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Short Note

On 3-perfect numbers

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Abstract: It is shown that every 3-perfect number in its prime factorization has the exponent of the number 2 which is greater than 1.

Keywords: k-perfect number, 3-perfect number

MSC: 11A25.

Let us recall some concepts, formulas and notations. The set of all natural numbers $\{1,2,3,\ldots\}$ will be denoted by $\mathbb{N}$. $\mathbb{R}^+$ is the set of all positive real numbers. We will use the notion (n,m) of greatest common divisor of two numbers $n,m \in \mathbb{N}$. If $n \in \mathbb{N}$ and $n > 1$, then its factorization is the representation of number n in the form $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$, where p_i is prime, $p_1 < p_2 < \ldots < p_s$ and $\alpha_i \in \mathbb{N}$ for any $i = \overline{1,s}$.

A function f is called arithmetic if it is defined on the set of all natural numbers. An arithmetic function f is called multiplicative if $f(1) = 1$ and for any numbers $a,b \in \mathbb{N}$ with $(a,b) = 1$ it follows that $f(ab) = f(a)f(b)$.

The arithmetic function σ is defined for any $n \in \mathbb{N}$ as the sum of all positive divisors of n, i.e. $\sigma(n) = \sum\limits_{d|n} d$. It is known that the function σ is multiplicative and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ is factorization of the number n, then

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdot \ldots \cdot \frac{p_s^{\alpha_s+1} - 1}{p_s - 1}.$$

In [1] for a natural number n the multiplicative function $h(n) = \dfrac{\sigma(n)}{n}$ and some of its properties are considered. Let us formulate and give a proof of one of them, which we will apply hereafter.

Lemma 1. [1] *Let $p > q$ be odd prime numbers, $\alpha \in \mathbb{N}$, then*

$$h(p^\alpha) < h(q). \tag{1}$$

Proof. Let $x, t \in \mathbb{R}^+$, $t > 1$ we consider the function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g(t^x) = \frac{t^{x+1} - 1}{t^{x+1} - t^x}.$$

Since the function $g(t^x)$ is continuous at every point of the set $\mathbb{R}^+$, then $g(t^x)$ is continuous on the entire set $\mathbb{R}^+$. Next, we will show that the function $g(t^x)$ is monotonically increasing on $\mathbb{R}^+$. Suppose $x_1, x_2 \in \mathbb{R}^+$ and $x_1 > x_2$. Since $t > 1$, it follows that $t^{x_1} > t^{x_2}$. We will show that $g(t^{x_1}) > g(t^{x_2})$. Let's assume the contrary, that is, $g(t^{x_2}) \geq g(t^{x_1})$. Then we have

$$\frac{t^{x_2+1} - 1}{t^{x_2+1} - t^{x_2}} \geq \frac{t^{x_1+1} - 1}{t^{x_1+1} - t^{x_1}},$$

$$\left(t^{x_2+1} - 1\right)\left(t^{x_1+1} - t^{x_1}\right) \geq \left(t^{x_1+1} - 1\right)\left(t^{x_2+1} - t^{x_2}\right),$$

$$t^{x_2+1}t^{x_1+1} - t^{x_2+1}t^{x_1} - t^{x_1+1} + t^{x_1} \geq t^{x_1+1}t^{x_2+1} - t^{x_1+1}t^{x_2} - t^{x_2+1} + t^{x_2},$$

$$\left(t^{x_1+1}t^{x_2} - t^{x_2+1}t^{x_1}\right) + t^{x_1}(1-t) \geq t^{x_2}(1-t),$$

$$t^{x_1}t^{x_2}\left(t-t\right)+t^{x_1}\left(1-t\right)\geq t^{x_2}\left(1-t\right), \text{ since } t>1, \text{ then}$$
$$t^{x_1}\leq t^{x_2}.$$

The obtained contradiction proves that the function $g(t^x)$ is monotonically increasing on $\mathbb{R}^+$.

Then

$$g(t^x)=\frac{t^{x+1}-1}{t^{x+1}-t^x}=\frac{t-\frac{1}{t^x}}{t-1}<\lim_{x\to\infty}g(t^x)=\frac{t}{t-1},$$

and $g(p^\alpha)<\frac{p}{p-1}$.

We obtain the inequality,

$$h(p^\alpha)=\frac{p^{\alpha+1}-1}{p^{\alpha+1}-p^\alpha}=g(p^\alpha)<\frac{p}{p-1}=1+\frac{1}{p-1}.$$

Since p and q are odd prime numbers and $p>q$, then $p-1$ is an even number such that $p-1>q$. Therefore

$$1+\frac{1}{p-1}<1+\frac{1}{q}=\frac{q+1}{q}=h(q).$$

Thus $h(p^\alpha)<h(q)$. □

Recall that a natural number n is called perfect if it is equal to the sum of its own divisors, i.e.

$$n=\sum_{d|n,\,d\geq 1,\,d\neq n}d.$$

Let $n,k\in\mathbb{N}$ such that $k,n>1$. A number n is called k-perfect if $\sigma(n)=kn$. The concept of a perfect number of multiplicity k is equivalent to the concept of a k-perfect number, the history of research of which can be found, e.g., in [2]. It follows from the definition of k-perfect number that the concepts of perfect and 2-perfect numbers coincide. For example, 6 and 28 are 2-perfect numbers, and 120 and 672 are 3-perfect numbers. Even perfect numbers were described by Euclid and Euler. Some necessary and sufficient conditions for the existence of odd perfect numbers were found in [3].

Lemma 2. *Let $k,n\in\mathbb{N}$ be such that $k,n>1$ and $(k,n)=1$. A number n is k-perfect if and only if kn is $\sigma(k)$-perfect.*

Proof. Let the number n be k-perfect. Let us show that kn is $\sigma(k)$-perfect. Indeed, $\sigma(kn)=\sigma(k)\sigma(n)=\sigma(k)\cdot kn$, i.e. $\sigma(kn)=\sigma(k)\cdot kn$.

Conversely. Let the number kn be $\sigma(k)$-perfect. Let us show that the number n is k-perfect. Indeed, $\sigma(k)\sigma(n)=\sigma(kn)=$ (because kn is $\sigma(k)$-perfect) $=\sigma(k)\cdot kn$. Hence, $\sigma(k)\sigma(n)=\sigma(k)\cdot kn$. Since $\sigma(k)\neq 0$ then, reducing both sides of the last equality by $\sigma(k)$, we get $\sigma(n)=kn$. □

Corollary 1. *Let odd perfect numbers exist. Then the odd number $n>1$ is perfect if and only if the number $2n$ is 3-perfect.*

Proof. To prove this corollary, it is sufficient to put $k=2$ in the formulation of Lemma 2. □

Further we will use the following statement proved in [4]. For completeness we will give its proof.

Lemma 3. [4] *Let $n\in\mathbb{N}$ be a 3-perfect number and $n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_s^{\alpha_s}$ be its factorization, then $s\geq 3$.*

Proof. Suppose the contrary, i.e. let $s=2$ (the case of $s=1$ is proved similarly). Then $n=p_1^{\alpha_1}p_2^{\alpha_2}$. Consider the expression

$$3p_1^{\alpha_1}p_2^{\alpha_2}=\sigma(n)=\frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1}<\frac{p_1^{\alpha_1+1}}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}}{p_2-1},$$

or

$$3<\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\leq\frac{2}{2-1}\cdot\frac{3}{3-1}=3.$$

The resulting contradiction proves this statement. □

Theorem 1. *Every even* 3*-perfect number has in its factorization the exponent of number 2 greater than 1.*

Proof. Let $n = 2^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ be its factorization of any even 3-perfect number. Then we have

$$3 \cdot 2^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s} = \sigma(n) = \left(2^{\alpha_1+1} - 1\right) \sigma\left(p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}\right).$$

Hence,

$$2^{\alpha_1} = \frac{1}{3}\left(2^{\alpha_1+1} - 1\right) \frac{\sigma\left(p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}\right)}{p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}} = \frac{1}{3}\left(2^{\alpha_1+1} - 1\right) t,$$

where

$$t = \frac{\sigma\left(p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}\right)}{p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}} = \prod_{i=2}^{s} h(p_i^{\alpha_i}),\ s \geq 3.$$

Then

$$2^{\alpha_1} = \frac{1}{2 - \frac{3}{t}}.$$

Consider the following cases.

1. Let $p_2 \neq 3$. Then according to inequality (1) it follows that $h(p_i^{\alpha_i}) < h(3)$ for any $i = \overline{2,s}$. Therefore, $t < h(3)^{s-1}$, where $s \geq 3$. Thus,

$$2^{\alpha_1} = \frac{1}{2 - \frac{3}{t}} > \frac{1}{2 - \frac{3}{h(3)^{s-1}}},\ \text{i.e.}$$

$$2^{\alpha_1} > \frac{1}{2 - \frac{3}{h(3)^{s-1}}}. \tag{2}$$

By the condition $\alpha_1 \geq 1$, then $\alpha_1 > 0.9$. Therefore, $2^{\alpha_1} > 2^{0.9}$ Next, we will find at what values of the variable s the right side of inequality (2) will be greater than $2^{0.9}$. We have

$$\frac{1}{2 - \frac{3}{h(3)^{s-1}}} > 2^{0.9}.$$

From this we obtain that

$$h(3)^{s-1} < \frac{3 \cdot 2^{0.9}}{2^{1.9} - 1} \approx 2.049. \tag{3}$$

Since $h(3) = 4/3$, then from inequality (3) we obtain that $s < 1 + \frac{\ln 2.049}{\ln \frac{4}{3}} \approx 3.5$. Hence, $s = 3$. Then we have

$$2^{\alpha_1} > \frac{1}{2 - \frac{3}{\left(\frac{4}{3}\right)^2}} = \frac{16}{5}.$$

So $\alpha_1 > 4 - \log_2 5 > 1$.

2. Let $p_2 = 3$. Then

$$t = \prod_{i=2}^{s} h(p_i^{\alpha_i}) = h(3^{\alpha_2}) \prod_{i=3}^{s} h(p_i^{\alpha_i}) < h(3^{\alpha_2}) h(3)^{s-2},$$

(since according to inequality (1) we have $h(p_i^{\alpha_i}) < h(3)$ for any $i = \overline{3, s}$). Thus,

$$2^{\alpha_1} = \frac{1}{2 - \frac{3}{t}} > \frac{1}{2 - \frac{3}{h(3^{\alpha_2})h(3)^{s-2}}}, \; s \geq 3. \tag{4}$$

As in case 1, we restrict the right-hand side of inequality (2) from below by a number $2^{0.9}$, i.e., we have

$$\frac{1}{2 - \frac{3}{h(3^{\alpha_2})h(3)^{s-2}}} > 2^{0.9}. \tag{5}$$

Then from inequality (3) we obtain that

$$h(3^{\alpha_2})h(3)^{s-2} < 2.049, \; s \geq 3.$$

Then we have

$$\left(1 + \frac{1}{3} + \ldots + \frac{1}{3^{\alpha_2}}\right)\left(\frac{4}{3}\right)^{s-2} < 2.049 \;\text{ or}$$

$$\left(1 + \frac{1}{3}\right)\left(\frac{4}{3}\right)^{s-2} + \left(\frac{1}{3^2} + \ldots + \frac{1}{3^{\alpha_2}}\right)\left(\frac{4}{3}\right)^{s-2} < 2.049.$$

Hence,

$$\left(1 + \frac{1}{3}\right)\left(\frac{4}{3}\right)^{s-2} < 2.049, \text{ or } \left(\frac{4}{3}\right)^{s-1} < 2.049.$$

Then $s = 3$ (see case 1). Then we have

$$2^{\alpha_1} = \frac{1}{2 - \frac{3}{t}} > \frac{1}{2 - \frac{3}{h(3^{\alpha_2})h(3)}} = \frac{1}{2 - \frac{3}{\frac{(3^{\alpha_2+1} - 1)}{3^{\alpha_2} \cdot (3-1)} \cdot \frac{4}{3}}} = \frac{2 \cdot 3^{\alpha_2+1} - 2}{3^{\alpha_2+1} - 4} =$$

$$= \frac{2(3^{\alpha_2+1} - 4) + 6}{3^{\alpha_2+1} - 4} = 2 + \frac{6}{3^{\alpha_2+1} - 4} > 2,$$

(since $\alpha_2 \geq 1$).) Therefore, $2^{\alpha_1} > 2$, i.e. $\alpha_1 > 1$. □

Corollary 2. *There are no odd perfect numbers.*

Proof. Suppose the contrary, i.e. let an odd perfect number n exists. Then the number $2n$ is 3-perfect by Corollary 2, which contradicts Theorem 1. □

Conflicts of Interest: This work does not have any conflicts of interest.

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