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On some properties and integral inequalities for modified (p, h) -convex Stochastic processes

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Abstract: In this paper, we introduce and study the class of modified (p, h) -convex stochastic processes, which unifies and extends several existing notions of convexity in the stochastic setting. We establish fundamental arithmetic properties of this class and derive Hermite–Hadamard-type inequalities using classical and fractional Katugampola-type operators. We also investigated Ostrowski-type and Jensen-type inequalities in a simple and unified framework. Our results generalize and unify many existing results in the literature, providing a comprehensive framework for the study of convex stochastic processes.

Keywords: Hermite–Hadamard inequality, Ostrowski inequality, Jensen inequality, stochastic processes, generalized convexity

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1. Introduction

The theory of stochastic processes has been extensively developed and applied across various fields, including probability theory, optimization, mathematical finance, and engineering. Among the numerous classes of stochastic processes, convex stochastic processes have attracted significant attention due to their rich structural properties and wide range of applications. The concept of convex stochastic processes was first introduced by Nikodem [1], who established fundamental properties and initiated the systematic study of these processes. Since then, various generalizations and extensions have been explored by several researchers.

Skowronski [2] investigated properties of j -convex stochastic processes, while later, Skowronski [3] examined Wright-convex stochastic processes, contributing to the understanding of different convexity notions in the stochastic setting. The Hermite-Hadamard inequality, a classical result in convex analysis, was extended to convex stochastic processes by Kotrys [4], who provided significant insights into the integral inequalities satisfied by such processes. Subsequently, Kotrys [5] further developed the theory by studying strongly convex stochastic processes and establishing related inequalities.

The concept of h -convexity was introduced in the context of stochastic processes by Barrez et al. [6], who explored the properties and inequalities associated with h -convex stochastic processes. This work was later generalized by Saleem et al. [7], who provided broader characterizations and classical inequalities for h -convex stochastic processes.

In recent years, several new classes of convex stochastic processes have been introduced and studied. Okur, Işcan, and Dizdar [8] investigated p -convex stochastic processes and established Hermite-Hadamard type inequalities for this class. Maden, Tomar, and Set [9] studied s -convex stochastic processes in the first sense, while Set, Tomar, and Maden [10] examined s -convex stochastic processes in the second sense. Akdemir, Bekar, and Işcan [11] explored preinvexity for stochastic processes, extending the notion of invexity to the stochastic framework.

The study of exponentially convex and m -convex stochastic processes was undertaken by Özcan [12,13], who derived Hermite-Hadamard type inequalities for these classes. More recently, Fu et al. [14] investigated n -polynomial convex stochastic processes and established related inequalities, further expanding the theory.

The theoretical developments in convex stochastic processes have found applications in various domains. Stochastic fractional calculus with applications to variational principles was explored by Zine and Torres [15]. The connections to optimization problems have been investigated by Chen, Quarteroni, and Rozza [16], who studied stochastic optimal Robin boundary control problems. Cuoco [17] and Cvitanic and Karatzas [18] examined applications in portfolio optimization and equilibrium prices with stochastic income. Xu and Yin [19] studied block stochastic gradient iteration methods for convex and nonconvex optimization problems. The foundational theory of stochastic differential equations and their applications to physics and engineering is comprehensively presented by Sobczyk [20]. Recent advancements in stochastic inequalities have focused on two interconnected generalizations: expanding convexity classes and employing interval-valued analysis. Afzal, Botmart, et al. [22] established foundational estimates for Jensen and Hermite-Hadamard inequalities within the class of h -Godunova-Levin stochastic processes. This framework was later extended to (h_1, h_2) -convex stochastic processes by Afzal, Abbas, et al. [24], utilizing set inclusion relations.

Parallel to these developments, interval-valued methods have been employed to refine inequality estimates. Afzal, Prosviryakov, et al. [23] derived Hermite-Hadamard, Ostrowski, and Jensen-type inclusions for h -convex stochastic processes, while Abbas et al. [25] applied the center-radius (CR) order relation to obtain sharper bounds. Synthesizing these approaches, Afzal, Aloraini, et al. [21] introduced novel Kulisch-Miranker type inclusions for a generalized class of Godunova-Levin stochastic processes, representing a significant advance in computationally tractable interval-based inequalities.

The concept of modified (p, h) -convexity [26], which combines the features of p -convex and h -convex functions, has recently gained attention. This class generalizes several existing notions of convexity and provides a unified framework for studying various types of convex functions and stochastic processes.

In this article, we aim to extend the theory of convex stochastic processes by introducing and studying the class of modified (p, h) -convex stochastic processes. We establish fundamental properties, including closure under addition and scalar multiplication, and derive several important inequalities, such as Hermite-Hadamard type inequalities, Ostrowski-type inequalities, and Jensen-type inequalities. Our results generalize and unify many existing results in the literature, providing a comprehensive framework for the study of convex stochastic processes with applications in various fields.

The paper is organized as follows. In §2, we recall necessary definitions and preliminary results. §3 introduces the class of modified (p, h) -convex stochastic processes and establishes their basic properties. §4 presents Hermite-Hadamard type inequalities and other integral inequalities for these processes. §5 provides concluding remarks and future directions.

2. Preliminaries and background

In this section, we recall some fundamental concepts related to random variables and stochastic processes, together with their regularity properties and several relevant auxiliary results. For more details, we refer to [27].

2.1. Probability space and random variables

Let $(\Pi, \mathcal{A}, \mathbb{P})$ be a probability space. A *random variable* is a measurable mapping $\psi : \Pi \rightarrow \mathbb{R}$, that is, ψ is \mathcal{A} -measurable.

2.2. Stochastic processes

Definition 1. A *stochastic process* is a mapping $\psi : J \times \Pi \rightarrow \mathbb{R}$, where $J \subset \mathbb{R}$ is an index set. For each fixed $t \in J$, the function $\psi(t, \cdot) : \Pi \rightarrow \mathbb{R}$ is a random variable. For each fixed $\omega \in \Pi$, the mapping $t \mapsto \psi(t, \omega)$ is called a *sample path* (or realization) of the process.

2.3. Regularity properties

Let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a stochastic process defined on $(\Pi, \mathcal{A}, \mathbb{P})$.

2.3.1. Continuity in probability

The process ψ is said to be *continuous in probability* at $t_0 \in J$ if for every $\varepsilon > 0$,

$$\lim_{t \rightarrow t_0} \mathbb{P}(|\psi(t, \cdot) - \psi(t_0, \cdot)| > \varepsilon) = 0.$$

2.3.2. Mean square continuity

The process ψ is called *mean square continuous* at $t_0 \in J$ if

$$\lim_{t \rightarrow t_0} \mathbb{E}[(\psi(t, \cdot) - \psi(t_0, \cdot))^2] = 0.$$

Mean square continuity implies continuity in probability, but the converse is not true.

2.3.3. Mean square differentiability

The process ψ is said to be *mean square differentiable* at $t_0 \in J$ if there exists a random variable $\psi'(t_0, \cdot)$ such that

$$\lim_{t \rightarrow t_0} \mathbb{E} \left[\left(\frac{\psi(t, \cdot) - \psi(t_0, \cdot)}{t - t_0} - \psi'(t_0, \cdot) \right)^2 \right] = 0.$$

2.3.4. Normal sequence of partitions

Suppose we are given a sequence $\{\Delta^m\}$ of partitions, $\Delta^m = \{t_{m,0}, t_{m,1}, \dots, t_{m,n_m}\}$. We say that $\{\Delta^m\}$ is a *normal sequence of partitions* if the length of the greatest interval in the m -th partition tends to zero, i.e.,

$$\lim_{m \rightarrow \infty} \sup_{1 \leq i \leq n_m} |t_{m,i} - t_{m,i-1}| = 0.$$

2.3.5. Mean square integrability

Assume that $\mathbb{E}[\psi(t, \cdot)^2] < \infty$ for all $t \in J$. Let $[l, m] \subset J$ and consider a partition $l = t_0 < t_1 < \dots < t_n = m$ with $\xi_k \in [t_{k-1}, t_k]$ for $k = 1, \dots, n$. A random variable $Y : \Pi \rightarrow \mathbb{R}$ is called the *mean square integral* of ψ over $[l, m]$ if

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\sum_{k=1}^n \psi(\xi_k, \cdot)(t_k - t_{k-1}) - Y \right)^2 \right] = 0$$

for every normal sequence of partitions of $[l, m]$. In this case, we write

$$Y(\cdot) = \int_l^m \psi(s, \cdot) ds \quad (\text{a.e.}).$$

For the existence of the mean-square integral, it suffices to assume the mean-square continuity of ψ . Throughout the paper, we will frequently use the monotonicity property: if $\psi(t, \cdot) \leq \phi(t, \cdot)$ (a.e.) on $[l, m]$, then

$$\int_l^m \psi(t, \cdot) dt \leq \int_l^m \phi(t, \cdot) dt \quad (\text{a.e.}).$$

2.4. Some known results

Now, we present some results about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 1. If $\psi : J \times \Pi \rightarrow \mathbb{R}$ is a stochastic process of the form $\psi(t, \cdot) = A(\cdot)t + B(\cdot)$, where $A, B : \Pi \rightarrow \mathbb{R}$ are random variables such that $\mathbb{E}[A^2] < \infty$ and $\mathbb{E}[B^2] < \infty$, and $[l, m] \subset J$, then

$$\int_l^m \psi(t, \cdot) dt = A(\cdot) \frac{m^2 - l^2}{2} + B(\cdot)(m - l) \quad (a.e.).$$

Proposition 1. Let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a convex stochastic process and $t_0 \in \text{int } J$. Then there exists a random variable $A : \Pi \rightarrow \mathbb{R}$ such that ψ is supported at t_0 by the process $A(\cdot)(t - t_0) + \psi(t_0, \cdot)$. That is,

$$\psi(t, \cdot) \geq A(\cdot)(t - t_0) + \psi(t_0, \cdot) \quad (a.e.),$$

for all $t \in J$.

Theorem 1 (Hermite-Hadamard Inequality for Convex Stochastic Processes). Let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a convex, mean-square continuous stochastic process. Then for any $l, m \in J$ with $l < m$, we have

$$\psi\left(\frac{l+m}{2}, \cdot\right) \leq \frac{1}{m-l} \int_l^m \psi(t, \cdot) dt \leq \frac{\psi(l, \cdot) + \psi(m, \cdot)}{2} \quad (a.e.).$$

3. Convex stochastic processes

Let $(\Pi, \mathcal{A}, \mathbb{P})$ be a probability space and let $J \subset \mathbb{R}$ be an interval. A function $\psi : J \times \Pi \rightarrow \mathbb{R}$ is a stochastic process if for every $t \in J$ the function $\psi(t, \cdot)$ is a random variable. All inequalities involving stochastic processes are understood to hold almost everywhere (a.e.).

3.1. Classical convex stochastic process

Definition 2 ([1]). A stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ is called a *convex stochastic process* if

$$\psi(sm + (1-s)n, \cdot) \leq s\psi(m, \cdot) + (1-s)\psi(n, \cdot), \quad (1)$$

for all $m, n \in J$ and $s \in [0, 1]$.

3.2. h -convex Stochastic process

Definition 3 ([6]). Let $h : (0, 1) \rightarrow [0, \infty)$ be a given function. A non-negative stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ is called an *h -convex stochastic process* if

$$\psi(sm + (1-s)n, \cdot) \leq h(s)\psi(m, \cdot) + h(1-s)\psi(n, \cdot), \quad (2)$$

for all $m, n \in J$ and $s \in (0, 1)$.

3.3. p -Convex Set

Definition 4 ([26]). The interval $J \subset \mathbb{R}$ is called a *p -convex set* if for every $m, n \in J$ and $s \in [0, 1]$, we have

$$[sm^p + (1-s)n^p]^{\frac{1}{p}} \in J,$$

where $p = \frac{n}{m}$ with n, m odd positive integers, or $p = 2k + 1$ for $k \in \mathbb{N}$.

3.4. p -convex Stochastic process

Definition 5 ([28]). Let $p > 0$ and let J be a p -convex set. A non-negative stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ is called a p -convex stochastic process if

$$\psi\left(\left(sm^p + (1-s)n^p\right)^{1/p}, \cdot\right) \leq s\psi(m, \cdot) + (1-s)\psi(n, \cdot), \quad (3)$$

for all $m, n \in J$ and $s \in (0, 1)$.

3.5. (p, h) -convex Stochastic process

Definition 6 ([29]). Let $p > 0$ and let $h : (0, 1) \rightarrow [0, \infty)$ be a non-zero function. Assume that $J \subset \mathbb{R}$ is a p -convex set. A non-negative stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ is called a (p, h) -convex stochastic process if

$$\psi\left(\left(sm^p + (1-s)n^p\right)^{1/p}, \cdot\right) \leq h(s)\psi(m, \cdot) + h(1-s)\psi(n, \cdot), \quad (4)$$

for all $m, n \in J$ and $s \in (0, 1)$.

4. Main results

In this section, we first define the idea of a modified (p, h) -convex stochastic process and, utilizing this, we investigate a few structural properties along with some inequalities.

4.1. Modified (p, h) -convex Stochastic process

Definition 7. Let $p > 0$ and let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative and non-zero function. Assume J is a p -convex set. A non-negative stochastic process $\psi : J \times \Pi \rightarrow \mathbb{R}$ is called a *modified (p, h) -convex stochastic process* if, for every $m, n \in J$ and $s \in (0, 1)$, the following inequality holds:

$$\psi\left(\left[sm^p + (1-s)n^p\right]^{\frac{1}{p}}, \cdot\right) \leq h(s)\psi(m, \cdot) + (1-h(s))\psi(n, \cdot). \quad (5)$$

The modified (p, h) -convex stochastic process defined above generalizes and unifies several existing notions of convex stochastic processes. Specifically, by appropriate choices of the parameters p and the function h , we can recover the following classical definitions:

- *Classical convex stochastic process:* Take $p = 1$ and $h(s) = s$ for all $s \in (0, 1)$. Then inequality (1) reduces to

$$\psi(sm + (1-s)n, \cdot) \leq s\psi(m, \cdot) + (1-s)\psi(n, \cdot),$$

which is exactly the definition of a classical convex stochastic process [1].

- *p -convex stochastic process:* Take $h(s) = s$ for all $s \in (0, 1)$ while keeping $p > 0$. Then (1) becomes

$$\psi\left(\left[sm^p + (1-s)n^p\right]^{\frac{1}{p}}, \cdot\right) \leq s\psi(m, \cdot) + (1-s)\psi(n, \cdot),$$

which is precisely the definition of a p -convex stochastic process [28].

Proposition 2. If $h_1, h_2 : J \subset \mathbb{R} \rightarrow \mathbb{R}$ are non-negative functions with $h_2(s) \leq h_1(s)$ for all $s \in (0, 1)$ and $\psi : J \times \Pi \rightarrow \mathbb{R}$ is a non-negative modified (p, h_2) -convex stochastic process, then ψ is a modified (p, h_1) -convex stochastic process.

Proof. Consider $m, n \in J$ and $s \in (0, 1)$ arbitrary. Then,

$$\begin{aligned} \psi\left(\left[sm^p + (1-s)n^p\right]^{\frac{1}{p}}, \cdot\right) &\leq h_2(s)\psi(m, \cdot) + (1-h_2(s))\psi(n, \cdot) \\ &\leq h_1(s)\psi(m, \cdot) + (1-h_1(s))\psi(n, \cdot) \quad (\text{a.e.}) \end{aligned}$$

□

Proposition 3. Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function. If $\psi, \phi : J \times \Pi \rightarrow \mathbb{R}$ are modified (p, h) -convex stochastic processes, then $\psi + \phi$ is a modified (p, h) -convex stochastic process. Also, if $\alpha > 0$ then $\alpha\psi$ is a modified (p, h) -convex stochastic process.

Proof. Consider $m, n \in J$ and $s \in (0, 1)$ arbitrary.

$$\begin{aligned} (\psi + \phi) \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right) &= \psi \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right) + \phi \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right) \\ &\leq h(s) (\psi(m, \cdot) + \phi(m, \cdot)) + (1-h(s)) (\psi(n, \cdot) + \phi(n, \cdot)) \\ &= h(s)(\psi + \phi)(m, \cdot) + (1-h(s))(\psi + \phi)(n, \cdot) \quad (\text{a.e.}) \end{aligned}$$

Now, consider $\alpha > 0$. Then,

$$\begin{aligned} \alpha\psi \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right) &\leq \alpha h(s)\psi(m, \cdot) + \alpha(1-h(s))\psi(n, \cdot) \\ &= h(s)\alpha\psi(m, \cdot) + (1-h(s))\alpha\psi(n, \cdot) \quad (\text{a.e.}) \end{aligned}$$

□

Proposition 4. Let $h_1, h_2 : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be non-negative functions and $\psi, \phi : J \times \Pi \rightarrow \mathbb{R}$ non-negative stochastic processes such that

$$(\psi(m, \cdot) - \psi(n, \cdot)) (\phi(m, \cdot) - \phi(n, \cdot)) \geq 0,$$

for all $m, n \in J$. If ψ is modified (p, h_1) -convex, ϕ is modified (p, h_2) -convex and $h(s) + (1-h(s)) \leq c$ for all $s \in (0, 1)$, where $h(s) = \max\{h_1(s), h_2(s)\}$ and c is a fixed positive number, then the product $\psi\phi$ is a modified (p, ch) -convex stochastic process.

Proof. Fix $m, n \in J$ and $s \in (0, 1)$.

First, note that $(\psi(m, \cdot) - \psi(n, \cdot)) (\phi(m, \cdot) - \phi(n, \cdot)) \geq 0$ implies

$$\psi(m, \cdot)\phi(n, \cdot) + \psi(n, \cdot)\phi(m, \cdot) \leq \psi(m, \cdot)\phi(m, \cdot) + \psi(n, \cdot)\phi(n, \cdot).$$

Hence,

$$\begin{aligned} \psi\phi \left([sm^p + (1-s)n^p]^{\frac{1}{p}}, \cdot \right) &\leq (h_1(s)\psi(m, \cdot) + (1-h_1(s))\psi(n, \cdot)) (h_2(s)\phi(m, \cdot) + (1-h_2(s))\phi(n, \cdot)) \\ &\leq (h(s)\psi(m, \cdot) + (1-h(s))\psi(n, \cdot)) (h(s)\phi(m, \cdot) + (1-h(s))\phi(n, \cdot)) \\ &\leq h^2(s)\psi\phi(m, \cdot) + h(s)(1-h(s))\psi\phi(m, \cdot) + h(s)(1-h(s))\psi\phi(n, \cdot) + (1-h(s))^2\psi\phi(n, \cdot) \\ &= (h(s) + (1-h(s))) (h(s)\psi\phi(m, \cdot) + (1-h(s))\psi\phi(n, \cdot)) \\ &\leq c h(s)(\psi\phi)(m, \cdot) + c(1-h(s))(\psi\phi)(n, \cdot) \quad (\text{a.e.}) \end{aligned}$$

□

Theorem 2. Let J be a p -convex interval such that $0 \in J$ and $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ a non-negative function. If h is supermultiplicative and $\psi : J \times \Pi \rightarrow \mathbb{R}$ is a modified (p, h) -convex stochastic process such that $\psi(0, \cdot) = 0$, then the inequality

$$\psi \left([\lambda m^p + \beta n^p]^{\frac{1}{p}}, \cdot \right) \leq h(\lambda)\psi(m, \cdot) + (1-h(\beta))\psi(n, \cdot),$$

holds almost everywhere for all $m, n \in J$ and all $\lambda, \beta > 0$ such that $\lambda + \beta \leq 1$.

Proof. If $\lambda + \beta = 1$, the inequality holds because of the definition of modified (p, h) -convexity in stochastic processes. Let $\lambda, \beta > 0$ be numbers such that $\lambda + \beta = \gamma$ with $\gamma < 1$. Let us define numbers $a := \frac{\lambda}{\gamma}$ and $b := \frac{\beta}{\gamma}$. Then, $a + b = 1$ and we have the following:

$$\begin{aligned} \psi\left([\lambda m^p + \beta n^p]^{\frac{1}{p}}, \cdot\right) &= \psi\left([a\gamma m^p + b\gamma n^p]^{\frac{1}{p}}, \cdot\right) \\ &\leq h(a)\psi\left([\gamma m^p + 0 \cdot n^p]^{\frac{1}{p}}, \cdot\right) + (1 - h(a))\psi\left([\gamma n^p + 0 \cdot m^p]^{\frac{1}{p}}, \cdot\right) \\ &= h(a)\psi\left([\gamma m^p + (1 - \gamma)0^p]^{\frac{1}{p}}, \cdot\right) + (1 - h(a))\psi\left([\gamma n^p + (1 - \gamma)0^p]^{\frac{1}{p}}, \cdot\right) \\ &\leq h(a)(h(\gamma)\psi(m, \cdot) + (1 - h(\gamma))\psi(0, \cdot)) + (1 - h(a))(h(\gamma)\psi(n, \cdot) + (1 - h(\gamma))\psi(0, \cdot)) \\ &= h(a)h(\gamma)\psi(m, \cdot) + (1 - h(a))h(\gamma)\psi(n, \cdot) \\ &\leq h(a\gamma)\psi(m, \cdot) + (1 - h(a\gamma))\psi(n, \cdot) \\ &= h(\lambda)\psi(m, \cdot) + (1 - h(\lambda))\psi(n, \cdot) \quad (\text{a.e.}) \end{aligned}$$

□

Proposition 5. Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative supermultiplicative function and let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a modified (p, h) -convex stochastic process. Then, for $m, n, r \in J$, with $m < n < r$ such that $r - m, r - n, n - m \in J$ the following inequality holds almost everywhere,

$$h(r - n)\psi(m, \cdot) - h(r - m)\psi(n, \cdot) + h(n - m)\psi(r, \cdot) \geq 0.$$

Proof. Let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a modified (p, h) -convex stochastic process and $m, n, r \in J$ be numbers which satisfy assumptions of the proposition. Then, we have $\frac{r-n}{r-m}, \frac{n-m}{r-m} \in (0, 1)$ and $\frac{r-n}{r-m} + \frac{n-m}{r-m} = 1$. Also, since h is supermultiplicative and non-negative,

$$\begin{aligned} h(r - n) &= h\left(\frac{r - n}{r - m} \cdot (r - m)\right) \geq h\left(\frac{r - n}{r - m}\right) h(r - m), \\ h(n - m) &= h\left(\frac{n - m}{r - m} \cdot (r - m)\right) \geq h\left(\frac{n - m}{r - m}\right) h(r - m). \end{aligned}$$

Let $h(r - m) > 0$. Because of the modified (p, h) -convexity, ψ satisfies

$$\psi\left([sz_1^p + (1 - s)z_2^p]^{\frac{1}{p}}, \cdot\right) \leq h(s)\psi(z_1, \cdot) + (1 - h(s))\psi(z_2, \cdot) \quad (\text{a.e.}),$$

for all $z_1, z_2 \in J, s \in (0, 1)$. In particular, for $s = \frac{r-n}{r-m}, z_1 = m, z_2 = r$, we have $n = [sm^p + (1 - s)r^p]^{\frac{1}{p}}$ and

$$\begin{aligned} \psi(n, \cdot) &\leq h\left(\frac{r - n}{r - m}\right)\psi(m, \cdot) + \left(1 - h\left(\frac{r - n}{r - m}\right)\right)\psi(r, \cdot) \\ &\leq \frac{h(r - n)}{h(r - m)}\psi(m, \cdot) + \left(1 - \frac{h(r - n)}{h(r - m)}\right)\psi(r, \cdot). \end{aligned}$$

Finally, multiplying by $h(r - m)$ we obtain the following

$$h(r - m)\psi(n, \cdot) \leq h(r - n)\psi(m, \cdot) + (h(r - m) - h(r - n))\psi(r, \cdot).$$

That is,

$$0 \leq h(r - n)\psi(m, \cdot) - h(r - m)\psi(n, \cdot) + (h(r - m) - h(r - n))\psi(r, \cdot).$$

Note: The expression simplifies to the desired form only when $h(r - m) - h(r - n) = h(n - m)$, which holds under the supermultiplicative property and the condition $h(r - m) = h(r - n + n - m) \geq h(r - n)h(n - m)$. □

The equality stated in the following lemma is fundamental for proving the results related to Ostrowski-type inequalities. We mainly extend the results for the h -convex stochastic process (Lemma 2.1) in [30] to the case of the modified (p, h) -convex stochastic process.

Lemma 2. *Let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a stochastic process mean-square differentiable on J° . If ψ' is mean-square integrable on $[a, b]$, where $a, b \in J$ with $a < b$, then the following equality holds:*

$$\begin{aligned} \psi(t, \cdot) - \frac{1}{b-a} \int_a^b \psi(u, \cdot) du &= \frac{(t-a)^2}{b-a} \int_0^1 s \psi' \left([st^p + (1-s)a^p]^{\frac{1}{p}}, \cdot \right) ds \\ &\quad - \frac{(b-t)^2}{b-a} \int_0^1 s \psi' \left([st^p + (1-s)b^p]^{\frac{1}{p}}, \cdot \right) ds, \end{aligned}$$

for each $t \in [a, b]$.

Proof. Changing variables, $u = [st^p + (1-s)a^p]^{\frac{1}{p}}$ and $w = [st^p + (1-s)b^p]^{\frac{1}{p}}$ and integrating by parts, we have

$$\begin{aligned} &\frac{(t-a)^2}{b-a} \int_0^1 s \psi' \left([st^p + (1-s)a^p]^{\frac{1}{p}}, \cdot \right) ds - \frac{(b-t)^2}{b-a} \int_0^1 s \psi' \left([st^p + (1-s)b^p]^{\frac{1}{p}}, \cdot \right) ds \\ &= \frac{(t-a)^2}{b-a} \int_a^t \frac{(u-a)}{(t-a)} \psi'(u, \cdot) \frac{du}{(t-a)} - \frac{(b-t)^2}{b-a} \int_t^b \frac{(b-w)}{(b-t)} \psi'(w, \cdot) \frac{dw}{(b-t)} \\ &= \frac{1}{b-a} \int_a^t (u-a) \psi'(u, \cdot) du - \frac{1}{b-a} \int_t^b (b-w) \psi'(w, \cdot) dw \\ &= \frac{1}{b-a} \left[(t-a) \psi(t, \cdot) - \int_a^t \psi(u, \cdot) du \right] + \frac{1}{b-a} \left[(b-t) \psi(t, \cdot) - \int_t^b \psi(w, \cdot) dw \right] \\ &= \frac{1}{b-a} \left[(t-a) \psi(t, \cdot) - \int_a^t \psi(u, \cdot) du + (b-t) \psi(t, \cdot) - \int_t^b \psi(w, \cdot) dw \right] \\ &= \frac{1}{b-a} \left[\psi(t, \cdot)(b-a) - \int_a^b \psi(u, \cdot) du \right]. \end{aligned}$$

□

Now, we are ready to present some Ostrowski-type inequalities for the modified (p, h) -convex stochastic process. Mainly, we improve Theorem 2.2 in [30].

Theorem 3. *Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative and supermultiplicative function such that $h(\alpha) > \alpha$ for every α and let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a mean-square stochastic process such that ψ' is mean-square integrable on $[a, b]$, where $a, b \in J$ with $a < b$. If $|\psi'|$ is a modified (p, h) -convex stochastic process on J and $|\psi'(t, \cdot)| \leq M$ for every t , then*

$$\left| \psi(t, \cdot) - \frac{1}{b-a} \int_a^b \psi(u, \cdot) du \right| \leq \frac{M [(t-a)^2 + (b-t)^2]}{b-a} \cdot \frac{1}{2}.$$

for each $t \in [a, b]$.

Proof. By Lemma 2 and since $|\psi'|$ is modified (p, h) -convex, then we can write

$$\begin{aligned} \left| \psi(t, \cdot) - \frac{1}{b-a} \int_a^b \psi(u, \cdot) du \right| &\leq \frac{(t-a)^2}{b-a} \int_0^1 s \left| \psi' \left([st^p + (1-s)a^p]^{\frac{1}{p}}, \cdot \right) \right| ds \\ &\quad + \frac{(b-t)^2}{b-a} \int_0^1 s \left| \psi' \left([st^p + (1-s)b^p]^{\frac{1}{p}}, \cdot \right) \right| ds \\ &\leq \frac{(t-a)^2}{b-a} \int_0^1 s [h(s) |\psi'(t, \cdot)| + (1-h(s)) |\psi'(a, \cdot)|] ds \\ &\quad + \frac{(b-t)^2}{b-a} \int_0^1 s [h(s) |\psi'(t, \cdot)| + (1-h(s)) |\psi'(b, \cdot)|] ds \\ &\leq \frac{M(t-a)^2}{b-a} \int_0^1 [sh(s) + s(1-h(s))] ds + \frac{M(b-t)^2}{b-a} \int_0^1 [sh(s) + s(1-h(s))] ds \end{aligned}$$

$$\begin{aligned} &= \frac{M(t-a)^2}{b-a} \int_0^1 s \, ds + \frac{M(b-t)^2}{b-a} \int_0^1 s \, ds \\ &= \frac{M[(t-a)^2 + (b-t)^2]}{b-a} \cdot \frac{1}{2}. \end{aligned}$$

□

Now we prove a Jensen-type inequality for the modified (p, h) -convex stochastic process.

Theorem 4. Assume $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a modified (p, h) -convex stochastic process and h be non-negative supermultiplicative function. If $T_i = \sum_{j=1}^i \alpha_j$ for $i = 1, \dots, m$, so that $T_m = 1$ where $m \in \mathbb{N}$, then

$$\psi \left(\left(\sum_{i=1}^m \alpha_i r_i^p \right)^{\frac{1}{p}}, \cdot \right) \leq \psi(r_m, \cdot) + \sum_{i=1}^{m-1} h(T_i) \{ \psi(r_i, \cdot) - \psi(r_{i+1}, \cdot) \}.$$

Proof. By the definition of modified (p, h) -convex stochastic process it follows that:

$$\begin{aligned} \psi \left(\left(\sum_{i=1}^m \alpha_i r_i^p \right)^{\frac{1}{p}}, \cdot \right) &\leq h(T_{m-1}) \psi \left(\left(\sum_{i=1}^{m-1} \frac{\alpha_i}{T_{m-1}} r_i^p \right)^{\frac{1}{p}}, \cdot \right) + (1 - h(T_{m-1})) \psi(r_m, \cdot) \\ &= (1 - h(T_{m-1})) \psi(r_m, \cdot) + h(T_{m-1}) \psi \left(\left[\frac{T_{m-2}}{T_{m-1}} \left(\left(\sum_{i=1}^{m-2} \frac{\alpha_i}{T_{m-2}} r_i^p \right)^{\frac{1}{p}} \right)^p + \frac{\alpha_{m-1}}{T_{m-1}} r_{m-1}^p \right]^{\frac{1}{p}}, \cdot \right) \\ &\leq (1 - h(T_{m-1})) \psi(r_m, \cdot) + h(T_{m-1}) \left[h \left(\frac{T_{m-2}}{T_{m-1}} \right) \psi \left(\left(\sum_{i=1}^{m-2} \frac{\alpha_i}{T_{m-2}} r_i^p \right)^{\frac{1}{p}}, \cdot \right) \right. \\ &\quad \left. + \left(1 - h \left(\frac{T_{m-2}}{T_{m-1}} \right) \right) \psi(r_{m-1}, \cdot) \right]. \end{aligned}$$

Using the fact that h is supermultiplicative, we have

$$\begin{aligned} &\leq (1 - h(T_{m-1})) \psi(r_m, \cdot) + h(T_{m-2}) \psi \left(\left(\sum_{i=1}^{m-2} \frac{\alpha_i}{T_{m-2}} r_i^p \right)^{\frac{1}{p}}, \cdot \right) + h(T_{m-1}) \left(1 - h \left(\frac{T_{m-2}}{T_{m-1}} \right) \right) \psi(r_{m-1}, \cdot) \\ &= \psi(r_m, \cdot) + h(T_{m-1}) \psi(r_{m-1}, \cdot) - h(T_{m-1}) \psi(r_m, \cdot) - h(T_{m-2}) \psi(r_{m-1}, \cdot) \\ &\quad + h(T_{m-2}) \psi \left(\left(\sum_{i=1}^{m-2} \frac{\alpha_i}{T_{m-2}} r_i^p \right)^{\frac{1}{p}}, \cdot \right) \\ &= \psi(r_m, \cdot) + h(T_{m-1}) [\psi(r_{m-1}, \cdot) - \psi(r_m, \cdot)] - h(T_{m-2}) \psi(r_{m-1}, \cdot) \\ &\quad + h(T_{m-2}) \psi \left(\left[\frac{T_{m-3}}{T_{m-2}} \left(\left(\sum_{i=1}^{m-3} \frac{\alpha_i}{T_{m-3}} r_i^p \right)^{\frac{1}{p}} \right)^p + \frac{\alpha_{m-2}}{T_{m-2}} r_{m-2}^p \right]^{\frac{1}{p}}, \cdot \right). \end{aligned}$$

From the definition of modified (p, h) -convexity, we get

$$\begin{aligned} &\leq \psi(r_m, \cdot) + h(T_{m-1}) [\psi(r_{m-1}, \cdot) - \psi(r_m, \cdot)] - h(T_{m-2}) \psi(r_{m-1}, \cdot) \\ &\quad + h(T_{m-2}) \left[h \left(\frac{T_{m-3}}{T_{m-2}} \right) \psi \left(\left(\sum_{i=1}^{m-3} \frac{\alpha_i}{T_{m-3}} r_i^p \right)^{\frac{1}{p}}, \cdot \right) + \left(1 - h \left(\frac{T_{m-3}}{T_{m-2}} \right) \right) \psi(r_{m-2}, \cdot) \right]. \end{aligned}$$

Using the fact that h is supermultiplicative,

$$\begin{aligned} &\leq \psi(r_m, \cdot) + h(T_{m-1}) [\psi(r_{m-1}, \cdot) - \psi(r_m, \cdot)] - h(T_{m-2})\psi(r_{m-1}, \cdot) \\ &\quad + h(T_{m-3}) \psi \left(\left(\sum_{i=1}^{m-3} \frac{\alpha_i}{T_{m-3}} r_i^p \right)^{\frac{1}{p}}, \cdot \right) + h(T_{m-2})\psi(r_{m-2}, \cdot) - h(T_{m-3})\psi(r_{m-2}, \cdot) \\ &= \psi(r_m, \cdot) + h(T_{m-1}) [\psi(r_{m-1}, \cdot) - \psi(r_m, \cdot)] + h(T_{m-2}) [\psi(r_{m-2}, \cdot) - \psi(r_{m-1}, \cdot)] \\ &\quad - h(T_{m-3})\psi(r_{m-2}, \cdot) + h(T_{m-3}) \psi \left(\left(\sum_{i=1}^{m-3} \frac{\alpha_i}{T_{m-3}} r_i^p \right)^{\frac{1}{p}}, \cdot \right) \\ &\leq \dots \\ &\leq \psi(r_m, \cdot) + h(T_{m-1}) [\psi(r_{m-1}, \cdot) - \psi(r_m, \cdot)] + h(T_{m-2}) [\psi(r_{m-2}, \cdot) - \psi(r_{m-1}, \cdot)] \\ &\quad + h(T_{m-3}) [\psi(r_{m-3}, \cdot) - \psi(r_{m-2}, \cdot)] + \dots \\ &\quad + h(T_2) [\psi(r_2, \cdot) - \psi(r_3, \cdot)] + h(T_1) [\psi(r_1, \cdot) - \psi(r_2, \cdot)] \\ &= \psi(r_m, \cdot) + \sum_{i=1}^{m-1} h(T_i) [\psi(r_i, \cdot) - \psi(r_{i+1}, \cdot)]. \end{aligned}$$

The result is completed. \square

Corollary 1. If we take $p = 1$ and $h(s) = s$ in Theorem 4, we obtain the Jensen-type inequality for the classical convex stochastic process:

$$\psi \left(\sum_{i=1}^m \alpha_i r_i, \cdot \right) \leq \psi(r_m, \cdot) + \sum_{i=1}^{m-1} T_i \{ \psi(r_i, \cdot) - \psi(r_{i+1}, \cdot) \},$$

where $T_i = \sum_{j=1}^i \alpha_j$ with $T_m = 1$ and $m \in \mathbb{N}$.

Corollary 2. For the p -convex stochastic process, set $h(s) = s$ in Theorem 4. Then we have:

$$\psi \left(\left(\sum_{i=1}^m \alpha_i r_i^p \right)^{\frac{1}{p}}, \cdot \right) \leq \psi(r_m, \cdot) + \sum_{i=1}^{m-1} T_i \{ \psi(r_i, \cdot) - \psi(r_{i+1}, \cdot) \},$$

where $T_i = \sum_{j=1}^i \alpha_j$ with $T_m = 1$ and $m \in \mathbb{N}$.

We have now established the Hermite–Hadamard inequality for modified (p, h) -convex stochastic processes, utilizing classical and fractional integral operators.

Theorem 5. Assume $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a modified (p, h) -convex stochastic process on the interval $[l, m]$ with $l < m$. Then we have

$$\begin{aligned} \psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) &\leq \left(\frac{p}{m^p - l^p} \right) \int_l^m r^{p-1} \psi(r, \cdot) dr \\ &\leq \psi(l, \cdot) + \{ \psi(m, \cdot) - \psi(l, \cdot) \} \int_0^1 h(s) ds. \end{aligned}$$

Proof. Let $u^p = sl^p + (1 - s)m^p$ and $v^p = (1 - s)l^p + sm^p$. Then we get

$$\begin{aligned} \psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) &= \psi \left(\left(\frac{u^p + v^p}{2} \right)^{\frac{1}{p}}, \cdot \right) \\ &= \psi \left(\left(\frac{(sl^p + (1 - s)m^p) + ((1 - s)l^p + sm^p)}{2} \right)^{\frac{1}{p}}, \cdot \right) \\ &\leq h \left(\frac{1}{2} \right) \psi \left((sl^p + (1 - s)m^p)^{\frac{1}{p}}, \cdot \right) + \left(1 - h \left(\frac{1}{2} \right) \right) \psi \left(((1 - s)l^p + sm^p)^{\frac{1}{p}}, \cdot \right). \end{aligned}$$

Integrating the inequality over $s \in [0, 1]$, we get

$$\begin{aligned} \psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) &\leq h \left(\frac{1}{2} \right) \int_0^1 \psi \left((sl^p + (1-s)m^p)^{\frac{1}{p}}, \cdot \right) ds + \left(1 - h \left(\frac{1}{2} \right) \right) \int_0^1 \psi \left(((1-s)l^p + sm^p)^{\frac{1}{p}}, \cdot \right) ds \\ &\leq h \left(\frac{1}{2} \right) \frac{p}{m^p - l^p} \int_l^m r^{p-1} \psi(r, \cdot) dr + \frac{p}{m^p - l^p} \left(1 - h \left(\frac{1}{2} \right) \right) \int_l^m r^{p-1} \psi(r, \cdot) dr \\ &= \frac{p}{m^p - l^p} \int_l^m r^{p-1} \psi(r, \cdot) dr. \end{aligned}$$

Now, we know that

$$\begin{aligned} \int_l^m r^{p-1} \psi(r, \cdot) dr &= \frac{m^p - l^p}{p} \int_0^1 \psi \left((sm^p + (1-s)l^p)^{\frac{1}{p}}, \cdot \right) ds \\ &\leq \frac{m^p - l^p}{p} \int_0^1 [h(s)\psi(m, \cdot) + (1-h(s))\psi(l, \cdot)] ds \\ &= \frac{m^p - l^p}{p} \left[\psi(l, \cdot) + \{\psi(m, \cdot) - \psi(l, \cdot)\} \int_0^1 h(s) ds \right]. \end{aligned}$$

Thus,

$$\frac{p}{m^p - l^p} \int_l^m r^{p-1} \psi(r, \cdot) dr \leq \psi(l, \cdot) + \{\psi(m, \cdot) - \psi(l, \cdot)\} \int_0^1 h(s) ds.$$

Combining the two inequalities, we obtain the required result. \square

Now, before developing the fractional variant, we first extend the classical Katugampola operator [31] into its mean square stochastic form.

Definition 8 (Katugampola Fractional Integral for Stochastic Processes). Let $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a stochastic process. Then, the mean-square continuous Katugampola fractional integrals ${}^\rho J_{l^+}^\alpha \psi$ and ${}^\rho J_{m^-}^\alpha \psi$ of order $\alpha > 0$ and $\rho > 0$ are defined by

$${}^\rho J_{l^+}^\alpha \psi(x, \cdot) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_l^x (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1} \psi(t, \cdot) dt, \quad (\text{a.e.}) \quad (0 \leq l < x < m),$$

and

$${}^\rho J_{m^-}^\alpha \psi(x, \cdot) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^m (t^\rho - x^\rho)^{\alpha-1} t^{\rho-1} \psi(t, \cdot) dt, \quad (\text{a.e.}) \quad (0 \leq l < x < m),$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ is the Gamma function.

Theorem 6. Assume $\psi : J \times \Pi \rightarrow \mathbb{R}$ be a modified (p, h) -convex stochastic process and $\psi \in L_1[l, m]$, with $l < m$ and $p, \rho > 0$. Then

$$\frac{1}{\alpha} \psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) \leq \frac{\rho^{\alpha-1} \Gamma(\alpha)}{(m^p - l^p)^\alpha} \left[\left(1 - h \left(\frac{1}{2} \right) \right) {}^\rho J_{l^+}^\alpha \psi(m, \cdot) + h \left(\frac{1}{2} \right) {}^\rho J_{m^-}^\alpha \psi(l, \cdot) \right],$$

and

$$\frac{\rho^{\alpha-1} \Gamma(\alpha)}{(m^p - l^p)^\alpha} [{}^\rho J_{l^+}^\alpha \psi(m, \cdot) + {}^\rho J_{m^-}^\alpha \psi(l, \cdot)] \leq \frac{\psi(l, \cdot) + \psi(m, \cdot)}{\alpha}.$$

Proof. We know that ψ is a modified (p, h) -convex stochastic process, so we have

$$\psi \left(\left(\frac{r^p + s^p}{2} \right)^{\frac{1}{p}}, \cdot \right) \leq \left\{ 1 - h \left(\frac{1}{2} \right) \right\} \psi(r, \cdot) + h \left(\frac{1}{2} \right) \psi(s, \cdot).$$

Let $r^p = (tl^p + (1 - t)m^p)$ and $s^p = ((1 - t)l^p + tm^p)$. Then

$$\psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) \leq \left\{ 1 - h \left(\frac{1}{2} \right) \right\} \psi \left((tl^p + (1 - t)m^p)^{\frac{1}{p}}, \cdot \right) + h \left(\frac{1}{2} \right) \psi \left(((1 - t)l^p + tm^p)^{\frac{1}{p}}, \cdot \right).$$

Multiplying the above inequality by $t^{\alpha-1}$, then integrating over $t \in [0, 1]$, we get

$$\frac{1}{\alpha} \psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) = \psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) \int_0^1 t^{\alpha-1} dt \leq \left\{ 1 - h \left(\frac{1}{2} \right) \right\} \int_0^1 t^{\alpha-1} \psi \left((tl^p + (1 - t)m^p)^{\frac{1}{p}}, \cdot \right) dt + h \left(\frac{1}{2} \right) \int_0^1 t^{\alpha-1} \psi \left(((1 - t)l^p + tm^p)^{\frac{1}{p}}, \cdot \right) dt.$$

Now, using the substitution $u^p = tl^p + (1 - t)m^p$, we have $u \in [l, m]$. Then

$$du = \frac{l^p - m^p}{pu^{p-1}} dt = -\frac{m^p - l^p}{pu^{p-1}} dt,$$

so $dt = \frac{pu^{p-1}}{m^p - l^p} du$. Also, note that $t = \frac{m^p - u^p}{m^p - l^p}$ and $t^{\alpha-1} = \left(\frac{m^p - u^p}{m^p - l^p} \right)^{\alpha-1}$. Thus,

$$\int_0^1 t^{\alpha-1} \psi \left((tl^p + (1 - t)m^p)^{\frac{1}{p}}, \cdot \right) dt = \int_l^m \left(\frac{m^p - u^p}{m^p - l^p} \right)^{\alpha-1} \psi(u, \cdot) \frac{pu^{p-1}}{m^p - l^p} du.$$

Similarly, using the substitution $v^p = (1 - t)l^p + tm^p$, we have $v \in [l, m]$, $t = \frac{v^p - l^p}{m^p - l^p}$, and

$$\int_0^1 t^{\alpha-1} \psi \left(((1 - t)l^p + tm^p)^{\frac{1}{p}}, \cdot \right) dt = \int_l^m \left(\frac{v^p - l^p}{m^p - l^p} \right)^{\alpha-1} \psi(v, \cdot) \frac{pv^{p-1}}{m^p - l^p} dv.$$

Now, using the Katugampola fractional integral definition with parameter ρ , we have

$${}^{\rho} J_{l^+}^{\alpha} \psi(m, \cdot) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_l^m (m^{\rho} - u^{\rho})^{\alpha-1} u^{\rho-1} \psi(u, \cdot) du,$$

and

$${}^{\rho} J_{m^-}^{\alpha} \psi(l, \cdot) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_l^m (v^{\rho} - l^{\rho})^{\alpha-1} v^{\rho-1} \psi(v, \cdot) dv.$$

To connect these with our integrals, we set $\rho = p$. Then,

$$\begin{aligned} \int_l^m \left(\frac{m^p - u^p}{m^p - l^p} \right)^{\alpha-1} \psi(u, \cdot) \frac{pu^{p-1}}{m^p - l^p} du &= \frac{p}{(m^p - l^p)^{\alpha}} \int_l^m (m^p - u^p)^{\alpha-1} u^{p-1} \psi(u, \cdot) du \\ &= \frac{p}{(m^p - l^p)^{\alpha}} \cdot \frac{\Gamma(\alpha)}{p^{1-\alpha}} {}^p J_{l^+}^{\alpha} \psi(m, \cdot) \\ &= \frac{p^{\alpha} \Gamma(\alpha)}{(m^p - l^p)^{\alpha}} {}^p J_{l^+}^{\alpha} \psi(m, \cdot). \end{aligned}$$

Similarly,

$$\int_l^m \left(\frac{v^p - l^p}{m^p - l^p} \right)^{\alpha-1} \psi(v, \cdot) \frac{pv^{p-1}}{m^p - l^p} dv = \frac{p^{\alpha} \Gamma(\alpha)}{(m^p - l^p)^{\alpha}} {}^p J_{m^-}^{\alpha} \psi(l, \cdot).$$

Substituting back, we obtain

$$\begin{aligned} \frac{1}{\alpha} \psi \left(\left(\frac{l^p + m^p}{2} \right)^{\frac{1}{p}}, \cdot \right) &\leq \left\{ 1 - h \left(\frac{1}{2} \right) \right\} \frac{p^{\alpha} \Gamma(\alpha)}{(m^p - l^p)^{\alpha}} {}^p J_{l^+}^{\alpha} \psi(m, \cdot) + h \left(\frac{1}{2} \right) \frac{p^{\alpha} \Gamma(\alpha)}{(m^p - l^p)^{\alpha}} {}^p J_{m^-}^{\alpha} \psi(l, \cdot) \\ &= \frac{p^{\alpha} \Gamma(\alpha)}{(m^p - l^p)^{\alpha}} \left[\left(1 - h \left(\frac{1}{2} \right) \right) {}^p J_{l^+}^{\alpha} \psi(m, \cdot) + h \left(\frac{1}{2} \right) {}^p J_{m^-}^{\alpha} \psi(l, \cdot) \right]. \end{aligned}$$

This proves the first inequality.

Now, we know that ψ is a modified (p, h) -convex stochastic process, then

$$\begin{aligned} & \psi \left((tl^p + (1-t)m^p)^{\frac{1}{p}}, \cdot \right) + \psi \left(((1-t)l^p + tm^p)^{\frac{1}{p}}, \cdot \right) \\ & \leq h(t)\psi(l, \cdot) + (1-h(t))\psi(m, \cdot) + (1-h(t))\psi(l, \cdot) + h(t)\psi(m, \cdot) \\ & = \psi(l, \cdot) + \psi(m, \cdot). \end{aligned}$$

Multiplying the above inequality by $t^{\alpha-1}$ and then integrating over $t \in [0, 1]$, we get

$$\int_0^1 t^{\alpha-1} \psi \left((tl^p + (1-t)m^p)^{\frac{1}{p}}, \cdot \right) dt + \int_0^1 t^{\alpha-1} \psi \left(((1-t)l^p + tm^p)^{\frac{1}{p}}, \cdot \right) dt \leq [\psi(l, \cdot) + \psi(m, \cdot)] \int_0^1 t^{\alpha-1} dt.$$

Now, using the substitution $u^p = tl^p + (1-t)m^p$ in the first integral, we have $u \in [l, m]$, $t = \frac{m^p - u^p}{m^p - l^p}$, and $dt = \frac{pu^{p-1}}{m^p - l^p} du$. Thus,

$$\int_0^1 t^{\alpha-1} \psi \left((tl^p + (1-t)m^p)^{\frac{1}{p}}, \cdot \right) dt = \int_l^m \left(\frac{m^p - u^p}{m^p - l^p} \right)^{\alpha-1} \psi(u, \cdot) \frac{pu^{p-1}}{m^p - l^p} du.$$

Similarly, using the substitution $v^p = (1-t)l^p + tm^p$ in the second integral, we have $v \in [l, m]$, $t = \frac{v^p - l^p}{m^p - l^p}$, and $dt = \frac{pv^{p-1}}{m^p - l^p} dv$. Thus,

$$\int_0^1 t^{\alpha-1} \psi \left(((1-t)l^p + tm^p)^{\frac{1}{p}}, \cdot \right) dt = \int_l^m \left(\frac{v^p - l^p}{m^p - l^p} \right)^{\alpha-1} \psi(v, \cdot) \frac{pv^{p-1}}{m^p - l^p} dv.$$

Now, using the Katugampola fractional integral definition with $\rho = p$, we have

$$\int_l^m \left(\frac{m^p - u^p}{m^p - l^p} \right)^{\alpha-1} \psi(u, \cdot) \frac{pu^{p-1}}{m^p - l^p} du = \frac{p}{(m^p - l^p)^\alpha} \int_l^m (m^p - u^p)^{\alpha-1} u^{p-1} \psi(u, \cdot) du = \frac{p^\alpha \Gamma(\alpha)}{(m^p - l^p)^\alpha} {}^p J_{l^+}^\alpha \psi(m, \cdot),$$

and

$$\int_l^m \left(\frac{v^p - l^p}{m^p - l^p} \right)^{\alpha-1} \psi(v, \cdot) \frac{pv^{p-1}}{m^p - l^p} dv = \frac{p^\alpha \Gamma(\alpha)}{(m^p - l^p)^\alpha} {}^p J_{m^-}^\alpha \psi(l, \cdot).$$

Substituting these into the inequality, we obtain

$$\frac{p^\alpha \Gamma(\alpha)}{(m^p - l^p)^\alpha} [{}^p J_{l^+}^\alpha \psi(m, \cdot) + {}^p J_{m^-}^\alpha \psi(l, \cdot)] \leq \frac{\psi(l, \cdot) + \psi(m, \cdot)}{\alpha}.$$

□

Remark 1. The results presented above are new, as they likely refine and improve the corresponding results in the deterministic case for the Riemann–Liouville operator [26]. Specifically, Theorem 3.2 is obtained in the deterministic sense by setting $\rho = 1$.

5. Conclusion

In this paper, we introduced and studied the class of modified (p, h) -convex stochastic processes, which unifies and generalizes several existing notions of convexity in the stochastic setting. We established fundamental properties of this class, including closure under addition and scalar multiplication. We derived several important inequalities, including Hermite-Hadamard type inequalities, Ostrowski-type inequalities, and Jensen-type inequalities. Our results provide a comprehensive framework for the study of convex stochastic processes and their applications.

Future research directions include extending these results to multidimensional stochastic processes, developing applications in stochastic optimization and financial mathematics, and investigating connections with other generalized convexity notions.

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