Open Journal of
Mathematical Sciences

PSR Press

Article

On several arctangent- and logarithmic-Hardy-Hilbert integral inequalities

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Academic Editor: Waqas Nazeer
Received: 09 October 2025; Accepted: 30 December 2025; Published: 19 January 2026

Abstract: In this paper, we establish several new arctangent- and logarithmic-Hardy-Hilbert integral inequalities. The approach combines fundamental principles with refined techniques from the theory of integral inequalities, leading to a range of original results. Complete proofs are presented together with a discussion of their sharpness and potential applications.

Keywords: Hardy-Hilbert integral inequality, arctangent function, logarithmic function, weighted L^p norm

MSC: 26D15.

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1. Introduction

A fundamental result in analysis is provided by the classical Hardy-Hilbert integral inequality. A possible formulation of this result is given below: Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx \right)^{1/p} \left(\int_0^{+\infty} g^q(x)dx \right)^{1/q}.$$

The constant factor, defined by $\pi/\sin(\pi/p)$, is optimal. In addition, the upper bound involves the classical (unweighted) L^p norm of f and the L^q norm of g. This inequality and its many extensions play a pivotal role in harmonic analysis, operator theory, and the study of function spaces. It is a fundamental tool for establishing norm upper bounds of integral operators defined with kernel functions. Over the past century, many authors have refined its structure, derived weighted forms and discrete analogues, and made various improvements. Comprehensive reviews can be found in the monographs [1–3], and in the survey paper [4]. Several recent contributions are also referred to in [5–10], showing how active this research area is.

In recent years, there has been a particular focus on Hardy-Hilbert-type integral inequalities involving special kernel functions, including logarithmic, trigonometric, and hyperbolic forms. This is due to their applications in the theory of special functions and partial differential equations. Building on these developments, this paper presents new Hardy-Hilbert integral inequalities that incorporate logarithmic and arctangent kernel functions. More precisely, we focus on upper bounds of the following double integrals:

$$\int_0^{+\infty} \int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy,$$

and

$$\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{x}{y}\right) f(x)g(y)dxdy,$$

where $f, g : [0, +\infty) \to [0, +\infty)$ are measurable functions, subject to integrability conditions. The resulting upper bounds are expressed in terms of a sharp constant, multiplied by appropriate weighted or unweighted integral norms of f and g. To the best of our knowledge, these results represent a novel contribution to the existing literature.

The remainder of the paper is organized as follows: In §2, we establish the arctangent-Hardy-Hilbert integral inequalities. §3 is devoted to the logarithmic-Hardy-Hilbert integral inequalities. Finally, §4 presents some concluding remarks.

2. Arctangent-Hardy-Hilbert integral inequalities

This section is devoted to establishing new arctangent-Hardy-Hilbert integral inequalities, featuring upper bounds expressed in terms of various integral norms.

2.1. Direct inequalities

The most basic arctangent-Hardy-Hilbert integral inequality is described in the theorem below.

Theorem 1. *Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} f(x)dx < +\infty, \quad \int_0^{+\infty} g(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \leq \frac{\pi}{2}\left(\int_0^{+\infty} f(x)dx\right)\left(\int_0^{+\infty} g(x)dx\right).$$

Proof. Using the elementary inequalities $0 \leq \arctan(z) \leq \pi/2$ for any $z > 0$, we derive

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy &\leq \int_0^{+\infty}\int_0^{+\infty} \frac{\pi}{2} f(x)g(y)dxdy \\ &= \frac{\pi}{2}\left(\int_0^{+\infty} f(x)dx\right)\left(\int_0^{+\infty} g(y)dxdy\right) \\ &= \frac{\pi}{2}\left(\int_0^{+\infty} f(x)dx\right)\left(\int_0^{+\infty} g(x)dx\right).\end{aligned}$$

This completes the proof. □

The constant factor $\pi/2$ arises naturally, and the corresponding upper bound involves the L^1 norms of the functions. However, it is not claimed to be sharp.

Another upper bound is provided by the theorem below, which incorporates weighted L^1 norms.

Theorem 2. *Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} xf(x)dx < +\infty, \quad \int_0^{+\infty} \frac{1}{x}g(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \leq \left(\int_0^{+\infty} xf(x)dx\right)\left(\int_0^{+\infty} \frac{1}{x}g(x)dx\right).$$

Proof. Using the elementary inequalities $0 \leq \arctan(z) \leq z$ for any $z > 0$, we derive

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy &\leq \int_0^{+\infty}\int_0^{+\infty} \frac{x}{y} f(x)g(y)dxdy \\ &= \left(\int_0^{+\infty} xf(x)dx\right)\left(\int_0^{+\infty} \frac{1}{y}g(y)dxdy\right)\end{aligned}$$

$$= \left(\int_0^{+\infty} xf(x)dx\right)\left(\int_0^{+\infty} \frac{1}{x}g(x)dx\right).$$

This concludes the proof. □

Consequently, the constant factor is 1, with various weight functions considered for the weighted L^1 norms.

The remainder of this section is devoted to more technical arctangent-Hardy-Hilbert integral inequalities. Prior to presenting these results, we introduce two preliminary propositions.

2.2. Intermediary propositions

The proposition below provides an upper bound for a specific L^p norm of a single function, expressed as a linear combination of weighted and truncated L^1 norms of that function.

Proposition 1. *Let $p > 1$. Let $\alpha \in (0, 1/p)$ and $\beta > 1/p$. Let $f : [0, +\infty) \to [0, +\infty)$ be a measurable function such that*

$$\int_0^{+\infty} x^\alpha f(x)dx < +\infty, \quad \int_1^{+\infty} x^\beta f(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du \le \frac{1}{1-\alpha p}\left(\int_0^{+\infty} x^\alpha f(x)dx\right)^p + \frac{1}{\beta p - 1}\left(\int_1^{+\infty} x^\beta f(x)dx\right)^p.$$

Proof. Using the Chasles integral relation, basic majorations, and power primitives taking into account the facts that $\alpha \in (0, 1/p)$ and $\beta > 1/p$, we get

$$\begin{aligned}
\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du &= \int_0^{1}\left(\int_u^{+\infty} f(x)dx\right)^p du + \int_1^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du \\
&\le \int_0^{1}\left(\int_u^{+\infty} \left(\frac{x}{u}\right)^\alpha f(x)dx\right)^p du + \int_1^{+\infty}\left(\int_u^{+\infty} \left(\frac{x}{u}\right)^\beta f(x)dx\right)^p du \\
&= \int_0^{1} u^{-\alpha p}\left(\int_u^{+\infty} x^\alpha f(x)dx\right)^p du + \int_1^{+\infty} u^{-\beta p}\left(\int_u^{+\infty} x^\beta f(x)dx\right)^p du \\
&\le \int_0^{1} u^{-\alpha p}\left(\int_0^{+\infty} x^\alpha f(x)dx\right)^p du + \int_1^{+\infty} u^{-\beta p}\left(\int_1^{+\infty} x^\beta f(x)dx\right)^p du \\
&= \left(\int_0^{1} u^{-\alpha p}du\right)\left(\int_0^{+\infty} x^\alpha f(x)dx\right)^p + \left(\int_1^{+\infty} u^{-\beta p}du\right)\left(\int_1^{+\infty} x^\beta f(x)dx\right)^p \\
&= \frac{1}{1-\alpha p}\left(\int_0^{+\infty} x^\alpha f(x)dx\right)^p + \frac{1}{\beta p - 1}\left(\int_1^{+\infty} x^\beta f(x)dx\right)^p.
\end{aligned}$$

This completes the proof. □

The proposition below establishes an upper bound for a particular L^p norm of a single function, expressed in terms of a weighted L^1 norm of that function.

Proposition 2. *Let $p > 1$. Let $f : [0, +\infty) \to [0, +\infty)$ be a measurable function such that*

$$\int_0^{+\infty} f(x)dx < +\infty, \quad \int_0^{+\infty} (xf(x))^p dx < +\infty.$$

Then we have

$$\left(\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du\right)^{1/p} \le p\left(\int_0^{+\infty} (xf(x))^p dx\right)^{1/p}.$$

Proof. For any $u > 0$, let us set

$$S(u) = \int_u^{+\infty} f(x)dx,$$

so that

$$\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du = \int_0^{+\infty} S^p(u)du.$$

We have $S'(u) = -f(u)$ almost everywhere, $\lim_{u\to 0} uS(u)^p = 0$ and $\lim_{u\to+\infty} S(u) = 0$. An integration by parts yields

$$\begin{aligned}\int_0^{+\infty} S(u)^p du &= \left[uS(u)^p\right]_0^{+\infty} - p\int_0^{+\infty} uS(u)^{p-1}S'(u)du\\ &= p\int_0^{+\infty} uS(u)^{p-1}f(u)du.\end{aligned}$$

Applying the Hölder integral inequality at the exponents p and $q = p/(p-1)$, we get

$$\int_0^{+\infty} uS(u)^{p-1}f(u)du \le \left(\int_0^{+\infty}(uf(u))^p du\right)^{1/p}\left(\int_0^{+\infty} S(u)^p du\right)^{(p-1)/p}.$$

Combining the above equalities and inequalities, we have

$$\int_0^{+\infty} S(u)^p du \le p\left(\int_0^{+\infty}(xf(x))^p dx\right)^{1/p}\left(\int_0^{+\infty} S(u)^p du\right)^{1-1/p}.$$

Dividing both sides by $\int_0^{+\infty} S(u)^p du$, we get

$$\left(\int_0^{+\infty} S(u)^p du\right)^{1/p} \le p\left(\int_0^{+\infty}(xf(x))^p du\right)^{1/p},$$

so that

$$\left(\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du\right)^{1/p} \le p\left(\int_0^{+\infty}(xf(x))^p dx\right)^{1/p}.$$

This ends the proof. □

The constant factor in this case is p, which is straightforward. This inequality may be regarded as a variant of the classical Hardy integral inequality.

2.3. Technical integral inequalities

We are now in a position to present the technical arctangent-Hardy-Hilbert inequalities. The theorem below provides an upper bound expressed in terms of a specific L^p norm of the function f.

Theorem 3. *Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du < +\infty, \quad \int_0^{+\infty} g^p(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \le \frac{\pi}{2\sin(\pi/(2q))}\left(\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.$$

Proof. Using the arctangent primitive and applying the Tonelli integral theorem (which is possible because the integrand is non-negative), we get

$$\int_0^{+\infty}\int_0^{+\infty}\arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty}\left(\int_0^{x/y}\frac{1}{1+t^2}dt\right) f(x)g(y)dxdy$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^{x/y} \frac{1}{1+t^2} f(x)g(y)dtdxdy \\
&= \int_0^{+\infty}\int_0^{+\infty}\int_{ty}^{+\infty} \frac{1}{1+t^2} f(x)g(y)dxdtdy \\
&= \int_0^{+\infty} \frac{1}{1+t^2}\left(\int_0^{+\infty} g(y)\left(\int_{ty}^{+\infty} f(x)dx\right)dy\right)dt.
\end{aligned}
$$

Applying the Hölder integral inequality with respect to y, we obtain

$$
\int_0^{+\infty} g(y)\left(\int_{ty}^{+\infty} f(x)dx\right)dy \leq \left(\int_0^{+\infty}\left(\int_{ty}^{+\infty} f(x)dx\right)^p dy\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.
$$

Making the change of variables $u = ty$, we get

$$
\int_0^{+\infty}\left(\int_{ty}^{+\infty} f(x)dx\right)^p dy = \frac{1}{t}\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du.
$$

Combining the above equalities and inequalities, we get

$$
\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \leq \left(\int_0^{+\infty} \frac{t^{-1/p}}{1+t^2}dt\right)\left(\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.
$$

Let us now examine the first integral. Making the change of variables $s = t^2$, so $t = s^{1/2}$ and $dt = (1/2)s^{-1/2}ds$, we obtain

$$
\int_0^{+\infty} \frac{t^{-1/p}}{1+t^2}dt = \frac{1}{2}\int_0^{+\infty} \frac{s^{a-1}}{1+s}ds,
$$

with

$$
a = \frac{1}{2}\left(1 - \frac{1}{p}\right) = \frac{1}{2q} \in (0,1).
$$

Using the following well-known formula:

$$
\int_0^{+\infty} \frac{s^{\alpha-1}}{1+s}ds = \frac{\pi}{\sin(\pi\alpha)},
$$

with $\alpha \in (0,1)$ (see [11, 3.241 2]), we have

$$
\int_0^{+\infty} \frac{t^{-1/p}}{1+t^2}dt = \frac{1}{2} \times \frac{\pi}{\sin(\pi a)} = \frac{\pi}{2\sin(\pi/(2q))}.
$$

Therefore, we have

$$
\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \leq \frac{\pi}{2\sin(\pi/(2q))}\left(\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.
$$

This completes the proof. □

We emphasize the constant factor $\pi/(2\sin(\pi/(2q)))$, which has been obtained as accurately as possible in our context. It can also be expressed as

$$
\frac{\pi}{2\sin(\pi/(2q))} = \frac{\pi}{2\cos(\pi/(2p))},
$$

so that

$$\int_0^{+\infty}\int_0^{+\infty}\arctan\left(\frac{x}{y}\right)f(x)g(y)dxdy \le \frac{\pi}{2\cos(\pi/(2p))}\left(\int_0^{+\infty}\left(\int_u^{+\infty}f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

The special L^p norm appearing in the upper bound is the one considered in Propositions 1 and 2, which in turn enable the derivation of more conventional upper bounds.

The theorem below exhibits such an upper bound.

Theorem 4. *Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $\alpha \in (0, 1/p)$ and $\beta > 1/p$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} x^{\alpha} f(x)dx < +\infty, \quad \int_1^{+\infty} x^{\beta} f(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\arctan\left(\frac{x}{y}\right)f(x)g(y)dxdy$$
$$\le \frac{\pi}{2\sin(\pi/(2q))}\left(\frac{1}{1-\alpha p}\left(\int_0^{+\infty}x^{\alpha}f(x)dx\right)^p + \frac{1}{\beta p-1}\left(\int_1^{+\infty}x^{\beta}f(x)dx\right)^p\right)^{1/p} \times \left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

Proof. Applying Theorem 3 and Proposition 1, we get

$$\int_0^{+\infty}\int_0^{+\infty}\arctan\left(\frac{x}{y}\right)f(x)g(y)dxdy$$
$$\le \frac{\pi}{2\sin(\pi/(2q))}\left(\int_0^{+\infty}\left(\int_u^{+\infty}f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}$$
$$\le \frac{\pi}{2\sin(\pi/(2q))}\left(\frac{1}{1-\alpha p}\left(\int_0^{+\infty}x^{\alpha}f(x)dx\right)^p + \frac{1}{\beta p-1}\left(\int_1^{+\infty}x^{\beta}f(x)dx\right)^p\right)^{1/p} \times \left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

This concludes the proof. □

The theorem below gives a more conventional upper bound, with a weighted L^p norm of f.

Theorem 5. *Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} f(x)dx < +\infty, \quad \int_0^{+\infty}(xf(x))^p dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\arctan\left(\frac{x}{y}\right)f(x)g(y)dxdy \le \frac{\pi p}{2\sin(\pi/(2q))}\left(\int_0^{+\infty}(xf(x))^p dx\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

Proof. Applying Theorem 3 and Proposition 2, we get

$$\int_0^{+\infty}\int_0^{+\infty}\arctan\left(\frac{x}{y}\right)f(x)g(y)dxdy \le \frac{\pi}{2\sin(\pi/(2q))}\left(\int_0^{+\infty}\left(\int_u^{+\infty}f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}$$
$$\le \frac{\pi p}{2\sin(\pi/(2q))}\left(\int_0^{+\infty}(xf(x))^p dx\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

This ends the proof. □

Another formulation of this inequality is

$$\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \le \frac{\pi p}{2\cos\left(\pi/(2p)\right)} \left(\int_0^{+\infty} (xf(x))^p dx\right)^{1/p} \left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.$$

The theorem below provides another upper bound, with a weighted L^p norm of f and a weighted L^q norm of g.

Theorem 6. *Let $p \in (1,2)$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $f, g : [0,+\infty) \to [0,+\infty)$ be measurable functions such that*

$$\int_0^{+\infty} xf^p(x)dx < +\infty, \quad \int_0^{+\infty} xg^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \le \Upsilon \left(\int_0^{+\infty} xf^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} xg^q(x)dx\right)^{1/q},$$

where

$$\Upsilon = \int_0^{\pi/2} \theta \sin^{-2/p}(\theta)\cos^{-2/q}(\theta)d\theta. \quad (1)$$

Proof. Making the polar change of variables $x = \rho\sin(\theta)$ and $y = \rho\cos(\theta)$, using $\arctan(\tan(z)) = z$ for $z \in \mathbb{R}$, and applying the Tonelli integral theorem (which is possible because the integrand is non-negative), we obtain

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy &= \int_0^{\pi/2}\int_0^{+\infty} \arctan\left(\frac{\rho\sin(\theta)}{\rho\cos(\theta)}\right) f(\rho\sin(\theta))g(\rho\cos(\theta))\rho d\rho d\theta \\ &= \int_0^{\pi/2} \theta \left(\int_0^{+\infty} \rho f(\rho\sin(\theta))g(\rho\cos(\theta))d\rho\right) d\theta.\end{aligned}$$

Applying the Hölder integral inequality at the exponents p and $q = p/(p-1)$, we have

$$\begin{aligned}\int_0^{+\infty} \rho f(\rho\sin(\theta))g(\rho\cos(\theta))d\rho &= \int_0^{+\infty} \rho^{1/p+1/q} f(\rho\sin(\theta))g(\rho\cos(\theta))d\rho \\ &\le \left(\int_0^{+\infty} \rho f^p(\rho\sin(\theta))d\rho\right)^{1/p} \left(\int_0^{+\infty} \rho g^q(\rho\cos(\theta))d\rho\right)^{1/q}.\end{aligned}$$

Making the changes of variables $u = \rho\sin(\theta)$ and $v = \rho\cos(\theta)$, we find that

$$\begin{aligned}&\left(\int_0^{+\infty} \rho f^p(\rho\sin(\theta))d\rho\right)^{1/p} \left(\int_0^{+\infty} \rho g^q(\rho\cos(\theta))d\rho\right)^{1/q} \\ &= \left(\frac{1}{\sin^2(\theta)}\int_0^{+\infty} uf^p(u)du\right)^{1/p} \left(\frac{1}{\cos^2(\theta)}\int_0^{+\infty} vg^q(v)dv\right)^{1/q} \\ &= \sin^{-2/p}(\theta)\cos^{-2/q}(\theta) \left(\int_0^{+\infty} xf^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} xg^q(x)dx\right)^{1/q}.\end{aligned}$$

Combining the above equalities and inequalities, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \arctan\left(\frac{x}{y}\right) f(x)g(y)dxdy \\ &\le \int_0^{\pi/2} \left(\theta \sin^{-2/p}(\theta)\cos^{-2/q}(\theta) \left(\int_0^{+\infty} xf^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} xg^q(x)dx\right)^{1/q}\right) d\theta \\ &= \left(\int_0^{\pi/2} \theta \sin^{-2/p}(\theta)\cos^{-2/q}(\theta)d\theta\right) \left(\int_0^{+\infty} xf^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} xg^q(x)dx\right)^{1/q}\end{aligned}$$

$$= \Upsilon \left(\int_0^{+\infty} x f^p(x) dx \right)^{1/p} \left(\int_0^{+\infty} x g^q(x) dx \right)^{1/q},$$

where Υ is given in Eq. (1). This ends the proof. □

Note that we restrict our study to $p \in (1,2)$ to ensure the existence of the constant factor Υ. While this constant does not have a closed-form expression, it can be accurately approximated using numerical methods.

To the best of our knowledge, the established arctangent-Hardy-Hilbert integral inequalities constitute a new finding within the existing body of literature. These inequalities have applications in harmonic analysis, approximation theory, operator theory and the study of boundary value problems in partial differential equations.

3. Logarithmic Hardy-Hilbert integral inequalities

We now turn our attention to certain logarithmic Hardy-Hilbert integral inequalities.

3.1. A direct inequality

The most basic logarithmic-Hardy-Hilbert integral inequality is described below:

Theorem 7. *Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} x f(x) dx < +\infty, \quad \int_0^{+\infty} \frac{1}{x} g(x) dx < +\infty.$$

Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{x}{y}\right) f(x) g(y) dx dy \leq \left(\int_0^{+\infty} x f(x) dx \right) \left(\int_0^{+\infty} \frac{1}{x} g(x) dx \right).$$

Proof. Using the elementary inequalities $0 \leq \log(1+z) \leq z$ for any $z > 0$, we derive

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{x}{y}\right) f(x) g(y) dx dy &\leq \int_0^{+\infty} \int_0^{+\infty} \frac{x}{y} f(x) g(y) dx dy \\ &= \left(\int_0^{+\infty} x f(x) dx \right) \left(\int_0^{+\infty} \frac{1}{y} g(y) dx dy \right) \\ &= \left(\int_0^{+\infty} x f(x) dx \right) \left(\int_0^{+\infty} \frac{1}{x} g(x) dx \right). \end{aligned}$$

This completes the proof. □

Therefore, the constant factor is 1, and different weight functions are considered for the weighted L^1 norms.

3.2. Technical inequalities

We are now in a position to present a technical logarithmic-Hardy-Hilbert inequality, featuring an upper bound expressed in terms of a specific L^p norm of the function f. The proof follows a similar approach to that of Theorem 3.

Theorem 8. *Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} \left(\int_u^{+\infty} f(x) dx \right)^p du < +\infty, \quad \int_0^{+\infty} g^p(x) dx < +\infty.$$

Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{x}{y}\right) f(x) g(y) dx dy \leq \frac{\pi}{\sin(\pi/q)} \left(\int_0^{+\infty} \left(\int_u^{+\infty} f(x) dx \right)^p du \right)^{1/p} \left(\int_0^{+\infty} g^q(x) dx \right)^{1/q}.$$

Proof. Using the logarithmic primitive and applying the Tonelli integral theorem (which is possible because the integrand is non-negative), we get

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{x}{y}\right)f(x)g(y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\left(\int_0^{x/y}\frac{1}{1+t}dt\right)f(x)g(y)dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{x/y}\frac{1}{1+t}f(x)g(y)dtdxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\int_{ty}^{+\infty}\frac{1}{1+t}f(x)g(y)dxdtdy\\ &= \int_0^{+\infty}\frac{1}{1+t}\left(\int_0^{+\infty}g(y)\left(\int_{ty}^{+\infty}f(x)dx\right)dy\right)dt.\end{aligned}$$

Applying the Hölder integral inequality with respect to y, we obtain

$$\int_0^{+\infty}g(y)\left(\int_{ty}^{+\infty}f(x)dx\right)dy \le \left(\int_0^{+\infty}\left(\int_{ty}^{+\infty}f(x)dx\right)^p dy\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

Making the change of variables $u = ty$, we have

$$\int_0^{+\infty}\left(\int_{ty}^{+\infty}f(x)dx\right)^p dy = \frac{1}{t}\int_0^{+\infty}\left(\int_u^{+\infty}f(x)dx\right)^p du.$$

Combining the above equalities and inequalities, we get

$$\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{x}{y}\right)f(x)g(y)dxdy \le \left(\int_0^{+\infty}\frac{t^{-1/p}}{1+t}dt\right)\left(\int_0^{+\infty}\left(\int_u^{+\infty}f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

Let us now examine the first integral. We have

$$\int_0^{+\infty}\frac{t^{-1/p}}{1+t}dt = \int_0^{+\infty}\frac{t^{1/q-1}}{1+t}dt = \frac{\pi}{\sin(\pi/q)}.$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{x}{y}\right)f(x)g(y)dxdy \le \frac{\pi}{\sin(\pi/q)}\left(\int_0^{+\infty}\left(\int_u^{+\infty}f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

This completes the proof. □

We emphasize the constant factor $\pi/(\sin(\pi/q))$, which has been obtained as accurately as possible in our context. It can also be expressed as

$$\frac{\pi}{\sin(\pi/q)} = \frac{\pi}{\sin(\pi/p)},$$

so that

$$\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{x}{y}\right)f(x)g(y)dxdy \le \frac{\pi}{\sin(\pi/p)}\left(\int_0^{+\infty}\left(\int_u^{+\infty}f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty}g^q(x)dx\right)^{1/q}.$$

Note that the upper bound also involves the specific L^p norm introduced in Propositions 1 and 2.

The theorem below presents a logarithmic-Hardy-Hilbert inequality, providing an explicit upper bound in terms of weighted L^1 norms of f and the L^q norm of g.

Theorem 9. *Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $\alpha \in (0, 1/p)$ and $\beta > 1/p$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} x^\alpha f(x)dx < +\infty, \quad \int_1^{+\infty} x^\beta f(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{x}{y}\right) f(x)g(y)dxdy$$
$$\leq \frac{\pi}{\sin(\pi/q)}\left(\frac{1}{1-\alpha p}\left(\int_0^{+\infty} x^\alpha f(x)dx\right)^p + \frac{1}{\beta p - 1}\left(\int_1^{+\infty} x^\beta f(x)dx\right)^p\right)^{1/p} \times \left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.$$

Proof. Applying Theorem 8 and Proposition 1, we get

$$\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{x}{y}\right) f(x)g(y)dxdy$$
$$\leq \frac{\pi}{\sin(\pi/q)}\left(\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}$$
$$\leq \frac{\pi}{\sin(\pi/q)}\left(\frac{1}{1-\alpha p}\left(\int_0^{+\infty} x^\alpha f(x)dx\right)^p + \frac{1}{\beta p - 1}\left(\int_1^{+\infty} x^\beta f(x)dx\right)^p\right)^{1/p} \times \left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.$$

This concludes the proof. □

The theorem below provides a more conventional upper bound, expressed in terms of the weighted L^p norm of f and the L^q norm of g.

Theorem 10. *Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} f(x)dx < +\infty, \quad \int_0^{+\infty} (xf(x))^p dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{x}{y}\right) f(x)g(y)dxdy \leq \frac{\pi p}{\sin(\pi/q)}\left(\int_0^{+\infty}(xf(x))^p dx\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.$$

Proof. Applying Theorem 8 and Proposition 2, we get

$$\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{x}{y}\right) f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/q)}\left(\int_0^{+\infty}\left(\int_u^{+\infty} f(x)dx\right)^p du\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}$$
$$\leq \frac{\pi p}{\sin(\pi/q)}\left(\int_0^{+\infty}(xf(x))^p dx\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.$$

This completes the proof. □

Another formulation of this inequality is

$$\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{x}{y}\right) f(x)g(y)dxdy \leq \frac{\pi p}{\sin(\pi/p)}\left(\int_0^{+\infty}(xf(x))^p dx\right)^{1/p}\left(\int_0^{+\infty} g^q(x)dx\right)^{1/q}.$$

The theorem below provides another upper bound, with a weighted L^p norm of f and a weighted L^q norm of g.

Theorem 11. *Let $p > 1$ and q be its Hölder conjugate, i.e., $1/p + 1/q = 1$. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that*

$$\int_0^{+\infty} x f^p(x) dx < +\infty, \quad \int_0^{+\infty} x g^q(x) dx < +\infty.$$

Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{x}{y}\right) f(x) g(y) dx dy \leq \Xi \left(\int_0^{+\infty} x f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} x g^q(x) dx\right)^{1/q},$$

where

$$\Xi = \int_0^{\pi/2} \log(1 + \tan(\theta)) \sin^{-2/p}(\theta) \cos^{-2/q}(\theta) d\theta, \tag{2}$$

provided that it exists.

Proof. Making the polar change of variables $x = \rho \sin(\theta)$ and $y = \rho \cos(\theta)$, and applying the Tonelli integral theorem (which is possible because the integrand is non-negative), we obtain

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{x}{y}\right) f(x) g(y) dx dy &= \int_0^{\pi/2} \int_0^{+\infty} \log\left(1 + \frac{\rho \sin(\theta)}{\rho \cos(\theta)}\right) f(\rho \sin(\theta)) g(\rho \cos(\theta)) \rho d\rho d\theta \\ &= \int_0^{\pi/2} \log(1 + \tan(\theta)) \left(\int_0^{+\infty} \rho f(\rho \sin(\theta)) g(\rho \cos(\theta)) d\rho\right) d\theta. \end{aligned}$$

Applying the Hölder integral inequality at the exponents p and $q = p/(p-1)$, we have

$$\begin{aligned} \int_0^{+\infty} \rho f(\rho \sin(\theta)) g(\rho \cos(\theta)) d\rho &= \int_0^{+\infty} \rho^{1/p + 1/q} f(\rho \sin(\theta)) g(\rho \cos(\theta)) d\rho \\ &\leq \left(\int_0^{+\infty} \rho f^p(\rho \sin(\theta)) d\rho\right)^{1/p} \left(\int_0^{+\infty} \rho g^q(\rho \cos(\theta)) d\rho\right)^{1/q}. \end{aligned}$$

Making the changes of variables $u = \rho \sin(\theta)$ and $v = \rho \cos(\theta)$, we find that

$$\begin{aligned} &\left(\int_0^{+\infty} \rho f^p(\rho \sin(\theta)) d\rho\right)^{1/p} \left(\int_0^{+\infty} \rho g^q(\rho \cos(\theta)) d\rho\right)^{1/q} \\ &= \left(\frac{1}{\sin^2(\theta)} \int_0^{+\infty} u f^p(u) du\right)^{1/p} \left(\frac{1}{\cos^2(\theta)} \int_0^{+\infty} v g^q(v) dv\right)^{1/q} \\ &= \sin^{-2/p}(\theta) \cos^{-2/q}(\theta) \left(\int_0^{+\infty} x f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} x g^q(x) dx\right)^{1/q}. \end{aligned}$$

Combining the above equalities and inequalities, we get

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{x}{y}\right) f(x) g(y) dx dy \\ &\leq \int_0^{\pi/2} \left(\log(1 + \tan(\theta)) \sin^{-2/p}(\theta) \cos^{-2/q}(\theta) \left(\int_0^{+\infty} x f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} x g^q(x) dx\right)^{1/q}\right) d\theta \\ &= \left(\int_0^{\pi/2} \log(1 + \tan(\theta)) \sin^{-2/p}(\theta) \cos^{-2/q}(\theta) d\theta\right) \left(\int_0^{+\infty} x f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} x g^q(x) dx\right)^{1/q} \\ &= \Xi \left(\int_0^{+\infty} x f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} x g^q(x) dx\right)^{1/q}, \end{aligned}$$

where Ξ is given in Eq. (2). This ends the proof. □

The constant factor Ξ does not admit a closed-form expression, but it can be accurately approximated using numerical methods

To the best of our knowledge, the established logarithmic-Hardy-Hilbert integral inequalities constitute a new finding. These inequalities have applications in harmonic analysis, approximation theory, operator theory, and the study of boundary value problems in partial differential equations.

4. Conclusion

We have derived new Hardy-Hilbert-type integral inequalities involving arctangent and logarithmic kernel functions. This extends the classical framework and enriches the family of known results. Potential future work could include discrete analogues, multidimensional extensions and applications to singular integral equations and boundary value problems in partial differential equations.

Conflicts of Interest: The author declares no conflict of interest.

Data Availability: No data is required for this research.

Funding Information: No funding is available for this research.

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