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On arithmetic cordial labeling of some graphs

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Abstract: Let η be a fixed positive integer. Let S be a subset of $\mathbb{Z}$, $\star : S \times S \to \mathbb{Z}$ be a binary function, and $\zeta_\eta : \{\xi \in \mathbb{Z} : \gcd(\xi, \eta) = 1\} \to \{0, 1\}$ be a function. For a simple connected graph G of order n, a bijective function $f : V(G) \to S$ (where $|S| = n$) is called an arithmetic cordial labeling modulo η under the arithmetic structure $\langle S, \zeta_\eta, \star\rangle$ if the induced function $f^*_\eta : E(G) \to \{0, 1\}$, defined by $f^*_\eta(ab) = 1$ whenever $\gcd(f(a) \star f(b), \eta) = 1$ and $\zeta_\eta(f(a) \star f(b)) = 1$; otherwise, $f^*_\eta(ab) = 0$, satisfies the condition $|e_{f^*_\eta}(0) - e_{f^*_\eta}(1)| \le 1$, where $e_{f^*_\eta}(i)$ is the number of edges with label i $(i = 0, 1)$. In this paper, we explore the arithmetic cordial labeling of some graphs under conditions imposed on the function ζ_η. The graphs included are star graphs, ladder graphs, alternate cycle snake graphs, join graphs, corona graphs, and tensor product graphs.

Keywords: binary function, graph, arithmetic cordial labeling, arithmetic structure

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1. Introduction

A simple graph $G = (V, E)$ is an ordered pair, where $V = V(G)$ is called the *vertex set* and $E = E(G)$ is called the *edge set*. The elements of E are unordered pairs of distinct elements of V. The elements of V and E are called *vertices* and *edges*, respectively. If G has n vertices and m edges, then G has *order* n and *size* m. Let $v \in V(G)$. Then the *degree* of v, denoted by $\deg(v)$, is the number of vertices adjacent to v. If $\deg(v) = 1$, then v is called a *pendant vertex* of G.

Graph labeling is a well-known concept in graph theory that studies the properties of assigning integers to the vertices and/or edges of a graph under specific conditions [1]. In 1987, I. Cahit [2] introduced the concept of *cordial labeling*. This graph labeling assigns integers 0 and 1 to the vertices of a graph, where the label of each edge is the absolute difference of the labels of its endpoints. If the numbers of vertices labeled 0 and 1 differ by at most 1, and the numbers of edges labeled 0 and 1 also differ by at most 1, then the graph is said to admit a cordial labeling. This concept inspired the introduction of several variants of cordial labeling, including Legendre cordial labeling [3], Euler cordial labeling [4], Legendre product cordial labeling [5], logarithmic cordial labeling [6], and (a, b)-Fibonacci-Legendre cordial labeling [7]. These variants use different concepts from number theory, such as the Legendre symbol, Euler's Theorem, discrete logarithms (indices), and the (a, b)-Fibonacci sequence. In this paper, we define a much more general concept that encompasses the above-mentioned variants of cordial labeling, which we call *arithmetic cordial labeling*.

Let η be a fixed positive integer. In addition, let S be a subset of $\mathbb{Z}$, $\star : S \times S \to \mathbb{Z}$ be a binary function, and $\zeta_\eta : \{\xi \in \mathbb{Z} : \gcd(\xi, \eta) = 1\} \to \{0, 1\}$ be a function. We call the triple $\langle S, \zeta_\eta, \star\rangle$ an *arithmetic structure*. For a simple connected graph G of order n, a bijective function $f : V(G) \to S$ (where $|S| = n$) is called an *arithmetic cordial labeling modulo η under $\langle S, \zeta_\eta, \star\rangle$* if the induced function $f^*_\eta : E(G) \to \{0, 1\}$, defined by

$$f^*_\eta(ab) = \begin{cases} 1 & \text{if } \gcd(f(a) \star f(b), \eta) = 1 \text{ and } \zeta_\eta(f(a) \star f(b)) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the condition $|e_{f^*_\eta}(0) - e_{f^*_\eta}(1)| \le 1$, where $e_{f^*_\eta}(i)$ denotes the number of edges labeled i $(i = 0, 1)$. A graph that admits this labeling is called an *arithmetic cordial graph modulo η under $\langle S, \zeta_\eta, \star\rangle$*.

While the proposed definition for arithmetic cordial labeling is inspired by the classical cordial labeling, it is important to note a key distinction. In classical setting, both vertex and edge labels must satisfy the balance condition $|v(0) - v(1)| \leq 1$ and $|e(0) - e(1)| \leq 1$. In this framework, the mapping $f : V(G) \to S$ is bijective, rather than binary mapping (i.e., $f : V(G) \to \{0,1\}$). Consequently, the "cordial" nature of the labeling is preserved specifically through the equitability of the induced edge label f^*_η by applying the condition $|e_{f^*_\eta}(0) - e_{f^*_\eta}(1)| \leq 1$. This distinction allows for a richer integration of number-theoretic properties while maintaining the core cordial principle of edge-label balance.

Let G be a simple connected graph of order n. Suppose that (a/p) is the Legendre symbol of a over an odd prime p (see Definition 2.8 in [3]), $\text{ind}_{\varpi,\eta}(a)$ is the discrete logarithm of a to the base ϖ modulo η (where ϖ is a fixed primitive root of η; see Definition 2.4 in [6]), and F_i is the ith (a,b)-Fibonacci number (see [7]). Also, consider the expression $\frac{a^{\phi(\eta)}-1}{\eta}$ (from Euler's Theorem; see Theorem 2.12 in [4]), where η is an odd positive integer with $\eta \geq 3$ and $\phi(\eta)$ is the Euler phi-function. Thus, the following are arithmetic cordial labeling with their corresponding arithmetic structures $\langle S, \zeta_\eta, \star \rangle$.

• Legendre cordial labeling modulo p:

$$S = \{1, 2, \ldots, n\}; \quad \zeta_p(a) = \frac{1 + (a/p)}{2}; \quad x \star y = x + y.$$

•Euler cordial labeling modulo η:

$$S = \{1, 2, \ldots, n\}, \quad \zeta_\eta(a) = \frac{a^{\phi(\eta)} - 1}{\eta}, \quad x \star y = x + y.$$

•Legendre product cordial labeling modulo p:

$$S = \{1, 2, \ldots, n\}; \quad \zeta_p(a) = \frac{1 + (a/p)}{2}; \quad x \star y = xy.$$

•Logarithmic cordial labeling modulo η:

$$S = \{1, 2, \ldots, n\}; \quad \text{ind}_{\varpi,\eta}(a) \equiv \zeta_\eta(a) \pmod 2; \quad x \star y = x + y.$$

•(a,b)-Fibonacci–Legendre cordial labeling modulo p:

$$S = \{0, 1, \ldots, n-1\}; \quad \zeta_p(a) = \frac{1 + (a/p)}{2}; \quad x \star y = F_x + F_y.$$

Now, suppose that the following properties hold for ζ_p where p is an odd prime:

Property 1. Let $\gcd(\theta_1, p) = \gcd(\theta_2, p) = 1$. If $\theta_1 \equiv \theta_2 \pmod p$, then $\zeta_p(\theta_1) = \zeta_p(\theta_2)$.

Property 2. If $A_i = \{a : \zeta_p(a) = i,\ 1 \leq a \leq p-1\}$ for $i = 0, 1$, then $|A_0| = |A_1|$. Hence, $|A_0| = |A_1| = \frac{p-1}{2}$.

Property 3. Let $\gcd(\theta, p) = 1$. Assume that $\chi_p : \{\xi \in \mathbb{Z} : \gcd(\xi, p) = 1\} \to \{-1, 1\}$ is a function (e.g., the Legendre symbol) and suppose that

$$\zeta_p(\theta) = \frac{1 + \chi_p(\theta)}{2}.$$

Thus, $\chi_p(\theta_1\theta_2) = \chi_p(\theta_1)\chi_p(\theta_2)$.

For Legendre cordial labeling and Legendre product cordial labeling, observe that if

$$\zeta^1_p(a) = \frac{1 + \chi_p(a)}{2}, \tag{1}$$

where $\chi_p(a) = (a/p)$, then Properties 1, 2, and 3 hold for ζ^1_p (see Theorem 2.9 in [4] and Theorem 2.10 in [5]). Additionally, for logarithmic cordial labeling, notice that if

$$\text{ind}_{\varpi_1,p}(a) \equiv \zeta^2_p(a) \pmod 2, \tag{2}$$

where ϖ_1 is a fixed primitive root of p, then ζ_p^2 satisfies Properties 1 and 2 (see Remarks 2 and 3 in [6] with $\eta = p$). Furthermore, let

$$\zeta_p^3(a) = \frac{1 + \chi_p(a)}{2}, \tag{3}$$

with $\chi_p(a) = (-1)^{\text{ind}_{\varpi_2,p}(a)}$, where ϖ_2 is a fixed primitive root of p. If $a_1 \equiv a_2 \pmod{p}$ with $\gcd(a_1, p) = \gcd(a_2, p) = 1$, then by Remark 2 in [6], we have $\text{ind}_{\varpi_2,p}(a_1) = \text{ind}_{\varpi_2,p}(a_2)$, which means $(-1)^{\text{ind}_{\varpi_2,p}(a_1)} = (-1)^{\text{ind}_{\varpi_2,p}(a_2)}$. Obviously, $\zeta_p^3(a_1) = \zeta_p^3(a_2)$, and so Property 1 holds. Now, assume that

$$A_i^2 = \{a : \zeta_p^2(a) = i,\ 1 \le a \le p-1\}, \text{ and}$$

$$A_i^3 = \{a : \zeta_p^3(a) = i,\ 1 \le a \le p-1\},$$

for $i = 0, 1$ with $\varpi_1 = \varpi_2$, where ζ_p^2 is defined in (2). If $a \in A_0^2$, then $\text{ind}_{\varpi_2,p}(a) \equiv 0 \pmod{2}$, and so $\chi_p(a) = 1$. Hence, $a \in A_0^3$. Conversely, if $a \in A_0^3$, then

$$\begin{aligned} 1 &= \frac{1 + \chi_p(a)}{2} && \text{(by (3))} \\ 2 &= 1 + \chi_p(a) \\ \chi_p(a) &= 1 \\ (-1)^{\text{ind}_{\varpi_2,p}(a)} &= 1. \end{aligned}$$

Clearly, $\text{ind}_{\varpi_2,p}(a)$ is even. Therefore, $a \in A_0^2$. Thus, $A_i^2 = A_i^3$ for $i = 0, 1$, and it follows that ζ_p^3 satisfies Property 2 since Property 2 holds for ζ_p^2. Lastly, assume that $\gcd(a_1, p) = \gcd(a_2, p) = 1$. Using Theorem 9.16 (ii) in [8] (page 369) with $m = p$ and $r = \varpi_2$,

$$\text{ind}_{\varpi_2,p}(a_1 a_2) \equiv \text{ind}_{\varpi_2,p}(a_1) + \text{ind}_{\varpi_2,p}(a_2) \pmod{(p-1)}.$$

Since $p - 1$ is even, we have

$$\text{ind}_{\varpi_2,p}(a_1 a_2) \equiv \text{ind}_{\varpi_2,p}(a_1) + \text{ind}_{\varpi_2,p}(a_2) \pmod{2},$$

which means that $\text{ind}_{\varpi_2,p}(a_1 a_2)$ and $\text{ind}_{\varpi_2,p}(a_1) + \text{ind}_{\varpi_2,p}(a_2)$ have the same parity. Consequently,

$$\begin{aligned} \chi_p(a_1 a_2) &= (-1)^{\text{ind}_{\varpi_2,p}(a_1 a_2)} \\ &= (-1)^{\text{ind}_{\varpi_2,p}(a_1) + \text{ind}_{\varpi_2,p}(a_2)} \\ &= (-1)^{\text{ind}_{\varpi_2,p}(a_1)} (-1)^{\text{ind}_{\varpi_2,p}(a_2)} \\ &= \chi_p(a_1) \chi_p(a_2). \end{aligned}$$

Hence, Property 3 holds. Additionally, the function

$$\zeta_p^4(a) = \frac{1 - \chi(a)}{2}, \tag{4}$$

where $\chi_p(a) = (a/p)$, also satisfies Properties 1, 2, and 3 with the same explanation as ζ_p^1 defined in (1).

As we observed, several functions satisfy Properties 1, 2, and/or 3. Motivated by this, this paper explores the arithmetic cordial labeling of some graphs. In §4 and §5, we assume that $\eta = p$ is an odd prime.

2. Basic concepts

Definition 1. A *path graph* P_n of order n is obtained from vertices $v_1, v_2, \ldots, v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. Also, a *cycle graph* C_n of order n is created from a path graph P_n such that two pendant vertices of P_n are adjacent.

Definition 2. A *star graph* Star_n of order n is obtained from vertices $x_0, v_1, v_2, \ldots, v_{n-1}$ such that x_0 is adjacent to v_i for $i = 1, 2, \ldots, n-1$. The vertex x_0 is called the *central vertex* of Star_n.

Definition 3. A *complete graph* K_n of order n is a graph where two distinct vertices are adjacent. In addition, an *empty graph* $\overline{K}_n$ of order n is a graph with no edges.

Definition 4. A *ladder graph* L_n of order $2n$ is obtained from two path graphs P_n^1 and P_n^2 such that the ith vertex of P_n^1 is adjacent to the ith vertex of P_n^2.

Definition 5. A *kayak paddle graph* $KP_{n,m,k}$ of order $n + m + k$ is created from two cycle graphs C_n and C_k, and path graph P_{m+2}, for which a pendant vertex of P_{m+2} is a vertex of C_n and the other pendant vertex is a vertex of C_k.

Definition 6. An *alternate cycle snake graph* $A_n(C_m)$ of order nm is obtained from cycle graphs $C_m^1, C_m^2, \ldots, C_m^n$, with $V(C_m^i) = \{v_1^i, v_2^i, \ldots, v_m^i\}$ for $i = 1, 2, \ldots, n$, for which v_m^j is adjacent to v_1^{j+1} for $j = 1, 2, \ldots, n-1$.

Definition 7. A graph G is called a *bipartite graph* if its vertex set $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 such that the edges of G have one end in V_1 and one end in V_2. The sets V_1 and V_2 are called *partite sets* of G.

Definition 8. Let G and H be graphs.

i. A *join graph* $G + H$ is a graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{ab : a \in V(G) \text{ and } b \in V(H)\}$.
ii. A *corona graph* $G \circ H$ is obtained from $|V(G)|$ copies of H and one copy of G such that every vertex of the ith copy of H is adjacent to the ith vertex of G.
iii. A *tensor product graph* $G \times H$ is a graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(a_1, b_1)(a_2, b_2) : a_1a_2 \in E(G) \text{ and } b_1b_2 \in E(H)\}$.

Definition 9. An arithmetic structure $\langle S_1, \zeta_{\eta_1}^1, \star_1 \rangle$ is *equivalent* to another arithmetic structure $\langle S_2, \zeta_{\eta_2}^2, \star_2 \rangle$, denoted by $\langle S_1, \zeta_{\eta_1}^1, \star_1 \rangle \cong \langle S_2, \zeta_{\eta_2}^2, \star_2 \rangle$, if there exists a bijective function $\psi : S_1 \to S_2$ such that for all $a, b \in S_1$, $a \neq b$,

i. $\gcd(a \star_1 b, \eta_1) = 1$ if and only if $\gcd(\psi(a) \star_2 \psi(b), \eta_2) = 1$, and
ii. $\zeta_{\eta_1}^1(a \star_1 b) = \zeta_{\eta_2}^2(\psi(a) \star_2 \psi(b))$ whenever $\gcd(a \star_1 b, \eta_1) = 1$.

3. General results

Theorem 1. *Let G be a simple connected graph and let η_1 and η_2 be fixed integers. Assume that $\langle S_1, \zeta_{\eta_1}^1, \star_1 \rangle \cong \langle S_2, \zeta_{\eta_2}^2, \star_2 \rangle$. Then G is an arithmetic cordial graph modulo η_1 under $\langle S_1, \zeta_{\eta_1}^1, \star_1 \rangle$ if and only if it is an arithmetic cordial graph modulo η_2 under $\langle S_2, \zeta_{\eta_2}^2, \star_2 \rangle$.*

Proof. Suppose that G is an arithmetic cordial graph modulo η_1 under $\langle S_1, \zeta_{\eta_1}^1, \star_1 \rangle$. So, there is a bijective function $f : V(G) \to S_1$ such that the induced edge label

$$f_{\eta_1}^*(ab) = \begin{cases} 1 & \text{if } \gcd(f(a) \star_1 f(b), \eta_1) = 1 \text{ and } \zeta_{\eta_1}^1(f(a) \star_1 f(b)) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the condition $|e_{f_{\eta_1}^*}(0) - e_{f_{\eta_1}^*}(1)| \leq 1$. Since $\langle S_1, \zeta_{\eta_1}^1, \star_1 \rangle \cong \langle S_2, \zeta_{\eta_2}^2, \star_2 \rangle$, there exists a bijective function $\psi : S_1 \to S_2$ such that for all $x, y \in S_1$,

i. $\gcd(x \star_1 y, \eta_1) = 1$ if and only if $\gcd(\psi(x) \star_2 \psi(y), \eta_2) = 1$, and
ii. $\zeta_{\eta_1}^1(x \star_1 y) = \zeta_{\eta_2}^2(\psi(x) \star_2 \psi(y))$ whenever $\gcd(x \star_1 y, \eta_1) = 1$.

Define a function $g : V(G) \to S_2$ by

$$g(v) = \psi(f(v)), \tag{5}$$

for all $v \in V(G)$. Obviously, g is a bijective function. Observe that the induced edge label for g is

$$g^*_{\eta_2}(ab) = \begin{cases} 1 & \text{if } \gcd(g(a) \star_2 g(b), \eta_2) = 1 \text{ and } \zeta^2_{\eta_2}(g(a) \star_2 g(b)) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Case 1. Suppose that $\gcd(f(a) \star_1 f(b), \eta_1) \neq 1$. By (i) and equation (5), it is obvious that $\gcd(f(a) \star_1 f(b), \eta_1) \neq 1$ if and only if $\gcd(g(a) \star_2 g(b), \eta_2) \neq 1$. So, $f^*_{\eta_1}(ab) = 0 = g^*_{\eta_2}(ab)$.

Case 2. Let $\gcd(f(a) \star_1 f(b), \eta_1) = 1$. Using Case 1, we have $\gcd(g(a) \star_2 g(b), \eta_2) = 1$. By (ii) and equation (5), observe that

$$\zeta^2_{\eta_2}(g(a) \star_2 g(b)) = \zeta^2_{\eta_2}(\psi(f(a)) \star_2 \psi(f(b))) = \zeta^1_{\eta_1}(f(a) \star_1 f(b)).$$

Hence, $f^*_{\eta_1}(ab) = g^*_{\eta_2}(ab)$.

Combining Cases 1 and 2, we have $g^*_{\eta_2}(ab) = f^*_{\eta_1}(ab)$ for all $ab \in E(G)$. Consequently, $e_{f^*_{\eta_1}}(i) = e_{g^*_{\eta_2}}(i)$ for $i = 0, 1$, and it follows that $|e_{g^*_{\eta_2}}(0) - e_{g^*_{\eta_2}}(1)| \leq 1$. Therefore, G is an arithmetic cordial graph modulo η_2 under $\langle S_2, \zeta^2_{\eta_2}, \star_2 \rangle$.

For the converse, by using the inverse of ψ, the proof is analogous. □

Let $\eta \geq 3$ be a positive integer and let $\varphi(\eta) = \{\xi \in \mathbb{Z} : \gcd(\xi, \eta) = 1,\ 1 \leq \xi < \eta\}$. In addition, suppose that $\phi(\eta)$ is the Euler phi-function of η. Clearly, $|\varphi(\eta)| = \phi(\eta)$. Consider the following properties:

Property 4. (General version of Property 1) Assume that $\gcd(\theta_1, \eta) = \gcd(\theta_2, \eta) = 1$. If $\theta_1 \equiv \theta_2 \pmod{\eta}$, then $\zeta_\eta(\theta_1) = \zeta_\eta(\theta_2)$.

Property 5. (General version of Property 2) If $A_i = \{a : \zeta_\eta(a) = i,\ a \in \varphi(\eta)\}$ for $i = 0, 1$, then $|A_0| = |A_1|$. So, $|A_0| = |A_1| = \frac{\phi(\eta)}{2}$.

Property 6. (General version of Property 3) Suppose that $\gcd(\theta, \eta) = 1$. Assume that $\chi_\eta : \{\xi \in \mathbb{Z} : \gcd(\xi, \eta) = 1\} \to \{-1, 1\}$ is a function and let

$$\zeta_\eta(\theta) = \frac{1 + \chi_\eta(\theta)}{2}.$$

Thus, $\chi_\eta(\theta_1\theta_2) = \chi_\eta(\theta_1)\chi_\eta(\theta_2)$.

Theorem 2. *Let $\eta \geq 3$ be a positive integer and let Properties 4, 5, and 6 hold for ζ^j_η, for $j = 1, 2$. Let $S = \{1, 2, \ldots, \eta m - 1\}$ where $m \geq 1$ is an integer. Then $\langle S, \zeta^1_\eta, \cdot \rangle \cong \langle S, \zeta^2_\eta, \cdot \rangle$.*

Proof. Let $A^j_i = \{a : \zeta^j_\eta(a) = i,\ a \in \varphi(\eta)\}$ for $i = 0, 1$ and $j = 1, 2$. Let $T = \{1, 2, \ldots, \eta\}$ and, by Property 5, suppose that

$$A^1_0 = \{q_i : 1 \leq i \leq \tfrac{\phi(\eta)}{2}\}, \quad A^1_1 = \{r_i : 1 \leq i \leq \tfrac{\phi(\eta)}{2}\},$$
$$A^2_0 = \{s_i : 1 \leq i \leq \tfrac{\phi(\eta)}{2}\}, \quad A^2_1 = \{t_i : 1 \leq i \leq \tfrac{\phi(\eta)}{2}\}.$$

Moreover, let

$$T - (A^1_0 \cup A^1_1) = \{w_i : 1 \leq i \leq \eta - 1 - \tfrac{\phi(\eta)}{2}\} \cup \{\eta\},$$
$$T - (A^2_0 \cup A^2_1) = \{x_i : 1 \leq i \leq \eta - 1 - \tfrac{\phi(\eta)}{2}\} \cup \{\eta\}.$$

Observe that S can be partitioned into m blocks:

$$\{i + (k-1)\eta : 1 \leq i \leq \eta\} \text{ for } k = 1, 2, \ldots, m-1, \quad \{i + (m-1)\eta : 1 \leq i \leq \eta - 1\}. \tag{6}$$

Thus, for any $s \in S$, s can be uniquely represented as $s = z + (k-1)\eta$ for some $z \in T$ and some values of k.

Define a function $\sigma : T \to T$ as follows:

$$\sigma(q_i) = s_i \text{ for } i = 1, 2, \ldots, \frac{\phi(\eta)}{2}$$
$$\sigma(r_i) = t_i \text{ for } i = 1, 2, \ldots, \frac{\phi(\eta)}{2}$$
$$\sigma(w_i) = x_i \text{ for } i = 1, 2, \ldots, \eta - 1 - \frac{\phi(\eta)}{2}$$
$$\sigma(\eta) = \eta.$$

Note that $A_0^1, A_1^1, T - (A_0^1 \cup A_1^1)$ form a partition on T. In addition, $A_0^2, A_1^2, T - (A_0^2 \cup A_1^2)$ also form a partition on T. Hence, σ is a bijective function. Moreover, observe that the elements of A_0^1 and A_1^1 are precisely map to the elements of A_0^2 and A_1^2, respectively. As a consequence, $z \in A_0^1 \cup A_1^1$ if and only if $\sigma(z) \in A_0^2 \cup A_1^2$ because all elements of A_i^j ($i = 0, 1$ and $j = 1, 2$) are relatively prime to η.

Let $\psi : S \to S$ be a function defined as follows:

For the first $m - 1$ blocks:

$$\psi(z + (k-1)\eta) = \sigma(z) + (k-1)\eta \text{ for } k = 1, 2, \ldots, m-1 \text{ and } z \in T;$$

For the last block:

$$\psi(z + (m-1)\eta) = \sigma(z) + (m-1)\eta \text{ for } z \in T - \{\eta\}. \tag{7}$$

Since σ is a bijective function from T to T and the m blocks indicated in (6) form a partition on S, it follows that ψ is a bijective function.

Case 1. Suppose that $a, b \in S$ with $a = z_1 + (k_1 - 1)\eta$ and $b = z_2 + (k_2 - 1)\eta$ where $z_1, z_2 \in A_0^1 \cup A_1^1 \cup [T - (A_0^1 \cup A_1^1)]$, for some values of k_1 and k_2. Let $\gcd(ab, \eta) = 1$. Thus,

$$\gcd([z_1 + (k_1-1)\eta][z_2 + (k_2-1)\eta], \eta) = \gcd(z_1 z_2, \eta) = 1.$$

As a result $z_1, z_2 \in A_0^1 \cup A_1^1$. So, $\sigma(z_1), \sigma(z_2) \in A_0^2 \cup A_1^2$ and it follows that

$$\gcd(\psi(a)\psi(b), \eta) = \gcd([\psi(z_1 + (k_1-1)\eta)][\psi(z_2 + (k_2-1)\eta)], \eta) = \gcd(\sigma(z_1)\sigma(z_2), \eta) = 1.$$

The converse follows similarly from the fact that the inverse function σ^{-1} maps from $A_0^2 \cup A_1^2$ to $A_0^1 \cup A_1^2$. Therefore, Definition 9 (i) holds.

Case 2. Let $\chi_\eta^j : \{\xi \in \mathbb{Z} : \gcd(\xi, \eta) = 1\} \to \{-1, 1\}$ and suppose that

$$\zeta_\eta^j(a) = \frac{1 + \chi_\eta^j(a)}{2},$$

for $j = 1, 2$. Suppose that $a, b \in S$ with $\gcd(ab, \eta) = 1$. Thus, $\gcd(\psi(a)\psi(b), \eta) = 1$ by Case 1. Note that $a = z_1 + (k_1 - 1)\eta$ and $b = z_2 + (k_2 - 1)\eta$ for some $z_1, z_2 \in A_0^1 \cup A_1^1$ and $k_1, k_2 \in \{1, 2, \ldots, m-1\}$. By Properties 4 and 6, we have

$$\zeta_\eta^1(ab) = \zeta_\eta^1(z_1 z_2) = \frac{1 + \chi_\eta^1(z_1)\chi_\eta^1(z_2)}{2},$$
$$\zeta_\eta^2(\psi(a)\psi(b)) = \zeta_\eta^2(\psi(z_1)\psi(z_2)) = \frac{1 + \chi_\eta^2(\sigma(z_1))\chi_\eta^2(\sigma(z_2))}{2}.$$

By the definition of σ, observe that $z \in A_i^1$ if and only if $\sigma(z) \in A_i^2$ for $i = 0, 1$. Thus,

$$\chi_\eta^1(z_j) = \chi_\eta^2(\sigma(z_j)) \text{ for } j = 1, 2.$$

Therefore, $\zeta_\eta^1(ab) = \zeta_\eta^2(\psi(a)\psi(b))$ and it follows that Definition 9 (ii) holds. □

The proof works identically if we define $S = \{1, 2, \ldots, \eta m\}$; in that case, one only needs to remove the condition $z \in T - \{\eta\}$ in equation (7) and replace it with $z \in T$. Combining this with Theorems 1 and 2, we obtain the following corollary.

Corollary 1. *Let $\eta \geq 3$ be an integer and let G be a graph order $\eta m + \varepsilon$, where m is a positive integer and $\varepsilon \in \{-1, 0\}$. Suppose that Properties 4, 5, and 6 hold for ζ_η^j, for $j = 1, 2$. Then G is an arithmetic cordial graph modulo η under $\langle S, \zeta_\eta^1, \cdot \rangle$ if and only if it is an arithmetic cordial graph modulo η under $\langle S, \zeta_\eta^2, \cdot \rangle$, where $S = \{1, 2, \ldots, \eta m + \varepsilon\}$.*

Using equations (1), (3), and (4) (where ζ_p^1, ζ_p^3, and ζ_p^4 satisfy Properties 1, 2, and 3), Corollary 1 shows that if G is a simple connected graph of order $mp + \varepsilon$, where p is an odd prime, m is a positive integer, and $\varepsilon \in \{-1, 0\}$, then the following statements are equivalent:

i. G is a Legendre product cordial graph modulo p;
ii. G is an arithmetic cordial graph modulo p under $\langle S, \zeta_p^3, \cdot \rangle$;
iii. G is an arithmetic cordial graph modulo p under $\langle S, \zeta_p^4, \cdot \rangle$.

Theorem 3. *(General version of Theorem 4.3 in [6]) Let $\eta \geq 3$ be an integer and let G be a graph of order n and size $m\phi(\eta)$, where m is a positive integer. Also, suppose that Properties 4 and 5 hold for ζ_η. Additionally, let $f : V(G) \to \{1, 2, \ldots, n\}$ be a bijective function and assume that $h(uv) = f(u) \star f(v)$ for every $uv \in E(G)$. If f satisfies the following conditions:*

i.
$$\{h(uv) : uv \in E(G)\} = \bigcup_{i \in T} \{j + i\eta : j \in \varphi(\eta)\},$$
where $\varphi(\eta) = \{\xi : \gcd(\xi, \eta) = 1,\ 0 < \xi < \eta\}$ and $T \subseteq \mathbb{Z}$ with $|T| = m$,
ii. *$h(uv) \neq h(ab)$ for every $uv \neq ab$,*

then G is an arithmetic cordial graph modulo η under $\langle S, \zeta_\eta, \star \rangle$, where $S \subseteq \mathbb{Z}$ with $|S| = n$.

Proof. Suppose that $\vartheta_i = \{j + i\eta : j \in \varphi(\eta)\}$ for $i \in T$. In addition, assume that $\vartheta_i \pmod{\eta} = \{\xi : a \equiv \xi \pmod{\eta},\ a \in \vartheta_i,\ 0 < \xi < \eta\}$ for $i \in T$. By Property 4, we have

$$\vartheta_i \pmod{\eta} = \varphi(\eta),$$

for $i \in T$. By conditions (i) and (ii), together with Property 5, we have

$$e_{f_\eta^*}(0) = e_{f_\eta^*}(1) = \frac{m\phi(\eta)}{2},$$

and so $|e_{f_\eta^*}(0) - e_{f_\eta^*}(1)| = 0$. Therefore, G is an arithmetic cordial graph modulo η under $\langle S, \zeta_\eta, \star \rangle$, where $S \subseteq \mathbb{Z}$ with $|S| = n$. □

Consider the Jacobi symbol (a/n), which is defined as follows. Let $n \geq 1$ be an odd integer and let a be an integer relatively prime to n. Then the Jacobi symbol is

$$(a/n) = \prod_{i=1}^{t} (a/p_i)^{e_i},$$

where $n = \prod_{i=1}^{t} p_i^{e_i}$ is the prime factorization of n, and the symbols on the right-hand side are Legendre symbols. This symbol generalizes the Legendre symbol by incorporating positive odd integers. Assume that

$$\zeta_\eta^5(a) = \frac{1 + (a/\eta)}{2}, \tag{8}$$

where (a/η) is a Jacobi symbol. It is evident that ζ_η^5 satisfies Properties 4 and 5 whenever η is not a perfect square (see Theorem 11.10 (i) in [8] (page 444) and Proposition 11 in [9]). In fact, the function ζ_p^1 defined in (1) is a special case of ζ_η^5 when $\eta = p$, where p is an odd prime. As an application of Theorem 3, we obtain the following result.

Theorem 4. *Let $\eta \geq 3$ be an integer that is not a perfect square, and suppose that c and m are positive integers. The graph $Star_{m\phi(\eta)+1}$ is an arithmetic cordial graph modulo η under $\langle S, \zeta_\eta^5, +\rangle$, where $S = \{c\eta\} \cup \left(\bigcup_{j=0}^{m-1}\{i + j\eta : i \in \varphi(\eta)\}\right)$ and the function ζ_η^5 is defined in (8).*

Proof. Note that the size of $\text{Star}_{m\phi(\eta)+1}$ is $m\phi(\eta)$. Let $V(\text{Star}_{m\phi(\eta)+1}) = \{v_0\} \cup \{v_i : i \in W\}$, where $W = \bigcup_{j=0}^{m-1}\{i + j\eta : i \in \varphi(\eta)\}$ and v_0 is the central vertex. Define a function $f : V(\text{Star}_{m\phi(\eta)+1}) \to S$ by

$$f(v_0) = c\eta,$$
$$f(v_i) = i \text{ for } i \in W,$$

where $S = \{c\eta\} \cup \left(\bigcup_{j=0}^{m-1}\{i + j\eta : i \in \varphi(\eta)\}\right)$. Thus, f is a bijective function.

For the edges, we have

$$f(v_0) + f(v_i) = c\eta + i + j\eta = i + (c + j)\eta,$$

for $i \in \varphi(\eta)$ and $j = 0, 1, \ldots, m - 1$. Let $T = \{c + j : j = 0, 1, \ldots, m - 1\}$. Hence,

$$\{f(v_0) + f(v_i) : i \in W\} = \bigcup_{k \in T}\{i + k\eta : i \in \varphi(\eta)\}.$$

It is clear that conditions (i) and (ii) of Theorem 3 are satisfied. Thus, $\text{Star}_{m\phi(\eta)+1}$ is an arithmetic cordial graph modulo η under $\langle S, \zeta_\eta^5, +\rangle$. □

4. Arithmetic Cordial labeling modulo p under $\langle S, \zeta_p, +\rangle$

In this section, we assume that Properties 1 and 2 hold for ζ_p.

Theorem 5. *The ladder graph L_p is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, +\rangle$ where $S = \{1, 2, \ldots, 2p\}$, for all $p \geq 3$.*

Proof. Assume that P_p^1 and P_p^2 are path graphs of L_p with $V(P_p^j) = \{v_1^j, v_2^j, \ldots, v_p^j\}$ for $j = 1, 2$. Let $f : V(G) \to S$ be a function defined by

$$f(v_i^j) = \begin{cases} i + \frac{p-1}{2} + (j-1)p & \text{for } i = 1, 2, \ldots, \frac{p+1}{2}, \\ i - \frac{p+1}{2} + (j-1)p & \text{for } i = \frac{p+3}{2}, \frac{p+5}{2}, \ldots, p, \end{cases}$$

for $j = 1, 2$, where $S = \{1, 2, \ldots, 2p\}$. Clearly, f is a bijective function.

For the edges of P_p^j,

$$f(v_i^j) + f(v_{i+1}^j) \equiv \begin{cases} 2i \pmod{p} & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ 2i - p \pmod{p} & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p - 1 \end{cases}$$

for $j = 1, 2$, and for the remaining edges,

$$f(v_i^1) + f(v_i^2) \equiv \begin{cases} 2i - 1 \pmod{p} & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ 0 \pmod{p} & \text{for } i = \frac{p+1}{2}, \\ 2i - p - 1 \pmod{p} & \text{for } i = \frac{p+3}{2}, \frac{p+5}{2}, \ldots, p. \end{cases}$$

In view of this labeling, let

$$\alpha_j = \bigcup_{i=1}^{p-1}\{\xi : f(v_i^j) + f(v_{i+1}^j) \equiv \xi \pmod{p}, 0 < \xi < p\},$$

for $j = 1, 2$, and let

$$\beta = \bigcup_{\substack{i=1 \\ i \neq (p+1)/2}}^{p} \{\xi : f(v_i^1) + f(v_i^2) \equiv \xi \pmod{p},\ 0 < \xi < p\}.$$

Evidently, $\alpha_1 = \alpha_2 = \beta = \{1, 2, \dots, p-1\}$. By Property 2, together with the fact that $f_p^*(v^1_{(p+1)/2} v^2_{(p+1)/2}) = 0$, we have

$$e_{f_p^*}(0) = 3\left(\frac{p-1}{2}\right) + 1 \text{ and } e_{f_p^*}(1) = 3\left(\frac{p-1}{2}\right).$$

Therefore, $|e_{f_p^*}(0) - e_{f_p^*}(1)| = 1$ and it follows that L_p is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, + \rangle$. □

Theorem 6. *The alternate cycle snake graph $A_n(C_p)$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, + \rangle$ where $\zeta_p(1) = 1$ and $S = \{1, 2, \dots, np\}$, for all $n \geq 2$ and $p \geq 3$.*

Proof. Assume that $C_p^1, C_p^2, \dots, C_p^n$ are cycle graphs of $A_n(C_p)$ with $V(C_p^j) = \{v_1^j, v_2^j, \dots, v_p^j\}$ for $j = 1, 2, \dots, n$. Now, define a function $f : V(A_n(C_p)) \to S$ by

$$f(v_i^j) = i + (j-1)p,$$

for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, n$, where $S = \{1, 2, \dots, np\}$. Hence, f is a bijective function.

For the edges of C_p^j, we have

$$f(v_i^j) + f(v_{i+1}^j) \equiv \begin{cases} 2i+1 \pmod{p} & \text{for } i = 1, 2, \dots, \frac{p-3}{2}, \\ 0 \pmod{p} & \text{for } i = \frac{p-1}{2}, \\ 2i - p + 1 \pmod{p} & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1, \end{cases}$$

and $f(v_1^j) + f(v_p^j) \equiv 1 \pmod{p}$, for $j = 1, 2, \dots, n$. Additionally, for the remaining edges,

$$f(v_p^k) + f(v_1^{k+1}) \equiv 1 \pmod{p},$$

for $k = 1, 2, \dots, n-1$.

In view of the above labeling, for each $j = 1, 2, \dots, n$, assume that

$$\alpha_j = \bigcup_{\substack{i=1 \\ i \neq (p-1)/2}}^{p-1} \{\xi : f(v_i^j) + f(v_{i+1}^j) \equiv \xi \pmod{p},\ 0 < \xi < p\},$$

and

$$\beta_j = \{\xi : f(v_1^j) + f(v_p^j) \equiv \xi \pmod{p},\ 0 < \xi < p\}.$$

Consequently, $\alpha_j \cup \beta_j = \{1, 2, \dots, p-1\}$ for $j = 1, 2, \dots, n$. By Property 2, there are $\frac{p-1}{2}$ edges of C_p^j with label 0 and $\frac{p-1}{2}$ edges with label 1, under f_p^*, for $j = 1, 2, \dots, n$. Moreover, observe that $f_p^*(v^j_{(p-1)/2} v^j_{(p+1)/2}) = 0$ for $j = 1, 2, \dots, n$, and $f_p^*(v_p^k v_1^{k+1}) = 1$ for $k = 1, 2, \dots, n-1$ (since $\zeta_p(1) = 1$). Therefore,

$$e_{f_p^*}(0) = n\left(\frac{p-1}{2}\right) + n \text{ and } e_{f_p^*}(1) = n\left(\frac{p-1}{2}\right) + n - 1.$$

So, $|e_{f_p^*}(0) - e_{f_p^*}(1)| = 1$. Hence, $A_n(C_p)$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, + \rangle$. □

Theorem 7. *Let G be a connected graph of order n and assume that $\varepsilon \in \{-1,0,1\}$. If G has size $n+\varepsilon$, then $G \circ P_{p-1}$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, +\rangle$, where $\zeta_p(1)=1$, $\zeta_p(2)=0$, and $S=\{1,2,\ldots,np\}$, for all $n \geq 2$ and $p \geq 3$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and assume that P^i_{p-1} is the ith copy of P_{p-1} with $V(P^i_{p-1}) = \{u^i_1, u^i_2, \ldots, u^i_{p-1}\}$ such that each vertex of P^i_{p-1} is adjacent to v_i, for $i=1,2,\ldots,n$. Now, let $f: V(G \circ P_{p-1}) \to S$ be a function defined by

$$f(u^i_j) = \begin{cases} j + \frac{p+1}{2} + (i-1)p & \text{for } j = 1,2,\ldots,\frac{p-1}{2}, \\ j - \frac{p-1}{2} + (i-1)p & \text{for } j = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1, \end{cases}$$

$$f(v_i) = \frac{p+1}{2} + (i-1)p,$$

for $i=1,2,\ldots,n$, where $S=\{1,2,\ldots,np\}$. So, f is a bijective function.

For the edges of P^i_{p-1}, we have

$$f(u^i_j) + f(u^i_{j+1}) \equiv \begin{cases} 2j+2 \pmod p & \text{for } j = 1,2,\ldots,\frac{p-3}{2}, \\ 2j+2-p \pmod p & \text{for } j = \frac{p-1}{2}, \frac{p+1}{2}, \ldots, p-2, \end{cases}$$

for $i=1,2,\ldots,n$. With this labeling, for each $i=1,2,\ldots,n$, let

$$\alpha_i = \bigcup_{j=1}^{p-2} \{\xi : f(u^i_j) + f(u^i_{j+1}) \equiv \xi \pmod p,\ 0 < \xi < p\},$$

and it follows that $\alpha_i = \{1\} \cup \{3,4,\ldots,p-1\}$. Since $2 \in A_0$ and $2 \notin \alpha_i$, by Property 2, there are $\frac{p-1}{2} - 1$ edges of P^i_{p-1} with label 0 and $\frac{p-1}{2}$ edges with label 1, under f^*_p, for each $i=1,2,\ldots,n$.

For the edges of the form $u^i_j v_i$, we have

$$f(u^i_j) + f(v_i) \equiv j+1 \pmod p \text{ for } j = 1,2,\ldots,p-1,$$

$$f(u^i_{p-1}) + f(v_i) \equiv 0 \pmod p,$$

for $i=1,2,\ldots,n$. In view of this labeling, let

$$\beta_i = \bigcup_{j=1}^{p-2} \{\xi : f(u^i_j) + f(v_i) \equiv \xi \pmod p,\ 0 < \xi < p\},$$

and it follows that $\beta_i = \{2,3,\ldots,p-1\}$, for $i=1,2,\ldots,n$. Note that $1 \in A_1$ but $1 \notin \beta_i$, for $i=1,2,\ldots,n$. In addition, observe that $f^*_p(u^i_{p-1}v_i) = 0$ for $i=1,2,\ldots,n$. Combining this with Property 2, we have $n\left(\frac{p-1}{2}\right)+n$ edges of the form $u^i_j v_i$ with label 0 and $n\left(\frac{p-1}{2}-1\right)$ edges with label 1, under f^*_p.

Lastly, for the edges of G, observe that

$$f(a) + f(b) \equiv 1 \pmod p,$$

for any $ab \in E(G)$. Since $\zeta_p(1)=1$, it follows that all edges of G have label 1, under f^*_p.

Therefore,

$$e_{f^*_p}(0) = n\left(\frac{p-1}{2} - 1\right) + n\left(\frac{p-1}{2}\right) + n, \text{ and },$$

$$e_{f^*_p}(1) = n\left(\frac{p-1}{2}\right) + n\left(\frac{p-1}{2} - 1\right) + n + \varepsilon.$$

Since $\varepsilon \in \{-1,0,1\}$, we have $|e_{f^*_p}(0) - e_{f^*_p}(1)| \leq 1$. So, $G \circ P_{p-1}$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, +\rangle$. □

Theorem 8. *Let G be a connected bipartite graph of order n. Then the tensor product graph $K_p \times G$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, + \rangle$ where $S = \{1, 2, \ldots, np\}$, for all $n \geq 2$ and $p \geq 3$.*

Proof. Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$ and let m be the size of G. Since G is a bipartite graph, its vertex set $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 with $V_j = \{u_1^j, u_2^j, \ldots, u_{n_j}^j\}$ for $j = 1, 2$, where $n_1 + n_2 = n$. Thus,

$$E(K_p \times G) = \bigcup_{u_x^1 u_y^2 \in E(G)} \{(v_i, u_x^1)(v_k, u_y^2) : i, k = 1, 2, \ldots, p,\ i \neq k\}.$$

Let $f : V(K_p \times G) \to S$ be a function, where $S = \{1, 2, \ldots, np\}$. Define f as follows: for each $t_j = 1, 2, \ldots, n_j$,

$$f((v_i, u_{t_j}^j)) = \begin{cases} i + (t_1 - 1)p & \text{for } j = 1 \text{ and } i = 1, 2, \ldots, p, \\ p - i + (n_1 + t_2 - 1)p & \text{for } j = 2 \text{ and } i = 1, 2, \ldots, p-1, \\ p + (n_1 + t_2 - 1)p & \text{for } j = 2 \text{ and } i = p. \end{cases}$$

Hence, f is a bijective function.

Let $x \in \{1, 2, \ldots, n_1\}$ and $y \in \{1, 2, \ldots, n_2\}$. For the edges, for each $u_x^1 u_y^2 \in E(G)$ and each $i = 1, 2, \ldots, p$ with $i \neq k$, we have

$$f((v_i, u_x^1)) + f((v_k, u_y^2)) \equiv \begin{cases} i - k \pmod p & \text{for } k = 1, 2, \ldots, p-1, \\ i \pmod p & \text{for } k = p. \end{cases}$$

In accordance with the above labeling, for each $u_x^1 u_y^2 \in E(G)$ and each $i = 1, 2, \ldots, p$, let

$$\alpha^i_{u_x^1 u_y^2} = \bigcup_{\substack{k=1 \\ k \neq i}}^{p} \{\xi : f((v_i, u_x^1)) + f((v_k, u_y^2)) \equiv \xi \pmod p,\ 0 < \xi < p\},$$

and as a result $\alpha^i_{u_x^1 u_y^2} = \{1, 2, \ldots, p-1\}$. By Property 2, we have

$$e_{f_p^*}(0) = e_{f_p^*}(1) = mp\left(\frac{p-1}{2}\right),$$

and it follows that $|e_{f_p^*}(0) - e_{f_p^*}(1)| = 0$. Therefore, $K_p \times G$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, + \rangle$. □

5. Arithmetic Cordial labeling modulo p under $\langle S, \zeta_p, \cdot \rangle$

In this section, we assume that Properties 1, 2, and 3 hold for ζ_p. Also, let $\zeta_p(a) = \frac{1+\chi_p(a)}{2}$ where $\chi_p : \{\xi \in \mathbb{Z} : \gcd(\xi, p) = 1\} \to \{-1, 1\}$ is a function. By Property 2, suppose that $s_1, s_2, \ldots, s_{(p-1)/2} \in A_0$ and $r_1, r_2, \ldots, r_{(p-1)/2} \in A_1$.

Remark 1. Let a and b be integers relatively prime to p. Then

$$\chi_p(ab) = \begin{cases} 1 & \text{if } \chi_p(a) = \chi_p(b) = 1 \text{ or } \chi_p(a) = \chi_p(b) = -1, \\ -1 & \text{otherwise.} \end{cases} \tag{9}$$

Thus,

$$\zeta_p(ab) = \begin{cases} 1 & \text{if } a, b \in A_0 \text{ or } a, b \in A_1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Immediately follows Properties 2 and 3. □

Theorem 9. *The join graph $\overline{K}_p + KP_{(p-1)/2,0,(p-1)/2}$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$ where $S = \{1, 2, \ldots, 2p-1\}$, for all $p \geq 7$.*

Proof. Let $C^1_{(p-1)/2}$ and $C^2_{(p-1)/2}$ be cycles, and P_2 be a path graph of $KP_{(p-1)/2,0,(p-1)/2}$. Suppose that

$$\begin{aligned} V(\overline{K}_p) &= \{v_1, v_2, \ldots, v_p\}, \\ V(C^1_{(p-1)/2}) &= \{u_1, u_2, \ldots, u_{(p-1)/2}\}, \\ V(C^2_{(p-1)/2}) &= \{w_1, w_2, \ldots, w_{(p-1)/2}\}, \\ V(P_2) &= \{u_{(p-1)/2}, w_1\}. \end{aligned}$$

Define the function $f : V(\overline{K}_p + KP_{(p-1)/2,0,(p-1)/2}) \to S$ by

$$\begin{aligned} f(v_i) &= \begin{cases} r_i & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ s_{i-(p-1)/2} & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1, \\ p & \text{for } i = p, \end{cases} \\ f(u_i) &= r_i + p \text{ for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ f(w_i) &= s_i + p \text{ for } i = 1, 2, \ldots, \frac{p-1}{2}, \end{aligned}$$

where $S = \{1, 2, \ldots, 2p-1\}$. Then f is a bijective function.

For the edges of $C^1_{(p-1)/2}$, $C^2_{(p-1)/2}$, and P_2,

$$\begin{aligned} f(u_i)f(u_{i+1}) &\equiv r_i r_{i+1} \pmod{p} \text{ for } i = 1, 2, \ldots, \frac{p-3}{2}, \\ f(u_1)f(u_{(p-1)/2}) &\equiv r_1 r_{(p-1)/2} \pmod{p}, \\ f(w_i)f(w_{i+1}) &\equiv s_i s_{i+1} \pmod{p} \text{ for } i = 1, 2, \ldots, \frac{p-3}{2}, \\ f(w_1)f(w_{(p-1)/2}) &\equiv s_1 s_{(p-1)/2} \pmod{p}, \\ f(u_{(p-1)/2})f(w_1) &\equiv r_{(p-1)/2} s_1 \pmod{p}. \end{aligned}$$

By Remark 1,

$$\begin{aligned} f^*_p(u_i u_{i+1}) &= 1 \text{ for } i = 1, 2, \ldots, \frac{p-3}{2}, \\ f^*_p(u_1 u_{(p-1)/2}) &= 1, \\ f^*_p(w_i w_{i+1}) &= 1 \text{ for } i = 1, 2, \ldots, \frac{p-3}{2}, \\ f^*_p(w_1 w_{(p-1)/2}) &= 1, \\ f^*_p(u_{(p-1)/2} w_1) &= 0. \end{aligned}$$

For the edges of the forms $v_i u_j$ and $v_i w_j$, for each $j = 1, 2, \ldots, \frac{p-1}{2}$,

$$\begin{aligned} f(v_i)f(u_j) &\equiv \begin{cases} r_i r_j \pmod{p} & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ s_{i-(p-1)/2} r_j \pmod{p} & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1, \\ 0 \pmod{p} & \text{for } i = p, \end{cases} \\ f(v_i)f(w_j) &\equiv \begin{cases} r_i s_j \pmod{p} & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ s_{i-(p-1)/2} s_j \pmod{p} & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1, \\ 0 \pmod{p} & \text{for } i = p. \end{cases} \end{aligned}$$

By Remark 1, for each $j = 1, 2, \ldots, \frac{p-1}{2}$,

$$f_p^*(v_i u_j) = \begin{cases} 1 & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ 0 & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p, \end{cases}$$

$$f_p^*(v_i w_j) = \begin{cases} 0 & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, p, \\ 1 & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1. \end{cases}$$

Therefore,

$$e_{f_p^*}(0) = 1 + 2\left(\frac{p+1}{2}\right)\left(\frac{p-1}{2}\right) = 1 + (p-1)\left(\frac{p+1}{2}\right), \text{ and}$$
$$e_{f_p^*}(1) = p - 1 + 2\left(\frac{p-1}{2}\right)^2 = (p-1)\left(\frac{p+1}{2}\right).$$

Clearly, $|e_{f_p^*}(0) - e_{f_p^*}(1)| = 1$ and so $\overline{K}_p + KP_{(p-1)/2,0,(p-1)/2}$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$. □

Theorem 10. *Let G_1 and G_2 be graphs of the same order $\frac{p-1}{2}$ and let $f : V(G_1 + G_2) \to S$ be a bijective function, where $S = \{1, 2, \ldots, p-1\}$. For each $i = 1, 2$ and each $j = -1, 1$, define*

$$\Omega_i^j = \{ab \in E(G_i) : \chi_p(f(a)f(b)) = j\},$$

and suppose that

$$B = \{a \in V(G_1) : \chi_p(f(a)) = 1\}.$$

In addition, let $\epsilon \in \{-1, 0, 1\}$. If

$$|\Omega_1^1| + |\Omega_2^1| = |\Omega_1^{-1}| + |\Omega_2^{-1}| + \left(2|B| - \frac{p-1}{2}\right)^2 + \epsilon, \tag{10}$$

then the join graph $G_1 + G_2$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$.

Proof. Note that $S = A_0 \cup A_1$. From the hypothesis, it is clear that $|\Omega_i^{-1}|$ edges of G_i are labeled 0 and $|\Omega_i^1|$ edges are labeled 1 under f_p^*, for $i = 1, 2$. Let $k = |B|$. Clearly, k vertices of G_1 have label r_i for some i. It means that the remaining vertices have label s_i, and there are $\frac{p-1}{2} - k$ such vertices. Consequently, k vertices of G_2 have label s_i and $\frac{p-1}{2} - k$ vertices have label r_i for some i. By Remark 1, there are $2k\left(\frac{p-1}{2} - k\right)$ edges labeled 1 and $k^2 + \left(\frac{p-1}{2} - k\right)^2$ edges labeled 0 of the form ab, where $a \in V(G_1)$ and $b \in V(G_2)$, under f_p^*.

Consequently,

$$e_{f_p^*}(1) = 2k\left(\frac{p-1}{2} - k\right) + |\Omega_1^1| + |\Omega_2^1|, \text{ and}$$
$$e_{f_p^*}(0) = k^2 + \left(\frac{p-1}{2} - k\right)^2 + |\Omega_1^{-1}| + |\Omega_2^{-1}|.$$

Given that

$$|\Omega_1^1| + |\Omega_2^1| = |\Omega_1^{-1}| + |\Omega_2^{-1}| + \left(2k - \frac{p-1}{2}\right)^2 + \epsilon,$$

where $k = |B|$ and $\epsilon \in \{-1, 0, 1\}$, we have

$$\begin{aligned}
e_{f_p^*}(0) - e_{f_p^*}(1) &= k^2 + \left(\frac{p-1}{2} - k\right)^2 + |\Omega_1^{-1}| + |\Omega_2^{-1}| \\
&\quad - \left[2k\left(\frac{p-1}{2} - k\right) + |\Omega_1^{-1}| + |\Omega_2^{-1}| + \left(2k - \frac{p-1}{2}\right)^2 + \epsilon\right] \\
&= k^2 + \left(\frac{p-1}{2}\right)^2 - 2\left[k\left(\frac{p-1}{2}\right)\right] + k^2 - 2k\left(\frac{p-1}{2}\right) + 2k^2 \\
&\quad - (2k)^2 + 2\left[2k\left(\frac{p-1}{2}\right)\right] - \left(\frac{p-1}{2}\right)^2 - \epsilon \\
&= 4k^2 - k\left(\frac{p-1}{2}\right) - k\left(\frac{p-1}{2}\right) - 4k^2 + 2k\left(\frac{p-1}{2}\right) - \epsilon \\
&= -\epsilon.
\end{aligned}$$

Since $\epsilon \in \{-1, 0, 1\}$, it follows that $|e_{f_p^*}(0) - e_{f_p^*}(1)| \leq 1$. Accordingly, $G_1 + G_2$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$. □

An an example of Theorem 10, consider the following result.

Theorem 11. *Assume that $\frac{p-1}{2} \equiv 0 \pmod 4$. The join graph $P_{(p-1)/2} + C_{(p-1)/2}$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$, where $S = \{1, 2, \ldots, p-1\}$ and $p \geq 17$.*

Proof. Since $\frac{p-1}{2} \equiv 0 \pmod 4$, we have $\frac{p-1}{2} = 4k$ for some integer k. We will use Theorem 10 with $G_1 = P_{(p-1)/2}$ and $G_2 = C_{(p-1)/2}$. Now, assume that $V(P_{4k}) = \{v_1, v_2, \ldots, v_{4k}\}$ and $V(C_{4k}) = \{u_1, u_2, \ldots, u_{4k}\}$. Define a function $f : V(P_{4k} + C_{4k}) \to \{1, 2, \ldots, 8k\}$ as

$$f(v_i) = \begin{cases} r_i & \text{for } i \equiv 1 \text{ or } 2 \pmod 4, \\ s_i & \text{for } i \equiv 0 \text{ or } 3 \pmod 4, \end{cases}$$

$$f(u_i) = \begin{cases} s_i & \text{for } i \equiv 1 \text{ or } 2 \pmod 4, \\ r_i & \text{for } i \equiv 0 \text{ or } 3 \pmod 4, \end{cases}$$

for $i = 1, 2, \ldots, 4k$. So, f is a bijective function and $|B| = 2k$.

For the edges of P_{4k},

$$f(v_i)f(v_{i+1}) = \begin{cases} r_i r_{i+1} & \text{for } i \equiv 1 \pmod 4, \\ r_i s_{i+1} & \text{for } i \equiv 2 \pmod 4, \\ s_i s_{i+1} & \text{for } i \equiv 3 \pmod 4, \\ s_i r_{i+1} & \text{for } i \equiv 0 \pmod 4, \end{cases}$$

for $i = 1, 2, \ldots, 4k-1$. By Remark 1, equation (9),

$$\Omega_1^1 = \bigcup_{\substack{i=1 \\ i \equiv 1 \text{ or } 3 \pmod 4}}^{4k-1} \{v_i v_{i+1}\} \text{ and } \Omega_1^{-1} = \bigcup_{\substack{i=1 \\ i \equiv 0 \text{ or } 2 \pmod 4}}^{4k-1} \{v_i v_{i+1}\}.$$

Hence, $|\Omega_1^1| = 2k$ and $|\Omega_1^{-1}| = 2k - 1$.

For the edges of C_{4k},

$$f(u_i)f(u_{i+1}) = \begin{cases} s_i s_{i+1} & \text{for } i \equiv 1 \pmod 4, \\ s_i r_{i+1} & \text{for } i \equiv 2 \pmod 4, \\ r_i r_{i+1} & \text{for } i \equiv 3 \pmod 4, \\ r_i s_{i+1} & \text{for } i \equiv 0 \pmod 4, \end{cases}$$

for $i = 1, 2, \ldots, 4k-1$, and

$$f(u_1)f(u_{4k}) \equiv s_1 r_{4k}.$$

Using Remark 1, equation (9), we have

$$\Omega_2^1 = \bigcup_{\substack{i=1 \\ i \equiv 1 \text{ or } 3 \pmod 4}}^{4k-1} \{u_i u_{i+1}\} \text{ and } \Omega_2^{-1} = \left[\bigcup_{\substack{i=1 \\ i \equiv 0 \text{ or } 2 \pmod 4}}^{4k-1} \{u_i u_{i+1}\} \right] \cup \{u_1 u_{4k}\}.$$

Thus, $|\Omega_2^1| = |\Omega_2^{-1}| = 2k$.

Applying Eq. (10) with $\epsilon = 1$, we have

$$\begin{aligned} |\Omega_1^1| + |\Omega_2^1| &= |\Omega_1^{-1}| + |\Omega_2^{-1}| + \left(2|B| - \frac{p-1}{2}\right)^2 + \epsilon \\ 2k + 2k &= 2k - 1 + 2k + [2(2k) - 4k]^2 + 1 \\ 4k &= 4k. \end{aligned}$$

So, $P_{(p-1)/2} + C_{(p-1)/2}$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$, where $S = \{1, 2, \ldots, p-1\}$. □

Theorem 12. *Let G be a connected graph of order $\frac{p-1}{2}$ and size $\frac{p-1}{2} + \epsilon$, where $\epsilon \in \{-1, 0, 1\}$. Then the corona graph $G \circ K_1$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$, where $S = \{1, 2, \ldots, p-1\}$, for all $p \geq 3$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_{(p-1)/2}\}$. Suppose that K_1^i is the ith copy of K_1 with $V(K_1^i) = \{u_i\}$ such that u_i is adjacent to v_i, for $i = 1, 2, \ldots, \frac{p-1}{2}$. Assume that $f : V(G \circ K_1) \to S$ is a function defined by

$$f(v_i) = r_i \text{ and } f(u_i) = s_i,$$

for $i = 1, 2, \ldots, \frac{p-1}{2}$, where $S = \{1, 2, \ldots, p-1\}$.

For the edges of G, if $v_i v_j \in E(G)$, for some $i, j \in \{1, 2, \ldots, \frac{p-1}{2}\}$, then

$$f(v_i)f(v_j) = r_i r_j.$$

By Remark 1, $f_p^*(e) = 1$ for every $e \in E(G)$.

For the rest of the edges,

$$f(u_i)f(v_i) = s_i r_i,$$

for $i = 1, 2, \ldots, \frac{p-1}{2}$. Thus, by Remark 1, $f_p^*(u_i v_i) = 0$ for $i = 1, 2, \ldots, \frac{p-1}{2}$.

Therefore,

$$e_{f_p^*}(0) = \frac{p-1}{2} \text{ and } e_{f_p^*}(1) = \frac{p-1}{2} + \epsilon.$$

Since $\epsilon \in \{-1, 0, 1\}$, it follows that $|e_{f_p^*}(0) - e_{f_p^*}(1)| \leq 1$. Hence, $G \circ K_1$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$. □

Theorem 13. *Let G be a connected graph of order $p-1$ and size $m(p-1) + \epsilon$, where $\epsilon \in \{-1, 0, 1\}$ and $m \geq 1$. In addition, suppose that H is a graph of order $p-1$ and size $p-1+m$. Then the corona graph $G \circ H$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$, where $S = \{1, 2, \ldots, p(p-1)\}$, for all $p \geq 3$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_{p-1}\}$. Suppose that H^i is the ith copy of H with $V(H^i) = \{u_1^i, u_2^i, \ldots, u_{p-1}^i\}$ where all vertices of H^i are adjacent to v_i for $i = 1, 2, \ldots, p-1$. Define a function $f : V(G \circ H) \to S$ as

$$f(u_j^i) = \begin{cases} r_i + (j-1)p & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ s_{i-(p-1)/2} + (j-1)p & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1, \end{cases}$$

for $j = 1, 2, \ldots, p-1$, and

$$f(v_i) = p + (i-1)p \text{ for } i = 1, 2, \ldots, p-1,$$

where $S = \{1, 2, \ldots, p(p-1)\}$. Thus, f is a bijective function.

For the edges of G, if $v_i v_j \in E(G)$, for some $i, j \in \{1, 2, \ldots, p-1\}$, then

$$f(v_i)f(v_j) \equiv 0 \pmod{p}.$$

Hence, $f_p^*(e) = 0$ for all $e \in E(G)$.

For the edges of H^i, if $u_a^i u_b^i \in E(H^i)$, for some $a, b \in \{1, 2, \ldots, p-1\}$, we have

$$f(u_a^i)f(u_b^i) \equiv \begin{cases} r_i^2 \pmod{p} & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, \\ s_{i-(p-1)/2}^2 \pmod{p} & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1. \end{cases}$$

By Remark 1, $f_p^*(e) = 1$ for all $e \in E(H^i)$, for $i = 1, 2, \ldots, p-1$.

For the remaining edges, for each $i, j = 1, 2, \ldots, p-1$,

$$f(v_i)f(u_j^i) \equiv 0 \pmod{p},$$

and it follows that $f_p^*(v_i u_j^i) = 0$.

Consequently,

$$e_{f_p^*}(0) = m(p-1) + \epsilon + (p-1)^2 \text{ and } e_{f_p^*}(1) = (p-1)(p-1+m) = (p-1)^2 + m(p-1).$$

Because $\epsilon \in \{-1, 0, 1\}$, $|e_{f_p^*}(0) - e_{f_p^*}(1)| \leq 1$. So, $G \circ H$ is an arithmetic cordial graph modulo p under $\langle S, \zeta_p, \cdot \rangle$. □

6. Conclusion

In this paper, we introduced the concept of arithmetic cordial labeling, providing a unified theoretical framework that generalizes several existing cordial labeling variants studied in [3–7]. The primary contribution of this work lies in the shift from treating these labelings as isolated instances to viewing them as specific cases of a unified arithmetic structure $\langle S, \zeta_\eta, \star \rangle$. By establishing the essential properties of these structures in Properties 1 to 3 and 4 to 5, we have created an "umbrella" theory that allows for the simultaneous study of diverse number-theoretic functions within a single cordiality constraint.

A significant mathematical advantage of this framework, as demonstrated in Theorems 1 and 2, is the transferability of results; proving that a graph admits an arithmetic cordial labeling in one structure automatically confirms its existence in all equivalent structures, thereby eliminating the need for redundant proofs across different labeling types. Furthermore, we demonstrated the structural robustness of this generalized approach by proving the existence of such labelings for a wide array of complex graph constructions, including star, ladder, alternate cycle snake, join, corona, and tensor product graphs.

Beyond its theoretical implications, the balanced distribution inherent in arithmetic cordial labeling offers potential real-world applications in network topology design and data security. Specifically, modular arithmetic-based vertex assignments can be utilized to ensure equitable load balancing in communication networks or to develop structured protocols for cryptographic authentication, where edge validity is determined by precise number-theoretic relations. This research provides a systematic template for integrating various number-theoretic functions into graph theory, offering a broader perspective on the interplay between algebraic structures and graph cordiality. Future research may explore the application of this generalized labeling to other graph operations or investigate structures where the balance condition is extended to larger moduli.

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