_n^*, B - w_n \rangle}{1 + \|w_n\|} \\ &\quad - \langle Jw_n - f^*, B - w_n \rangle - \langle Jw_n - f^*, w_n \rangle, \end{aligned} \tag{24}$$

for all n . It follows (by the definitions of $\psi_{\mathcal{M}}$ and $\psi_{\mathcal{N}}$) that

$$\frac{\langle \mathcal{M}_{\lambda_n}w_n, A - \mathcal{R}_{\lambda_n}w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n}w_n\|} \leq \psi_{\mathcal{M}}(A),$$

and

$$\frac{\langle g_n^*, B - w_n \rangle}{1 + \|w_n\|} \leq \psi_{\mathcal{N}}(B),$$

for all n . Consequently, we arrive at

$$\langle g_n^*, y + x_0 \rangle \geq -C_1 \psi_{\mathcal{M}}(A) - C_2 \psi_{\mathcal{N}}(B) - S(A, B), \quad y \in B(0, \varepsilon), \tag{25}$$

for all n , where C_1 and C_2 are upper bounds for $\{1 + \|\mathcal{R}_{\lambda_n}w_n\|\}$ and $\{1 + \|w_n\|\}$, respectively, and $S(A, B)$ is an upper bound for $\{\|Jw_n - f^*\| \|B - w_n\| + \|Jw_n - f^*\| \|w_n\|\}$. We observe that $y + x_0 \in B(x_0, \varepsilon)$ for all $y \in B(0, \varepsilon)$, where $x_0 = c - d$, $c \in D(\psi_{\mathcal{N}})$ and $d \in D(\psi_{\mathcal{M}}) \subseteq D(\psi_{\mathcal{N}})$. We observe that (17) gives

$$\langle g_n^*, w_n - d \rangle = -\langle \mathcal{M}_{\lambda_n} w_n, w_n - d \rangle - \langle Jw_n - f^*, w_n - d \rangle, \tag{26}$$

for all n . In addition, we see that

$$\begin{aligned} \langle g_n^*, y \rangle &= \langle g_n^*, (y + x_0) - x_0 \rangle \\ &= \langle g_n^*, y + x_0 \rangle - \langle g_n^*, x_0 \rangle \\ &\geq -C_1 \psi_{\mathcal{M}}(A) - C_2 \psi_{\mathcal{N}}(B) - S(A, B) - \langle g_n^*, x_0 \rangle, \end{aligned} \tag{27}$$

for all n and $y \in B(0, \varepsilon)$. Next (21) and (26) imply

$$\begin{aligned} \langle g_n^*, x_0 \rangle &= \langle g_n^*, c - d \rangle \\ &= \langle g_n^*, w_n - d \rangle - \langle g_n^*, w_n \rangle - \langle g_n^*, w_n - c \rangle + \langle g_n^*, w_n \rangle \\ &= \langle g_n^*, w_n - d \rangle - \langle g_n^*, w_n - c \rangle \\ &= -\langle \mathcal{M}_{\lambda_n} w_n, w_n - d \rangle - \langle Jw_n - f^*, w_n - d \rangle - \langle g_n^*, w_n - c \rangle \\ &= -\langle \mathcal{M}_{\lambda_n} w_n, w_n - \mathcal{R}_{\lambda_n} w_n \rangle + \langle \mathcal{M}_{\lambda_n} w_n, d - \mathcal{R}_{\lambda_n} w_n \rangle - \langle g_n^*, w_n - c \rangle - \langle Jw_n - f^*, w_n - d \rangle \\ &= -\langle \mathcal{M}_{\lambda_n} w_n, w_n - \mathcal{R}_{\lambda_n} w_n \rangle + (1 + \|\mathcal{R}_{\lambda_n} w_n\|) \frac{\langle \mathcal{M}_{\lambda_n} w_n, d - \mathcal{R}_{\lambda_n} w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n} w_n\|} \\ &\quad + (1 + \|w_n\|) \frac{\langle g_n^*, c - w_n \rangle}{1 + \|w_n\|} - \langle Jw_n - f^*, w_n - d \rangle \\ &\leq (1 + \|\mathcal{R}_{\lambda_n} w_n\|) \frac{\langle \mathcal{M}_{\lambda_n} w_n, d - \mathcal{R}_{\lambda_n} w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n} w_n\|} + (1 + \|w_n\|) \frac{\langle g_n^*, c - w_n \rangle}{1 + \|w_n\|} - \langle Jw_n - f^*, w_n - d \rangle, \end{aligned} \tag{28}$$

for all n . Since $c \in D(\psi_{\mathcal{N}})$ (i.e., $\psi_{\mathcal{N}}(c) \in \mathbf{R}$) and $d \in D(\psi_{\mathcal{N}}) \subseteq D(\psi_{\mathcal{M}})$ (i.e., $\psi_{\mathcal{M}}(d) \in \mathbf{R}$), the definitions of $\psi_{\mathcal{M}}$ and $\psi_{\mathcal{N}}$ imply

$$\frac{\langle g_n^*, c - w_n \rangle}{1 + \|w_n\|} \leq \psi_{\mathcal{N}}(c), \tag{29}$$

and

$$\frac{\langle \mathcal{M}_{\lambda_n} w_n, d - \mathcal{R}_{\lambda_n} w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n} w_n\|} \leq \psi_{\mathcal{M}}(d), \tag{30}$$

for all n . In addition, (28), (29) and (30) give

$$\begin{aligned} -\langle g_n^*, x_0 \rangle &\geq -(1 + \|\mathcal{R}_{\lambda_n} w_n\|) \psi_{\mathcal{M}}(d) - (1 + \|w_n\|) \psi_{\mathcal{N}}(c) - \|Jw_n - f^*\| \|w_n - d\| \\ &\geq -C_1 \psi_{\mathcal{M}}(d) - C_2 \psi_{\mathcal{N}}(c) - C_3, \end{aligned} \tag{31}$$

for all n , where C_1, C_2 and C_3 are upper bounds for $\{(1 + \|\mathcal{R}_{\lambda_n} w_n\|)\}$, $\{1 + \|w_n\|\}$ and $\{\|Jw_n - f^*\| \|w_n - d\|\}$, respectively. At last, by using (27) and (31), we arrive at

$$\langle g_n^*, y \rangle \geq -C_1 \psi_{\mathcal{M}}(A) - C_2 \psi_{\mathcal{N}}(B) - S(A, B) - C_1 \psi_{\mathcal{M}}(d) - C_2 \psi_{\mathcal{N}}(c) - C_3 := -K(x_0, y), \tag{32}$$

for all n and $y \in B(0, \varepsilon)$, where

$$K(x_0, y) = C_1 \psi_{\mathcal{M}}(A) + C_2 \psi_{\mathcal{N}}(B) + S(A, B) + C_1 \psi_{\mathcal{M}}(d) + C_2 \psi_{\mathcal{N}}(c) + C_3.$$

Thus using $-y \in B(0, \varepsilon)$ in place of $y \in B(0, \varepsilon)$, we see that $\langle g_n^*, y \rangle \leq K(x_0, -y)$ for all n . Thus for each $y \in B(0, \varepsilon)$, the sequence $\{\langle g_n^*, y \rangle\}$ is bounded. Then by (i) of Lemma 5, the sequence $\{g_n^*\}$ is bounded, i.e., $\{\mathcal{M}_{\lambda_n} w_n\}$ is bounded. Thus (ii) of Lemma 5 yields that $f^* \in R(\mathcal{M} + \mathcal{N} + J)$, i.e., $\mathcal{M} + \mathcal{N} + J$ is surjective, and hence $\mathcal{M} + \mathcal{N}$ is maximal monotone. The proof is complete. \square

In Theorem 2, we shall give a maximality result using the symmetric assumptions on $D(\psi_{\mathcal{M}})$ and $D(\psi_{\mathcal{N}})$ instead of $D(\psi_{\mathcal{N}}) \subseteq D(\psi_{\mathcal{M}})$. Clearly, these symmetric conditions imply that $D(\mathcal{N}) \cap D(\mathcal{M}) \neq \emptyset$ because

$0 \in D(\psi_{\mathcal{N}}) \cap D(\psi_{\mathcal{M}})$. Indeed, the convexity of $D(\psi_{\mathcal{N}})$ and $D(\psi_{\mathcal{M}})$, and their symmetric conditions imply that

$$0 = 2^{-1}(a) + 2^{-1}(-a) \in D(\psi_{\mathcal{N}}),$$

and

$$0 = 2^{-1}(b) + 2^{-1}(-b) \in D(\psi_{\mathcal{M}}),$$

for all $a \in D(\psi_{\mathcal{N}})$ and $b \in D(\psi_{\mathcal{M}})$.

Theorem 2. Let E and E^* be strictly convex reflexive Banach spaces. Suppose $\mathcal{M} : E \supseteq D(\mathcal{M}) \rightarrow 2^{E^*}$ and $\mathcal{N} : E \supseteq D(\mathcal{N}) \rightarrow 2^{E^*}$ are maximal monotone operators such that $D(\mathcal{M}) \cap D(\mathcal{N}) \neq \emptyset$, $D(\psi_{\mathcal{M}})$ and $D(\psi_{\mathcal{N}})$ are symmetric, and

$$\text{int}(D(\psi_{\mathcal{M}}) - D(\psi_{\mathcal{N}})) \neq \emptyset.$$

Then $\mathcal{N} + \mathcal{M}$ is maximal monotone.

Proof. Suppose $f^* \in E^*$, $\lambda_n \downarrow 0+$, $\{w_n\}$, $\{g_n^*\}$ and $\{\mathcal{M}_{\lambda_n} w_n\}$ be such that (17) is satisfied. It follows from Lemma 3 and arguments used in the proof of Theorem 1, that the sequences $\{w_n\}$, $\{Jw_n\}$ and $\{\mathcal{R}_{\lambda_n} w_n\}$ are bounded. Suppose there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq D(\psi_{\mathcal{M}}) - D(\psi_{\mathcal{N}})$, i.e., $x_0 = c - d$, where $c \in D(\psi_{\mathcal{M}})$ (i.e., $-c \in D(\psi_{\mathcal{M}})$ because $D(\psi_{\mathcal{M}})$ is symmetric) and $d \in D(\psi_{\mathcal{N}})$ (i.e., $-d \in D(\psi_{\mathcal{N}})$ because $D(\psi_{\mathcal{N}})$ is symmetric). Thus for $y \in B(0, \varepsilon)$, we have $y + x_0 \in B(x_0, \varepsilon) \subseteq D(\psi_{\mathcal{M}}) - D(\psi_{\mathcal{N}})$ with $y + x_0 = A - B$ (i.e., $A \in D(\psi_{\mathcal{M}})$ and $B \in D(\psi_{\mathcal{N}})$). We follow the arguments to get estimate (24) in the proof of Theorem 1, to conclude

$$\langle g_n^*, y + x_0 \rangle \geq -C_1 \psi_{\mathcal{M}}(A) - C_2 \psi_{\mathcal{N}}(B) - S(A, B),$$

where the constants in the right side of this inequality are those used in estimate (25). Additionally, we get

$$\langle g_n^*, y \rangle = \langle g_n^*, y + x_0 \rangle - \langle g_n^*, c - d \rangle \geq -C_1 \psi_{\mathcal{M}}(A) - C_2 \psi_{\mathcal{N}}(B) - S(A, B) - \langle g_n^*, c - d \rangle, \tag{33}$$

for all n . In order to get a lower bound for the sequence $\{-\langle g_n^*, c - d \rangle\}$, we shall use (17) and the symmetric conditions on $D(\psi_{\mathcal{M}})$ and $D(\psi_{\mathcal{N}})$. This step needs different arguments as compared with those used in the proof of Theorem 1. Indeed, (17)

$$\langle g_n^*, -c \rangle = -\langle \mathcal{M}_{\lambda_n} w_n, -c - \mathcal{R}_{\lambda_n} w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n \rangle - \langle Jw_n - f^*, -c \rangle, \tag{34}$$

and

$$-\langle g_n^*, w_n \rangle = \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle + \langle Jw_n - f^*, w_n \rangle, \tag{35}$$

for all n . We observe that $\mathcal{R}_{\lambda_n} w_n = w_n - \lambda_n J^{-1}(\mathcal{M}_{\lambda_n} w_n)$ for all n , and

$$\langle \mathcal{M}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n \rangle = \langle \mathcal{M}_{\lambda_n} w_n, w_n - \lambda_n J^{-1}(\mathcal{M}_{\lambda_n} w_n) \rangle = \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - \lambda_n \|\mathcal{M}_{\lambda_n} w_n\|^2,$$

for all n . Consequently, we shall use (34) and (35) to conclude that

$$\begin{aligned} -\langle g_n^*, x_0 \rangle &= -\langle g_n^*, c - d \rangle \\ &= \langle g_n^*, d \rangle + \langle g_n^*, -c \rangle \\ &= -\langle g_n^*, -d - w_n \rangle - \langle g_n^*, w_n \rangle + \langle g_n^*, -c \rangle \\ &= -\langle g_n^*, -d - w_n \rangle + \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle + \langle Jw_n - f^*, w_n \rangle \\ &\quad - \langle \mathcal{M}_{\lambda_n} w_n, -c - \mathcal{R}_{\lambda_n} w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n \rangle - \langle Jw_n - f^*, -c \rangle \\ &= -\langle g_n^*, -d - w_n \rangle + \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle + \langle Jw_n - f^*, w_n \rangle \\ &\quad - \langle \mathcal{M}_{\lambda_n} w_n, -c - \mathcal{R}_{\lambda_n} w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle + \lambda_n \|\mathcal{M}_{\lambda_n} w_n\|^2 - \langle Jw_n - f^*, -c \rangle \\ &\geq -(1 + \|w_n\|) \frac{\langle g_n^*, -d - w_n \rangle}{1 + \|w_n\|} + \langle Jw_n - f^*, w_n \rangle \\ &\quad - (1 + \|\mathcal{R}_{\lambda_n} w_n\|) \frac{\langle \mathcal{M}_{\lambda_n} w_n, -c - \mathcal{R}_{\lambda_n} w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n} w_n\|} - \langle Jw_n - f^*, -c \rangle, \end{aligned} \tag{36}$$

for all n . Next because of the symmetric conditions on $D(\psi_{\mathcal{M}})$ (i.e., $-c \in D(\psi_{\mathcal{M}})$ because $c \in D(\psi_{\mathcal{M}})$) and $D(\psi_{\mathcal{N}})$ ($-d \in D(\psi_{\mathcal{N}})$ because $d \in D(\psi_{\mathcal{N}})$), we apply the definitions of $\psi_{\mathcal{M}}$ and $\psi_{\mathcal{N}}$ to see that

$$\frac{\langle g_n^*, -d - w_n \rangle}{1 + \|w_n\|} \leq \psi_{\mathcal{N}}(-d) \quad \text{and} \quad \frac{\langle \mathcal{M}_{\lambda_n} w_n, -c - \mathcal{R}_{\lambda_n} w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n} w_n\|} \leq \psi_{\mathcal{M}}(-c),$$

for all n . Finally, applying these estimates and using (36), we arrive at

$$\begin{aligned} -\langle g_n^*, x_0 \rangle &\geq -(1 + \|w_n\|)\psi_{\mathcal{N}}(-d) - (1 + \|\mathcal{R}_{\lambda_n} w_n\|)\psi_{\mathcal{M}}(-c) + \langle Jw_n - f^*, w_n + c \rangle \\ &\geq -C_1\psi_{\mathcal{M}}(-c) - C_2\psi_{\mathcal{N}}(-d) - \Gamma_0, \end{aligned} \tag{37}$$

for all n , where C_1, C_2 and Γ_0 are upper bounds for the sequences $\{1 + \|\mathcal{R}_{\lambda_n} w_n\|\}$, $\{1 + \|w_n\|\}$ and $\{\langle Jw_n - f^*, w_n + c \rangle\}$, respectively. At the last step, inserting (37) in (33) (for each $y \in B(0, \varepsilon)$), we get

$$\langle g_n^*, y \rangle \geq -C_1\psi_{\mathcal{M}}(A) - C_2\psi_{\mathcal{N}}(B) - S(A, B) - C_1\psi_{\mathcal{M}}(-c) - C_2\psi_{\mathcal{N}}(-d) - \Gamma_0 = C(x_0, y),$$

for all n , i.e., for each $y \in B(0, \varepsilon)$, the sequence $\{\langle g_n^*, y \rangle\}$ is bounded. Thus (i) of Lemma 5 confirms the boundedness of $\{g_n^*\}$, i.e., $\{\mathcal{M}_{\lambda_n} w_n\}$ is bounded because of (17). In conclusion, (ii) of Lemma 5 proves the maximal monotonicity of $\mathcal{M} + \mathcal{N}$. The proof is complete. \square

It is essential to point out that Theorem 2 gives a positive answer for Simons problem [1, Problem 28.15] under the symmetric conditions on domains of $\psi_{\mathcal{M}}$ and $\psi_{\mathcal{N}}$.

Theorem 3 provides a new maximality result.

Theorem 3. *Let E and E^* be strictly convex reflexive Banach spaces. Let $\mathcal{M} : E \supseteq D(\mathcal{M}) \rightarrow 2^{E^*}$ be maximal monotone and $\Psi : E \supseteq D(\Psi) \rightarrow \mathbf{R} \cup \{\infty\}$ be a proper, convex and lower-semicontinuous function such that $D(\partial\Psi) \cap D(\mathcal{M}) \neq \emptyset$. Suppose $D(\Psi)$ and $D(\mathcal{M})$ are symmetric such that*

$$\text{int}(D(\mathcal{M}) - D(\Psi)) \neq \emptyset.$$

Then $\partial\Psi + \mathcal{M}$ is maximal monotone.

Proof. For each $\lambda_n \downarrow 0^+$, let \mathcal{M}_{λ_n} and \mathcal{R}_{λ_n} be the Yosida approximant and resolvent of \mathcal{M} , respectively. As in the proof of Theorem 1, let $w_n \in D(\partial\Psi)$ and $g_n^* \in \partial\Psi(w_n)$ such that

$$g_n^* + \mathcal{M}_{\lambda_n} w_n + Jw_n = f^*, \tag{38}$$

for all n . The boundedness of $\{w_n\}$ and $\{Jw_n\}$ follows based on the arguments in the proof of Theorem 1. Thus Lemma 3 confirms the boundedness of the sequence $\{\mathcal{R}_{\lambda_n} w_n\}$. Choose $x_0 \in \text{int}(D(\mathcal{M}) - D(\Psi))$, i.e., $x_0 = a - b$ for some $a \in D(\mathcal{M})$ and $b \in D(\Psi)$. Then there exists $R > 0$ such that $\bar{B}(x_0, R) \subseteq D(\mathcal{M}) - D(\Psi)$. Clearly, we see that $y + x_0 \in B(x_0, R)$ if and only if $y \in B(0, R)$. Set $y + x_0 = A - B$ for some $A \in D(\mathcal{M})$ and $B \in D(\Psi)$ (i.e., $\Psi(B) \in \mathbf{R}$). By the definition of $\partial\Psi(w_n)$ (subdifferential of Ψ at $w_n \in D(\partial\Psi)$ for all n), we see that $g_n^* \in \partial\Psi(w_n)$ if and only if

$$\langle g_n^*, w_n - y \rangle \geq \Psi(w_n) - \Psi(y), \tag{39}$$

for all n and $y \in E$. In particular, taking B in place of y , yields

$$\langle g_n^*, w_n - B \rangle \geq \Psi(w_n) - \Psi(B), \tag{40}$$

for all n . In addition, (38) yields

$$\langle \mathcal{M}_{\lambda_n} w_n, w_n - (A - B) \rangle = -\langle g_n^*, w_n - (A - B) \rangle - \langle Jw_n - f^*, w_n - (A - B) \rangle, \tag{41}$$

$$\langle \mathcal{M}_{\lambda_n} w_n, B - w_n \rangle = -\langle g_n^*, B - w_n \rangle - \langle Jw_n - f^*, B - w_n \rangle, \tag{42}$$

and

$$\langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle = -\langle g_n^*, w_n \rangle - \langle Jw_n - f^*, w_n \rangle, \tag{43}$$

for all n . Choosing $A^* \in \mathcal{M}(A)$ and noticing that

$$\mathcal{R}_{\lambda_n} w_n \in D(\mathcal{M}) \text{ and } \mathcal{M}_{\lambda_n} w_n \in \mathcal{M}(\mathcal{R}_{\lambda_n} w_n),$$

for all n , the monotonicity of \mathcal{M} shows that

$$\langle \mathcal{M}_{\lambda_n} w_n - A^*, \mathcal{R}_{\lambda_n} w_n - A \rangle \geq 0, \tag{44}$$

for all n . By using equations (40) through (44), we get the estimate

$$\begin{aligned} \langle g_n^*, y + x_0 \rangle &= -\langle g_n^*, w_n - (A - B) \rangle + \langle g_n^*, w_n \rangle \\ &= \langle \mathcal{M}_{\lambda_n} w_n, w_n - (A - B) \rangle + \langle Jw_n - f^*, w_n - (A - B) \rangle + \langle g_n^*, w_n \rangle \\ &= \langle \mathcal{M}_{\lambda_n} w_n, w_n - \mathcal{R}_{\lambda_n} w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, A - \mathcal{R}_{\lambda_n} w_n \rangle + \langle g_n^*, w_n \rangle \\ &\quad + \langle \mathcal{M}_{\lambda_n} w_n, B - w_n \rangle + \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle + \langle Jw_n - f^*, w_n - (A - B) \rangle \\ &= \lambda_n \|\mathcal{M}_{\lambda_n} w_n\|^2 + \langle \mathcal{M}_{\lambda_n} w_n - A^*, \mathcal{R}_{\lambda_n} w_n - A \rangle + \langle g_n^*, w_n \rangle + \langle A^*, \mathcal{R}_{\lambda_n} w_n - A \rangle \\ &\quad + \langle g_n^*, w_n - B \rangle + \langle Jw_n - f^*, w_n - B \rangle - \langle g_n^*, w_n \rangle - \langle Jw_n - f^*, w_n \rangle \\ &\quad + \langle Jw_n - f^*, w_n - (A - B) \rangle + \langle g_n^*, w_n \rangle \\ &\geq -\|A^*\| \|\mathcal{R}_{\lambda_n} w_n - A\| + \Psi(w_n) - \Psi(B) - \|f^* - Jw_n\| \|w_n - B\| - \|Jw_n - f^*\| \|w_n\| \\ &\quad - \|Jw_n - f^*\| \|w_n - (A - B)\|, \end{aligned} \tag{45}$$

for all $n \in \mathbb{Z}_+$. Since $B \in D(\Psi)$ (i.e., $\Psi(B) \in \mathbf{R}$), $w_n \in D(\Psi)$ (i.e., $\psi(w_n) \in \mathbf{R}$ for all n) and Ψ is bounded below by an affine function (cf., Barbu [8, Proposition 1.1]), there exists $h^* \in E^*$ and $q \in \mathbf{R}$ such that

$$\Psi(w_n) \geq \langle h^*, w_n \rangle + q, \tag{46}$$

for all n . As a result (40) and (46) imply

$$\langle g_n^*, w_n - B \rangle \geq \Psi(w_n) - \Psi(B) \geq -\|h^*\| \|w_n\| + q - \Psi(B), \tag{47}$$

for all n . Combining (45) and (47), we obtain that

$$\begin{aligned} \langle g_n^*, y + x_0 \rangle &\geq -\|A^*\| \|\mathcal{R}_{\lambda_n} w_n - A\| - \|h^*\| \|w_n\| + q - \Psi(B) \\ &\quad - \|f^* - Jw_n\| \|w_n - B\| - \|f^* - Jw_n\| \|w_n\| - \|Jw_n - f^*\| \|w_n - (A - B)\|, \end{aligned} \tag{48}$$

for all n . Since $\{w_n\}$, $\{Jw_n\}$ and $\{\mathcal{R}_{\lambda_n} w_n - A\}$ are bounded, there exists $Q_1 = Q_1(A, B) > 0$ such that

$$\langle g_n^*, y + x_0 \rangle \geq -Q_1,$$

for all n . We notice that $-b \in D(\Psi)$ (i.e., $\Psi(-b) \in \mathbf{R}$ because $D(\Psi)$ is symmetric and $b \in D(\Psi)$). Furthermore, $\psi_{\mathcal{M}}(-a) \in \mathbf{R}$ because $a, -a \in D(\mathcal{M}) \subseteq D(\psi_{\mathcal{M}})$. In addition, (38) along with the definitions of $\psi_{\mathcal{M}}(-a)$ (by adding and subtracting the term $\mathcal{R}_{\lambda_n} w_n$ whenever needed) gives

$$-\langle g_n^*, w_n - a \rangle = \langle \mathcal{M}_{\lambda_n} w_n, w_n - a \rangle + \langle Jw_n - f^*, w_n - a \rangle, \tag{49}$$

and

$$\begin{aligned} \langle g_n^*, w_n - a \rangle &= -\langle \mathcal{M}_{\lambda_n} w_n, w_n - a \rangle - \langle Jw_n - f^*, w_n - a \rangle \\ &= -\langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, -a - w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - \langle Jw_n - f^*, w_n - a \rangle \\ &= -\langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, -a - \mathcal{R}_{\lambda_n} w_n \rangle - \langle \mathcal{M}_{\lambda_n} w_n, \mathcal{R}_{\lambda_n} w_n - w_n \rangle \\ &\quad - \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - \langle Jw_n - f^*, w_n - a \rangle \\ &= -\langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - (1 + \|\mathcal{R}_{\lambda_n} w_n\|) \frac{\langle \mathcal{M}_{\lambda_n} w_n, -a - \mathcal{R}_{\lambda_n} w_n \rangle}{1 + \|\mathcal{R}_{\lambda_n} w_n\|} + \lambda_n \|\mathcal{M}_{\lambda_n} w_n\|^2 \end{aligned}$$

$$\begin{aligned}
 & - \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - \langle Jw_n - f^*, w_n - a \rangle \\
 \geq & -2 \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - (1 + \|\mathcal{R}_{\lambda_n} w_n\|) \psi_{\mathcal{M}}(-a) - \langle Jw_n - f^*, w_n - a \rangle,
 \end{aligned} \tag{50}$$

for all n . Moreover, the definition of $g_n^* \in \partial\Psi(w_n)$ for all n , gives

$$\langle g_n^*, w_n - (-b) \rangle \geq \Psi(w_n) - \Psi(-b), \tag{51}$$

for all n . Thus (43), (46)-(51) imply

$$\begin{aligned}
 -\langle g_n^*, x_0 \rangle &= \langle g_n^*, -a - (-b) \rangle \\
 &= \langle g_n^*, w_n - (-b) \rangle + \langle g_n^*, w_n - a \rangle - 2 \langle g_n^*, w_n \rangle \\
 &\geq \Psi(w_n) - \Psi(-b) - 2 \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle - (1 + \|\mathcal{R}_{\lambda_n} w_n\|) \psi_{\mathcal{M}}(-a) - \langle Jw_n - f^*, w_n - a \rangle \\
 &\quad + 2 \langle \mathcal{M}_{\lambda_n} w_n, w_n \rangle + 2 \langle Jw_n - f^*, w_n \rangle \\
 &\geq \Psi(w_n) - \Psi(-b) - (1 + \|\mathcal{R}_{\lambda_n} w_n\|) \psi_{\mathcal{M}}(-a) - \|Jw_n - f^*\| \|w_n - a\| - 2 \|Jw_n - f^*\| \|w_n\| \\
 &\geq -2 \|h^*\| \|w_n\| + q - \Psi(-b) - C_1 \psi_{\mathcal{M}}(-a) - C_2,
 \end{aligned} \tag{52}$$

for all n , where C_1 and C_2 are upper bounds for $\{1 + \|\mathcal{R}_{\lambda_n} w_n\|\}$ and $\{\|Jw_n - f^*\| \|w_n - a\| + 2 \|Jw_n - f^*\| \|w_n\|\}$, respectively. Then there exists $Q_2 = Q_2(a, b) > 0$ such that $-\langle g_n^*, x_0 \rangle \geq -Q_2$ for all n . Consequently, for each $y \in B(0, R)$ we have

$$\langle g_n^*, y \rangle = \langle g_n^*, y + x_0 \rangle - \langle g_n^*, x_0 \rangle \geq -Q_1 - Q_2,$$

for all n . Replacing y by $-y$, we get $\langle g_n^*, y \rangle \leq Q_1 + Q_2$ for all n , i.e., for each $y \in B(0, R)$, we get the boundedness of the sequence $\{\langle g_n^*, y \rangle\}$. Thus (i) of Lemma 5 concludes that $\{g_n^*\}$ is bounded, i.e., $\{\mathcal{M}_{\lambda_n} w_n\}$ is bounded. Consequently, (ii) of Lemma 5 proves the maximality of $\mathcal{M} + \partial\Psi$. The proof is complete. \square

Theorem 4 yields a new maximality result for the sum of three maximal monotone operators involving the sum of two subdifferentials of convex functions. Its proof is based on Lemma 5, Theorem 3 and arguments used in the proofs of Theorems 1-3.

Theorem 4. Let E and E^* be strictly convex reflexive Banach spaces. Let $\mathcal{A} : E \supseteq D(\mathcal{A}) \rightarrow 2^{E^*}$ be maximal monotone and $\Phi : E \supseteq D(\Phi) \rightarrow \mathbf{R} \cup \{\infty\}$ and $\Pi : E \supseteq D(\Pi) \rightarrow \mathbf{R} \cup \{\infty\}$ be proper, convex and lower semi-continuous functions such that $D(\partial\Pi) \subseteq D(\mathcal{A})$ and $D(\partial\Phi) \cap D(\partial\Pi) \neq \emptyset$. Assume, further, that

- (i) $D(\mathcal{A})$ and $D(\Pi)$ are symmetric;
- (ii)

$$0 \in \text{int}(D(\mathcal{A}) \cap D(\Phi) - D(\partial\Pi)). \tag{53}$$

Then $\mathcal{A} + \partial\Phi + \partial\Pi$ is maximal monotone.

Proof. Clearly, (53) implies $\text{int}(D(\mathcal{A}) - D(\Pi)) \neq \emptyset$ because $D(\mathcal{A}) \cap D(\Phi) \subseteq D(\mathcal{A})$ and $D(\partial\Pi) \subseteq D(\Pi)$. In addition (i) holds, and hence all conditions in Theorem 3 are satisfied. Thus Theorem 3 implies that $B = \mathcal{A} + \partial\Pi$ is maximal monotone. Let $\mathcal{P} = \partial\Phi$. For each $n \in \mathbf{Z}_+$ and $\lambda_n \downarrow 0^+$, it follows that $\mathcal{P} + B_{\lambda_n}$ is maximal monotone, i.e., $\mathcal{P} + B_{\lambda_n} + J$ is surjective for all n . Thus for each $f^* \in E^*$, there exists $w_n \in D(\mathcal{P})$ and $g_n^* \in \mathcal{P}w_n$ such that

$$g_n^* + B_{\lambda_n} w_n + Jw_n = f^* \text{ for all } n. \tag{54}$$

Let \mathcal{J}_{λ_n} be the Yosida resolvent of B . Based on similar arguments applied in the proof of Theorem 1, we conclude that $\{w_n\}$, $\{\mathcal{J}_{\lambda_n} w_n\}$ and $\{Jw_n\}$ are bounded. By using (53), we choose $R > 0$ such that

$$B(0, R) \subseteq D(\mathcal{A}) \cap D(\Phi) - D(\partial\Pi).$$

For each $y \in B(0, R)$, set $y = a - b$, where $a \in D(\mathcal{A}) \cap D(\Phi)$ and $b \in D(\partial\Pi)$. As a result of (54), we obtain that

$$\langle B_{\lambda_n} w_n, y \rangle = - \langle g_n^*, a - b \rangle - \langle Jw_n - f^*, a - b \rangle$$

$$\begin{aligned} &= \langle g_n^*, -a \rangle + \langle g_n^*, b \rangle - \langle Jw_n - f^*, a - b \rangle \\ &= \langle g_n^*, w_n - a \rangle - \langle g_n^*, w_n - b \rangle - \langle Jw_n - f^*, a - b \rangle, \end{aligned} \tag{55}$$

for all n . By the definition of $g_n^* \in \partial\Phi(w_n)$, we see that $\langle g_n^*, w_n - a \rangle \geq \Phi(w_n) - \Phi(a)$ for all n . Clearly, $\Phi(w_n) - \Phi(a) \in \mathbf{R}$ for all n , because $w_n \in D(\Phi)$ for all n and $a \in D(\Phi)$. In addition, there exists $\lambda^* \in E^*$ and $q \in \mathbf{R}$ such that

$$\Phi(w_n) \geq \langle \lambda^*, w_n \rangle + q \geq -\|\lambda^*\| \|w_n\| + q, \tag{56}$$

for all n , i.e., we have

$$\langle g_n^*, w_n - a \rangle \geq -\|\lambda^*\| \|w_n\| + q - \Phi(a) \text{ for all } n. \tag{57}$$

Furthermore, (54) shows that

$$\begin{aligned} -\langle g_n^*, w_n - b \rangle &= \langle B_{\lambda_n} w_n, w_n - b \rangle + \langle Jw_n - f^*, w_n - b \rangle \\ &= \langle B_{\lambda_n} w_n, w_n - \mathcal{J}_{\lambda_n} w_n \rangle + \langle B_{\lambda_n} w_n, \mathcal{J}_{\lambda_n} w_n - b \rangle + \langle Jw_n - f^*, w_n - b \rangle \\ &= \lambda_n \|B_{\lambda_n} w_n\|^2 + \langle B_{\lambda_n} w_n, \mathcal{J}_{\lambda_n} w_n - b \rangle + \langle Jw_n - f^*, w_n - b \rangle \\ &\geq \langle B_{\lambda_n} w_n, \mathcal{J}_{\lambda_n} w_n - b \rangle + \langle Jw_n - f^*, w_n - b \rangle, \end{aligned} \tag{58}$$

for all n . Since $b \in D(B) = D(\mathcal{A}) \cap D(\partial\Pi) = D(\partial\Pi)$, one can choose $b^* \in Bb$ and use the monotonicity of B and the fact that $(\mathcal{J}_{\lambda_n} w_n, B_{\lambda_n} w_n) \in G(B)$ for all n , to conclude that

$$\begin{aligned} \langle B_{\lambda_n} w_n, \mathcal{J}_{\lambda_n} w_n - b \rangle &= \langle B_{\lambda_n} w_n - b^*, \mathcal{J}_{\lambda_n} w_n - b \rangle + \langle b^*, \mathcal{J}_{\lambda_n} w_n - b \rangle \\ &\geq -\|b^*\| \|\mathcal{J}_{\lambda_n} w_n - b\|, \end{aligned} \tag{59}$$

for all n . Next inserting the estimates in (57), (58) and (59) in (55), we arrive at

$$\langle B_{\lambda_n} w_n, y \rangle \geq -\|\lambda^*\| \|w_n\| + q - \Phi(a) - \|b^*\| \|\mathcal{J}_{\lambda_n} w_n - b\| - \|Jw_n - f^*\| (\|a - b\| + \|w_n - b\|), \tag{60}$$

for all n , i.e., for each $y \in B(0, R)$, we get

$$\langle B_{\lambda_n} w_n, y \rangle \geq -D_1(a, b) + q - \Phi(a) \text{ for all } n,$$

where $D_1(a, b)$ is an appropriate upper bound for the sequence $\{\|\lambda^*\| \|w_n\| + \|b^*\| \|\mathcal{J}_{\lambda_n} w_n - b\| + \|Jw_n - f^*\| (\|a - b\| + \|w_n - b\|)\}$. Thus (i) of Lemma 5 shows that $\{B_{\lambda_n} w_n\}$ is bounded, i.e., $\{g_n^*\}$ is bounded. Then (ii) of Lemma 5 confirms the maximality of $B + \partial\Phi = \mathcal{A} + \partial\Phi + \partial\Pi$. The proof is complete. \square

4. An application

In this section, we shall prove an existence of solution in $Y = L^p(0, T; W)$ (where $p \geq 2$, $W = W_0^{1,p}(\Omega)$, Ω is a nonempty, open and bounded subset of \mathbf{R}^N) for problem (*). It is well-known that $W^* = W_0^{1,q}(\Omega)$, $Y^* = L^q(0, T; W^*)$ and $W \subseteq L^2(\Omega) \subseteq W^*$, where q is the conjugate exponent of p (i.e., $p^{-1} + q^{-1} = 1$). For $y \in Y$ (i.e., $y(t) \in W$ for all $t \in [0, T]$), the norm of y is defined by

$$\|y\|_Y^p = \int_0^T \|y(t)\|_W^p dt,$$

where for each $t \in [0, T]$, $\|y(t)\|_W$ denotes the norm of $y(t)$ in W . For each $w \in Y^*$ and $z \in Y$, the pairing $\langle w, z \rangle$, is understood as

$$\langle w, z \rangle_{Y^* \times Y} = \int_0^T \langle w(t), z(t) \rangle_{W^* \times W} dt,$$

where the pairing $\langle w(t), z(t) \rangle_{W^* \times W}$ ($t \in [0, T]$) is the duality pairing in $W^* \times W$. Whenever there is no confusion, we shall use the symbol $\langle w, z \rangle$ and $\langle w(t), z(t) \rangle$, to denote $\langle w, z \rangle_{Y^* \times Y}$ and $\langle w(t), z(t) \rangle_{W^* \times W}$, respectively.

Let $\mathcal{A} : Y \supseteq D(\mathcal{A}) \rightarrow Y^*$ be given by $\mathcal{A}w = w'$, where w' is understood in the sense of distribution, i.e., w' satisfies

$$\int_0^T w'(t)\phi(t)dt = - \int_0^T w(t)\phi'(t)dt, \phi \in C_0^\infty(0, T),$$

and $D(\mathcal{A}) = \{v \in Y : v' \in Y^*, v(0) = v(T)\}$. For each $w \in D(\mathcal{A})$, it follows that $\mathcal{A}w \in Y^*$ and

$$\langle \mathcal{A}w, \phi \rangle = \int_0^T \langle w'(t), \phi(t) \rangle dt, w \in D(\mathcal{A}), \phi \in Y. \tag{61}$$

Theorem 5. Let $Y = L^p(0, T; W)$, $W = W_0^{1,p}(\Omega)$, $p \geq 2$, q is conjugate exponent of p and Ω be a nonempty, bounded and open subset of \mathbf{R}^N , $N \in \mathbf{Z}_+$. Suppose Φ and $q_i (i = 0, 1, 2, \dots, N)$ satisfy corresponding conditions in $(c_1) - (c_4)$ and \mathcal{A} is given by (61). Assume, further, that \mathcal{K} is a nonempty, closed, convex and symmetric subset of $D(\mathcal{A})$, $D(\partial\Phi) \cap \mathcal{K} \neq \emptyset$ and

$$0 \in \text{int}(D(\mathcal{A}) \cap D(\Phi) - \mathcal{K}).$$

Then (*) admits at least one solution in $D(\partial\Phi) \cap \mathcal{K}$.

Proof. Let \mathcal{A} be as given in (61). It is well-known due to Proposition 32.10 in Zeidler [7], that \mathcal{A} is densely defined maximal monotone with closed graph. Clearly, $D(\mathcal{A})$ is symmetric because $D(\mathcal{A})$ is a linear space of Y . Thus all assumptions of Theorem 4 are fulfilled with maximal monotone operators \mathcal{A} , $\partial\Phi$ and $\partial I_{\mathcal{K}}$ (with $\Pi = I_{\mathcal{K}}$ in Theorem 4). Thus Theorem 4 confirms the maximality of $\mathcal{T} = \mathcal{A} + \partial\Phi + \partial I_{\mathcal{K}}$. Let $Q : Y \rightarrow Y^*$ be defined by (**). It is well-known (cf., Landes and Mustonnen [33]) that Q is bounded, continuous and pseudomonotone, and $-\Delta_p : Y \rightarrow Y^*$ is bounded, continuous, monotone (i.e., pseudomonotone). Thus $\mathcal{S} = -\Delta_p + Q : Y \rightarrow Y^*$, is bounded pseudomonotone (cf., Browder and Hess [29, Proposition 9]). Again, we notice that Φ is bounded below by an affine function, i.e., there exists $(\lambda^*, d) \in E^* \times \mathbf{R}$ such that

$$\Phi(w) \geq \langle \lambda^*, w \rangle + d \geq -\|\lambda^*\| \|w\| + d, \tag{62}$$

for all $w \in E$. It is clear to see that $\langle \mathcal{A}w, w \rangle \geq 0$ for all $w \in D(\mathcal{A})$, because $\mathcal{A}(0) = 0$, \mathcal{A} is linear and monotone. On the other hand, $0 \in \mathcal{K}$ because \mathcal{K} is nonempty and symmetric. The definitions of $\eta^* \in \partial I_{\mathcal{K}}(w)$ and $\chi^* \in \partial\Phi(w)$ imply $\langle \eta^*, w \rangle \geq 0$ and $\langle \chi^*, w \rangle \geq \Phi(w) - \Phi(0)$ for all $w \in \mathcal{K} \cap D(\partial\Phi)$. These facts along with definitions of Q and \mathcal{A} yield

$$\begin{aligned} \langle -\Delta_p w + Qw + \mathcal{A}w + \chi^* + \eta^*, w \rangle &= \langle -\Delta_p w, w \rangle + \langle Qw, w \rangle + \langle \mathcal{A}w, w \rangle + \langle \chi^*, w \rangle + \langle \eta^*, w \rangle \\ &= \int_0^T \int_{\Omega} |\nabla w|^{p-2} \nabla w \nabla w dx dt + \int_0^T \sum_{i=1}^N \int_{\Omega} q_i(x, t, w(x, t), \nabla w(x, t)) \frac{\partial w}{\partial x_i} dx dt \\ &\quad + \int_{\Gamma} q_0(x, t, w(x, t), \nabla w(x, t)) w(x, t) dx dt - \|\lambda^*\| \|w\|_Y + d - \Phi(0), \end{aligned} \tag{63}$$

for all $w \in D(\partial\Phi) \cap \mathcal{K}$, $\chi^* \in \partial\Phi(w)$ and $\eta^* \in \partial I_{\mathcal{K}}(w)$. Furthermore, the following holds.

(a) By the definition of $-\Delta_p$, it follows that

$$\langle -\Delta_p w, w \rangle = \int_0^T \int_{\Omega} |\nabla w|^{p-2} \nabla w \nabla w dx dt = \int_0^T \int_{\Omega} |\nabla w|^p dx dt = \int_0^T \|\nabla w(t)\|_{L^p(\Omega)}^p dt \text{ for all } w \in Y. \tag{64}$$

(b) Conditions (c_3) and (c_4) yield

$$\begin{aligned} \int_{\Gamma} q_0(x, t, w(x, t), \nabla w(x, t)) w(x, t) dx dt &\geq \theta \int_{\Gamma} |w(x, t)|^p dx dt - \pi |\Gamma| \\ &= \theta \int_0^T \|w(t)\|_{L^p(\Omega)}^p dt - \pi |\Gamma| \text{ for all } w \in Y, \end{aligned} \tag{65}$$

where $|\Gamma|$ is $N + 1$ -dimensional Lebesgue measure of Γ .

(c) An application of Lemma 1 (where p and q are conjugate exponents to each other such that $q(p - 1) = p$) shows that

$$\begin{aligned} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |w(x, t)|^{p-1} \left| \frac{\partial w}{\partial x_i} \right| dx \right) dt &\leq \frac{1}{q} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |w(x, t)|^{q(p-1)} dx \right) dt + \frac{1}{p} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt \\ &= \frac{1}{q} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |w(x, t)|^p dx \right) dt + \frac{1}{p} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt \\ &= \frac{N}{q} \int_0^T \left(\int_{\Omega} |w(x, t)|^p dx \right) dt + \frac{1}{p} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt, \quad w \in Y, \end{aligned} \tag{66}$$

and

$$\begin{aligned} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |\nabla w(x, t)|^{p-1} \left| \frac{\partial w}{\partial x_i} \right| dx \right) dt &\leq \frac{1}{q} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |\nabla w(x, t)|^{q(p-1)} dx \right) dt + \frac{1}{p} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt \\ &= \frac{1}{q} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |\nabla w(x, t)|^p dx \right) dt + \frac{1}{p} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt \\ &= \frac{N}{q} \int_0^T \left(\int_{\Omega} |\nabla w(x, t)|^p dx \right) dt + \frac{1}{p} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt, \quad w \in Y. \end{aligned} \tag{67}$$

Next we use (66) and (67), condition (c₃) and the definitions of norms in Y and W , to obtain the estimate

$$\begin{aligned} \sum_{i=1}^N \int_{\Gamma} q_i(x, t, w(x, t), \nabla w(x, t)) \frac{\partial w}{\partial x_i} dx dt &\geq - \sum_{i=1}^N \int_{\Gamma} |q_i(x, t, w(x, t), \nabla w(x, t))| \left| \frac{\partial w}{\partial x_i} \right| dx dt \\ &\geq -c \sum_{i=1}^N \int_{\Gamma} \left(|w(x, t)|^{p-1} + |\nabla w(x, t)|^{p-1} \right) \left| \frac{\partial w}{\partial x_i} \right| dx dt \\ &\geq -\frac{c}{q} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |w(x, t)|^p dx \right) dt - \frac{c}{p} \int_0^T \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt \\ &\quad - \frac{c}{q} \sum_{i=1}^N \int_0^T \left(\int_{\Omega} |\nabla w(x, t)|^p dx \right) dt - \frac{c}{p} \int_0^T \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial w}{\partial x_i} \right|^p dx \right) dt \\ &\geq -\frac{Nc}{q} \int_0^T \|w\|_W^p dt - \frac{2c}{p} \int_0^T \|w\|_W^p dt \\ &= -\frac{Nc}{q} \|w\|_Y^p - \frac{2c}{p} \|w\|_Y^p, \quad w \in Y. \end{aligned} \tag{68}$$

Consequently, (65) and (68) give

$$\begin{aligned} \langle Qw, w \rangle &= \sum_{i=1}^N \int_{\Gamma} q_i(x, t, w(x, t), \nabla w(x, t)) \frac{\partial w}{\partial x_i} dx dt + \int_{\Gamma} q_0(x, t, w(x, t), \nabla w(x, t)) w(x, t) dx dt \\ &\geq \frac{-Nc}{q} \|w\|_Y^p - \frac{2c}{p} \|w\|_Y^p + \int_0^T \|w(t)\|_{L^p(\Omega)}^p dt - \pi|\Gamma| \text{ for all } w \in Y. \end{aligned} \tag{69}$$

In conclusion, combining Eqs (63)-(69), we obtain

$$\begin{aligned} \langle -\Delta_p w + Qw + \mathcal{A}w + \chi^* + \eta^*, w \rangle &= \langle -\Delta_p w, w \rangle + \langle \mathcal{A}w, w \rangle + \langle Qw, w \rangle + \langle \chi^*, w \rangle + \langle \eta^*, w \rangle \\ &\geq \int_0^T \|\nabla w(t)\|_{L^p(\Omega)}^p dt + \int_0^T \|w(t)\|_{L^p(\Omega)}^p dt - \pi|\Gamma| - \frac{Nc}{q} \|w\|_Y^p - \frac{2c}{p} \|w\|_Y^p - \|\lambda^*\| \|w\|_Y + d - \Phi(0) \\ &= \int_0^T \left(\|\nabla w(t)\|_{L^p(\Omega)}^p + \|w(t)\|_{L^p(\Omega)}^p \right) dt - \pi|\Gamma| - \frac{Nc}{q} \|w\|_Y^p - \frac{c}{p} \|w\|_Y^p - \|\lambda^*\| \|w\|_Y + d - \Phi(0) \\ &= \int_0^T \|w(t)\|_W^p dt - \pi|\Gamma| - \frac{Nc}{q} \|w\|_Y^p - \frac{2c}{p} \|w\|_Y^p - \|\lambda^*\| \|w\|_Y + d - \Phi(0) \end{aligned}$$

$$= \mathcal{C} \|w\|_Y^p - \pi |\Gamma| - \|\lambda^*\| \|w\|_Y + d - \Phi(0), \quad (70)$$

for all $w \in D(\partial\Phi) \cap \mathcal{K}$, $\chi^* \in \partial\Phi(w)$ and $\eta^* \in \partial I_{\mathcal{K}}(w)$, where $\mathcal{C} = 1 - \frac{Nc}{q} - \frac{2c}{p}$. Since $c \in (0, pq(Np + 2q)^{-1})$, it follows that $\mathcal{C} > 0$. Since $p \geq 2$, (70) shows that

$$\inf_{w \in D(\partial\Phi) \cap \mathcal{K}, \chi^* \in \partial\Phi(w), \eta^* \in \partial I_{\mathcal{K}}} \left(\frac{\langle -\Delta_p w + Qw + \mathcal{A}w + \chi^* + \eta^*, w \rangle}{\|w\|_Y} \right) \rightarrow \infty,$$

as $\|w\|_Y \rightarrow \infty$, i.e., $\mathcal{T} + \mathcal{S}$ is coercive. Thus Lemma 4 confirms the surjectivity of $\mathcal{T} + \mathcal{S}$, i.e., for each $f^* \in E^*$, (*) admits at least one solution in $D(\partial\Phi) \cap \mathcal{K}$. The proof is complete. \square

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