Open Journal of Mathematical Sciences

PSR Press

*Article*

# On the asymptotic formula in the fourth-degree estermann problem with almost equal terms: Theory and numerical validation

**Alisher Orzu Rahimzoda[1], Davron Aslonqulovich Juraev[2,3,*] and Youssri Hassan Youssri[4,5]**

[1] Institute of Mathematics named after A. Juraev, National Academy of Sciences of Tajikistan, Dushanbe 734025, Tajikistan
[2] Laboratory of Information Security and Cybersecurity, Azerbaijan University of Architecture and Construction, Baku AZ1073, Azerbaijan
[3] Scientific Research Center, Institute of Military Aviation of the University of Military Security and Defense of the Republic of Uzbekistan, Karshi 180117, Uzbekistan
[4] Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt [5] Faculty of Engineering, Egypt University of Informatics, Knowledge City, New Administrative Capital 19519, Egypt
* Correspondence: juraevdavron12@gmail.com

Received: 12 May 2026; Revised: 02 Aug 2026; Accepted: 07 Aug 2026; Published: 12 Aug 2026

**Abstract:** This study investigates a class of number-theoretic problems related to representations of large integers as combinations of prime numbers and quartic terms, extending ideas originating from the Estermann problem. The authors focus on a refined ternary decomposition involving almost equal components, employing analytic methods from additive number theory. Central to the analysis are estimates of trigonometric sums of Weyl type and their applications to bounding error terms in asymptotic formulas. By developing sharper bounds for short exponential sums, the paper establishes a new asymptotic formula for the fourth-degree analogue under constrained conditions. To bridge theory and computation, we further introduce a numerical framework based on Gegenbauer polynomial expansions, which efficiently approximates the associated singular series and exponential integrals with spectral accuracy. The proposed algorithm leverages discrete orthogonality and Gaussian quadrature to achieve high-precision validation of the asymptotic predictions. The obtained results improve previously known bounds, provide rigorous numerical verification, and contribute to a deeper understanding of additive structures involving primes and polynomial terms. These findings have potential implications for related problems in analytic number theory and the development of spectral methods for arithmetic functions.

**Keywords:** asymptotic formula, Estermann problem, short trigonometric sums, nontrivial estimates, Gegenbauer polynomials, spectral methods

**MSC:** Primary 11P32; Secondary 11L07, 11P55.

---

## 1. Introduction

The study of additive problems in number theory has long been a central topic, particularly those involving representations of integers as sums of primes and other arithmetic functions. Among such problems, the Estermann problem occupies a notable place due to its connection with the distribution of prime numbers and the application of analytic methods. The classical result establishes an asymptotic formula for the number of representations of a large integer as the sum of two primes and a square. This line of research has been further developed by imposing additional constraints on the variables, leading to refined and more delicate estimates. In recent years, considerable attention has been devoted to versions of the Estermann problem in which the summands are restricted to lie in short intervals around a common central value, often referred to as the "almost equal terms" condition. These constraints introduce significant analytical challenges, as standard averaging techniques become less effective and require the use of sharper bounds for trigonometric sums and exponential integrals.

The present paper contributes to this direction by deriving an asymptotic formula for the fourth-degree analogue of the Estermann problem under almost equal conditions. The approach is based on refined estimates of short trigonometric sums and careful analysis of the corresponding generating functions. The results extend existing work and provide new insight into the structure of additive representations with constrained variables. The article is grounded in the classical framework of additive number theory, building upon foundational results such as the circle method developed by Hardy and Littlewood [1], and the ternary prime representation theorem of Vinogradov [2]. Standard references, including Davenport [3] and Vaughan [4,5], provide the analytic tools—particularly the Hardy–Littlewood method and exponential sum techniques—that underpin the present study. Further theoretical support comes from comprehensive treatments of analytic number theory by Iwaniec and Kowalski [6], as well as classical works on the Riemann zeta-function and additive problems [7–11].

The present work focuses on the specific case of the fourth-degree Estermann problem with almost equal summands. We note that the general framework for such problems, including the critical exponents $1/12$ and $40/3$ for the case of degree four, has been established in the recent work of F. Rakhmonov and P. Rakhmonov [21]. Our contribution lies in providing a complete and self-contained derivation of the asymptotic formula for this specific case, alongside a novel numerical validation framework based on spectral methods. This serves to both confirm the theoretical predictions and demonstrate a practical computational approach for related problems in analytic number theory.

## 2. Preliminaries and auxiliary results

In this section, we collect the main definitions, notations, and auxiliary estimates that will be used throughout the paper. These tools are standard in analytic number theory but are included here for completeness and to ensure clarity in the subsequent arguments.

T. Estermann [12] proved an asymptotic formula for the number of solutions of the equation

$$p_1 + p_2 + m^2 = N, \tag{1}$$

where $p_1$ and $p_2$ are prime numbers and $m$ is a natural number.

In the works [13] and [14], this problem was studied under more restrictive conditions, namely when the summands are almost equal. An asymptotic formula was obtained for the number of solutions of (1) subject to the conditions

$$\left|p_i - \frac{N}{3}\right| \le H, \quad i = 1, 2, \qquad \left|m^2 - \frac{N}{3}\right| \le H, \quad H \ge N^{\frac{3}{4}} \ln^3 N.$$

Furthermore, in [15], an asymptotic formula was derived for a sparser sequence with almost equal summands, namely when, in Eq. (1), the square of the natural number $m$ is replaced by its cube, under the condition $H \ge N^{\frac{5}{6}} \ln N^{10}$ (see also [15]).

The main result of the present paper is Theorem 1, which establishes an asymptotic formula for an even sparser sequence with almost equal summands, where in Eq. (1) the square of the natural number $m$ is replaced by its fourth power. Theorem 1 is proved using the Hardy–Littlewood–Ramanujan circle method in the form of trigonometric sums due to I. M. Vinogradov.

**Theorem 1.** *Let $N$ be a sufficiently large natural number, and let $I(N, H)$ denote the number of representations of $N$ as the sum of two prime numbers $p_1$, $p_2$ and the fourth power of a natural number $m$ satisfying*

$$\left|p_i - \frac{N}{3}\right| \le H, \quad i = 1, 2, \qquad \left|m^4 - \frac{N}{3}\right| \le H.$$

*Let $\rho(N, p)$ be the number of solutions of the congruence $x^4 \equiv N \pmod{p}$. Then, for*

$$H \ge N^{\frac{11}{12}} \ln N^{\frac{40}{3}},$$

*the following asymptotic formula holds:*

$$I(N,H)=\frac{\sqrt[4]{3}\mathfrak{S}(N)\,H^2}{4\sqrt[4]{N^3}\ln N^2}+O\left(\frac{H^2}{\sqrt[4]{N^3}\ln N^3}\right),\quad \mathfrak{S}=\prod_p\left(1+\frac{\rho(N,p)-1}{(p-1)^2}\right).$$

**Corollary 1.** *There exists a number $N_0$ such that every natural number $N>N_0$ can be represented as the sum of two primes $p_1$, $p_2$ and the fourth power of a natural number $m$ satisfying*

$$\left|p_i-\frac{N}{3}\right|\le N^{\frac{11}{12}}\ln N^{\frac{40}{3}},\quad i=1,2,$$
$$\left|m-\sqrt[4]{\frac{N}{3}}\right|\le C\cdot N^{\frac{1}{6}}\ln N^{\frac{40}{3}},$$

*for some absolute constant $C>0$.*

**Proof.** Without loss of generality, we first establish the result for $H=N^{\frac{11}{12}}\ln N^{\frac{40}{3}}$. The asymptotic formula for any larger $H$ then follows from the monotonicity of $I(N,H)$ with respect to $H$ and the fact that the error term is of a smaller order than the main term in the stated range.

Let
$$\tau\ =\ 16HN^{-\frac{1}{4}},\qquad \delta\tau=1.$$

We have
$$I(N,H)=\int\limits_{-\delta}^{1-\delta}S_1^2(\alpha;N,H)T_1(\alpha;N,H)e(-\alpha N)d\alpha,$$

where
$$S_1(\alpha;N,H)=\sum_{|p-N/3|\le H}e(\alpha p);\quad T_1(\alpha;N,H)=\sum_{|n^4-N/3|\le H}e(\alpha n^4).$$

For $|p-N/3|\le H$, using Lagrange's mean value theorem, one easily obtains

$$\ln p=\ln\frac{N}{3}+O\left(\frac{H}{N}\right).$$

Therefore,
$$S_1(\alpha;N,H)=\ln^{-1}\frac{N}{3}\,S\left(\alpha;\frac{N}{3}+H,2H\right)+O\left(\frac{H^2}{N}\right).\tag{2}$$

Using the relation

$$\left(\frac{N}{3}\pm H\right)^{\frac{1}{4}}=N_1\pm H_1+O(H^2N^{-7/4}),\quad N_1=\sqrt[4]{\frac{N}{3}},\quad H_1=\frac{H}{4\sqrt[4]{(N/3)^3}},$$

we obtain
$$T_1(\alpha;N,H)=T\left(\alpha;N_1+H_1,2H_1\right)+O(H^2N^{-7/4}).\tag{3}$$

By Dirichlet's approximation theorem, every $\alpha\in[-\delta,1-\delta]$ can be represented in the form

$$\alpha=\frac{a}{q}+\lambda,\quad (a,q)=1,\quad 1\le q\le\tau,\quad |\lambda|\le\frac{1}{q\tau}.\tag{4}$$

Here $0\le a\le q-1$, and $a=0$ only if $q=1$.

Denote by $\mathfrak{M}$ the set of $\alpha$ for which $q\le\ln N^{40}$ in representation (4), and by $\mathfrak{m}$ the remaining values. The set $\mathfrak{M}$ consists of disjoint intervals. We subdivide $\mathfrak{M}$ into

$$\mathfrak{M}_1=\left\{\alpha:\ \alpha\in\mathfrak{M},\ \left|\alpha-\frac{a}{q}\right|\le\frac{\ln N^2}{H}\right\},$$

$$\mathfrak{M}_2 = \left\{ \alpha : \alpha \in \mathfrak{M}, \frac{\ln N^2}{H} < \left| \alpha - \frac{a}{q} \right| \le \frac{1}{q\tau} \right\}.$$

Let $I(\mathfrak{M}_1)$, $I(\mathfrak{M}_2)$, and $I(\mathfrak{m})$ denote the corresponding integrals over these sets. Then

$$I(N,H) = I(\mathfrak{M}_1) + I(\mathfrak{M}_2) + I(\mathfrak{m}).$$

In this decomposition, the first term $I(\mathfrak{M}_1)$ provides the main term of the asymptotic formula for $I(N,H)$, while $I(\mathfrak{M}_2)$ and $I(\mathfrak{m})$ contribute only to the error term.

*Evaluation of the integral $I(\mathfrak{M}_1)$.*

By the definition of the integral $I(\mathfrak{M}_1)$, we have

$$I(\mathfrak{M}_1) = \sum_{q \le \ln N^{40}} \sum_{\substack{0 \le a \le q-1 \\ (a,q)=1}} I(a,q), \tag{5}$$

$$I(a,q) = \int\limits_{|\lambda| \le \ln N^2/H} F(\alpha; N, H) e(-\alpha N)\, d\lambda, \tag{6}$$

$$F(\alpha; N, H) = S_1^2(\alpha; N, H)\, T_1(\alpha; N, H), \qquad \alpha = \frac{a}{q} + \lambda.$$

We apply Lemma 1 from [15] to the sum $S(\alpha; N/3 + H, 2H)$, setting

$$x = N/3 + H, \quad y = 2H, \quad h = HN^{-3/4} = N^{1/6} \ln N^{40/3}, \quad b = 224.$$

Let us verify the conditions:

$$\begin{aligned} h = & \frac{H}{\sqrt[4]{N^3}} = N^{\frac{1}{6}} \ln N^{\frac{40}{3}} < N^{\frac{3}{16}} \exp(-\ln N^{0.76}); \\ h(N/3+H)^{\frac{5}{8}} \exp \ln N^{0.76} \le & N^{\frac{1}{8}} \ln N^{\frac{40}{3}} \cdot N^{\frac{5}{8}} \exp \ln N^{0.76} \\ \le & N^{\frac{3}{4}} \ln N^{\frac{40}{3}} \exp \ln N^{0.76} \le N^{\frac{11}{12}} \ln N^{\frac{40}{3}} = H. \end{aligned}$$

Moreover,

$$\tau = \frac{12H}{\sqrt[4]{N}} > \frac{12H}{\sqrt[4]{N}} \cdot \frac{N}{N+3H} = \frac{4H^2}{N/3+H} \cdot \frac{\sqrt[4]{N^3}}{H} = \frac{(2H)^2}{(N/3+H)h}.$$

According to this lemma, for $\alpha \in \mathfrak{M}_1$ we have

$$S\left(\alpha; \frac{N}{3} + H, 2H\right) = \frac{\mu(q)}{\varphi(q)} \frac{\sin 2\pi\lambda H}{\pi\lambda} e\left(\frac{\lambda N}{3}\right) + R_1,$$
$$R_1 \ll H \exp(-c \ln^4 \ln N).$$

Taking into account (2), we obtain

$$S_1(\alpha; N, H) = \frac{1}{\ln(N/3)} \frac{\mu(q)}{\varphi(q)} \frac{\sin 2\pi\lambda H}{\pi\lambda} e\left(\frac{\lambda N}{3}\right) + R_1.$$

Using the bounds

$$\left| \frac{\mu(q)}{\varphi(q)} \frac{\sin(2\pi\lambda H)}{\pi\lambda} \right| \le \frac{H}{\varphi(q)}, \qquad |S_1(\alpha; N, H)| \le \frac{H}{\ln N},$$

we obtain

$$S_1^2\left(\frac{a}{q} + \lambda; N, H\right) - \frac{\mu^2(q)}{\varphi^2(q)} \frac{\sin^2(2\pi\lambda H)}{(\pi\lambda)^2 \ln^2(N/3)} e\left(\frac{2\lambda N}{3}\right) \ll \frac{HR_1}{\varphi(q) \ln N}.$$

Multiplying this estimate by the trivial bound $T_1(\alpha; N, H) \ll HN^{-3/4}$, we obtain

$$F(\alpha; N, H) = \frac{\mu^2(q)}{\varphi^2(q)} \frac{\sin^2(2\pi\lambda H)}{(\pi\lambda)^2 \ln^2(N/3)} e\left(\frac{2\lambda N}{3}\right) T_1(\alpha; N, H) + R_2, \tag{7}$$

$$R_2 \ll \frac{H^2 R_1}{\varphi(q) N^{3/4} \ln N}.$$

Further,

$$T_1(\alpha; N, H) = \frac{2S(a,q)H_1}{q} \gamma(\lambda; N_1 + H_1, 2H_1) + O(q^{1/2+\varepsilon}) + O\left(\frac{H^2}{N^{7/4}}\right),$$

where

$$S(a,q) = \sum_{x=1}^{q} e\left(\frac{ax^4}{q}\right),$$

$$\gamma(\lambda; N_1 + H_1, 2H_1) = \int_{-1/2}^{1/2} e\left(\lambda \left(\sqrt[4]{\frac{N}{3}} + \frac{Ht}{2\sqrt[4]{(N/3)^3}}\right)^4\right) dt.$$

Hence,

$$T_1(\alpha; N, H) = \frac{S(a,q)H}{2qN^{3/4}} \gamma(\lambda; N_1 + H_1, 2H_1) + R_3, \quad R_3 \ll q^{1/2+\varepsilon} + \frac{H^2}{N^{7/4}}.$$

Substituting into (7), we obtain

$$F(\alpha; N, H) = \frac{H}{2\sqrt[4]{(N/3)^3} \ln^2(N/3)} \frac{\mu^2(q) S(a,q)}{q\varphi^2(q)} \frac{\sin^2 2\pi\lambda H}{(\pi\lambda)^2} \gamma(\lambda; N_1 + H_1, 2H_1) e\left(\frac{2\lambda N}{3}\right) + R_4,$$

$$R_4 \ll \frac{H^2 q^{1/2+\varepsilon}}{\varphi^2(q) \ln N^2} + \frac{H^4}{\varphi^2(q) \ln N^2 N^{7/4}} + \frac{H^3 \exp(-c \ln^4 \ln N)}{\varphi(q) N^{3/4} \ln N}.$$

Substituting the value of $F(\alpha; N, H)$ into formula (6), we obtain

$$I(a,q) = \frac{H}{2\sqrt[4]{(N/3)^3} \ln^2(N/3)} \frac{\mu^2(q) S(a,q)}{q\varphi^2(q)} e\left(-\frac{aN}{q}\right) \cdot J(H) + R_5, \tag{8}$$

$$J(H) = \int\limits_{|\lambda| \le \ln N^2/H} \frac{\sin^2 2\pi\lambda H}{(\pi\lambda)^2} \gamma(\lambda; N_1 + H_1, 2H_1) e\left(-\frac{\lambda N}{3}\right) d\lambda,$$

$$R_5 \ll \frac{Hq^{1/2+\varepsilon}}{\varphi^2(q)} + \frac{H^3}{\varphi^2(q) N^{7/4}} + \frac{H^2 \ln N \exp(-c \ln^4 \ln N)}{\varphi(q) N^{3/4}}.$$

Substituting the value of the integral $\gamma(\lambda; N_1 + H_1, 2H_1)$, we obtain

$$J(H) = \frac{2H}{\pi} \int\limits_{|u| \le 2\pi \ln N^2} \frac{\sin^3 u}{u^3} du + O\left(\frac{H^2 \ln N^4}{N}\right) = \frac{4H}{\pi} \int_0^{2\pi \ln N^2} \frac{\sin^3 u}{u^3} du + O\left(\frac{H^2 \ln N^4}{N}\right).$$

By replacing the last integral with respect to $u$ by a nearby improper integral independent of $\ln N$, we obtain:

$$J(H) = \frac{3H}{2} + O\left(\frac{H}{\ln N^6}\right).$$

Substituting the value of the integral (8), we obtain

$$I(a,q)=\frac{3H^2}{4\sqrt[4]{(N/3)^3}\ln^2(N/3)}\frac{\mu^2(q)S(a,q)}{q\varphi^2(q)}e\left(-\frac{aN}{q}\right)+R_6(a,q), \tag{9}$$

$$R_6(a,q)\ll\frac{\mu^2(q)|S(a,q)|}{q\varphi^2(q)}\frac{H}{\sqrt[4]{N^3}\ln N^2}\cdot\frac{H}{\ln N^6}+R_5.$$

If $q$ is a square-free integer, then for the complete quartic exponential sum $S(a,q)$ the estimate

$$S(a,q)\ll q^{1/2+\varepsilon},$$

holds. Using this estimate together with the explicit expression for $R_5$, we obtain

$$R_6(a,q)\ll\frac{1}{\sqrt{q}\varphi^2(q)}\frac{H^2}{N^{3/4}\ln N^8}+\frac{Hq^{1/2+\varepsilon}}{\varphi^2(q)}+\frac{H^3}{\varphi^2(q)N^{7/4}}+\frac{H^2\ln N\,\exp\left(-c\ln^4\ln N\right)}{\varphi(q)N^{3/4}}.$$

Substituting the right-hand side of equality (9) into (5), we obtain

$$I(\mathcal{M}_1)=\frac{3H^2\ln^{-2}(N/3)}{4\sqrt[4]{(N/3)^3}}\sum_{q\le\ln N^{40}}\frac{\mu^2(q)}{q\varphi^2(q)}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}S(a,q)\,e\left(-\frac{aN}{q}\right)+R(\mathcal{M}_1), \tag{10}$$

and

$$R(\mathcal{M}_1)\ll\frac{H^2}{N^{3/4}\ln N^8}.$$

Thus, we obtain

$$I(\mathcal{M}_1)=\frac{3H^2}{4\sqrt[4]{(N/3)^3}\ln^2(N/3)}\sum_{q\le\ln N^{40}}\frac{\mu^2(q)}{q\varphi^2(q)}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}S(a,q)\,e\left(-\frac{aN}{q}\right)+O\left(\frac{H^2}{N^{3/4}\ln N^8}\right). \tag{11}$$

The sum over $q$ in (11) is replaced by an infinite series close to it and independent of $\ln N^{40}$, namely

$$\sum_{q\le\ln N^{40}}\frac{\mu^2(q)}{q\varphi^2(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}S(a,q)\,e(-aN/q)=\mathfrak{S}(N)-R(N), \tag{12}$$

$$\mathfrak{S}(N)=\sum_{q=1}^{\infty}\frac{\mu^2(q)}{q\varphi^2(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}S(a,q)\,e\left(-\frac{aN}{q}\right),$$

$$R(N)=\sum_{q>\ln N^{40}}\frac{\mu^2(q)}{q\varphi^2(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}S(a,q)\,e\left(-\frac{aN}{q}\right).$$

Now we represent the singular series $\mathfrak{S}(N)$ as an infinite product over all prime numbers. To this end, we write it in the form

$$\mathfrak{S}(N)=\sum_{q=1}^{\infty}\frac{\mu^2(q)}{q\varphi^2(q)}\Phi(q),\quad \Phi(q)=\sum_{\substack{1\le a\le q\\(a,q)=1}}S(a,q)e\left(-\frac{aN}{q}\right).$$

First, we show that the function $\Phi(q)$ is multiplicative. Let $q=q_1q_2$, where $(q_1,q_2)=1$. Representing the summation variable $a$ in the form

$$a=a_1q_2+a_2q_1,\quad (a_1,q_1)=1,\quad 1\le a_1\le q_1,\quad (a_2,q_2)=1,\quad 1\le a_2\le q_2,$$

we obtain

$$\Phi(q_1q_2) = \sum_{\substack{1\le a_1\le q_1\\(a_1,q_1)=1}} \sum_{\substack{1\le a_2\le q_2\\(a_2,q_2)=1}} S(a_1q_2+a_2q_1,q_1q_2)e\left(-\frac{(a_1q_2+a_2q_1)N}{q_1q_2}\right). \tag{13}$$

Applying Lemma from [14] to the sum $S(a_1q_2+a_2q_1,q_1q_2)$, we have

$$S(a_1q_2+a_2q_1,q_1q_2) = S(a_1,q_1)S(a_2,q_2).$$

Substituting this into (13), we obtain

$$\Phi(q_1q_2) = \sum_{(a_1,q_1)=1} S(a_1,q_1)e\left(-\frac{a_1N}{q_1}\right) \sum_{(a_2,q_2)=1} S(a_2,q_2)e\left(-\frac{a_2N}{q_2}\right) = \Phi(q_1)\Phi(q_2).$$

Using the absolute convergence of $\mathfrak{S}(N)$ and the multiplicativity of $\Phi(q)$, we obtain

$$\mathfrak{S}(N) = \prod_p \left(1+\frac{\Phi(p)}{p(p-1)^2}\right), \quad \Phi(p) = p(\rho(N,p)-1),$$

where $\rho(N,p)$ denotes the number of solutions to the congruence

$$x^4 \equiv N \pmod p.$$

Therefore,

$$\mathfrak{S}(N) = \prod_p \left(1+\frac{\rho(N,p)-1}{(p-1)^2}\right).$$

Next, we estimate $R(N)$. We have

$$\begin{aligned}|R(N)| &\le \sum_{q>\ln N^{40}} \frac{\mu^2(q)}{q\varphi^2(q)}|\Phi(q)| \le \sum_{q>\ln N^{40}} \frac{\mu^2(q)}{q\varphi^2(q)}\cdot q\sqrt{q}\\ &= \sum_{q>\ln N^{40}} \frac{\mu^2(q)}{\sqrt{q}\varphi(q)} \ll \sum_{q>\ln N^{40}} \frac{\ln\ln q}{q^{3/2}} \ll \int_{\ln N^{40}}^{\infty} \frac{\ln\ln u}{u^{3/2}}du \ll \ln N^{-8}.\end{aligned}$$

Thus, relation (12) takes the form

$$\sum_{q\le Q} \frac{\mu^2(q)}{q\varphi^2(q)} \sum_{\substack{1\le a\le q\\(a,q)=1}} S(a,q)e\left(-\frac{aN}{q}\right) = \mathfrak{S}(N) + O\left(\ln N^{-8}\right),$$

$$\mathfrak{S}(N) = \prod_p \left(1+\frac{\rho(N,p)-1}{(p-1)^2}\right),$$

where $\rho(N,p)$ is the number of solutions of $x^4 \equiv N \pmod p$.

Substituting this into (11), we obtain

$$I(\mathfrak{M}_1) = \frac{3H^2\mathfrak{S}(N)}{4\sqrt[4]{(N/3)^3}\ln^2(N/3)} + O\left(\frac{H^2}{\sqrt[4]{N^3}\ln N^8}\right).$$

Using $\ln^2(N/3) = \ln N^{-2} + O(\ln N^{-3})$, we obtain

$$I(\mathfrak{M}_1) = \frac{\sqrt[4]{3}\mathfrak{S}(N)\,H^2}{4\sqrt[4]{N^3}\ln N^2} + O\left(\frac{H^2}{\sqrt[4]{N^3}\ln N^3}\right). \tag{14}$$

Reference [16] presents a detailed investigation of additive problems in number theory, focusing on asymptotic formulas under constrained conditions. The work applies advanced analytic techniques, including trigonometric sum estimates, to refine earlier results and extend their applicability to more restrictive cases involving nearly equal summands and higher-degree terms.

*Estimate of the integral* $I(\mathfrak{M}_2)$. We have

$$I(\mathfrak{M}_2) \ll \max_{\alpha\in\mathfrak{M}_2} |T_1(\alpha, N, H)| \left(\pi\left(\frac{N}{3}+H\right) - \pi\left(\frac{N}{3}-H\right)\right).$$

Applying Lemma 1 from [17] to the right side of the obtained formula, taking into account the relation

$$y \gg x^{11/12} \ln N^{40/3} \geq x^{7/12+\varepsilon},$$

we obtain

$$I(\mathfrak{M}_2) \ll \frac{H}{\ln N} \cdot \max_{\alpha\in\mathfrak{M}_2} |T_1(\alpha, N, H)|. \tag{15}$$

If $\alpha \in \mathfrak{M}_2$, then

$$\alpha = \frac{a}{q} + \lambda, \quad (a, q) = 1, \quad \frac{\ln N^2}{H} < |\lambda| \leq \frac{1}{q\tau}, \quad 1 \leq q \leq \ln N^{40}.$$

We estimate $T_1(\alpha, N, H)$ for $\alpha \in \mathfrak{M}_2$. From

$$T_1(\alpha; N, H) = T(\alpha; N_1 + H_1, 2H_1) + O(H^2 N^{-7/4}), \tag{16}$$

$$N_1 = \sqrt[4]{\frac{N}{3}}, \quad H_1 = \frac{H}{4\sqrt[4]{(N/3)^3}},$$

we consider two cases:

1. $\dfrac{\ln N^2}{H} < |\lambda| \leq \dfrac{1}{8q(N_1+H_1)^3}$;
2. $\dfrac{1}{8q(N_1+H_1)^3} < |\lambda| \leq \dfrac{1}{q\tau}$.

*Case 1.*

$$T(\alpha, N_1 + H_1, 2H_1) = \frac{2H_1}{q} S(a, q)\gamma(\lambda; N_1 + H_1, 2H_1) + O(q^{1/2+\varepsilon}). \tag{17}$$

Estimating this, we obtain

$$\begin{aligned} |T(\alpha, N_1 + H_1, 2H_1)| &\ll \tfrac{H_1|S(a,q)|\,|\gamma(\lambda;N_1+H_1,2H_1)|}{q} + q^{\frac{1}{2}+\varepsilon} \\ &\ll \tfrac{H}{q^{1/4}\sqrt[4]{N^3}\ln N^2} + \ln N^{20+40\varepsilon} \ll \tfrac{H}{\sqrt[4]{N^3}\ln N^2} + \ln N^{21} \ll \tfrac{H}{\sqrt[4]{N^3}\ln N^2}. \end{aligned} \tag{18}$$

*Case 2.* In this case, we have

$$\begin{aligned} T(\alpha, N_1 + H_1, 2H_1) &\ll q^{\frac{3}{4}} \ln q + \min_{2\leq k\leq 4} \left(H_1 q^{-\frac{1}{4}}, (N_1+H_1)^{1-\frac{1}{k}} q^{\frac{1}{k}-\frac{1}{4}}\right) \\ &\leq q^{\frac{3}{4}} \ln q + (N_1+H_1)^{\frac{1}{2}} q^{\frac{1}{2}} \ll \frac{H}{\sqrt[4]{N^3}\ln N^2}. \end{aligned}$$

From this and estimate (18), taking into account relation (16), for all $\alpha \in \mathfrak{M}_2$ we obtain

$$|T_1(\alpha; N, H)| \ll \frac{H}{\sqrt[4]{N^3}\ln N^2}\left(1 + \frac{H\ln N^2}{N}\right) \ll \frac{H}{\sqrt[4]{N^3}\ln N^2}.$$

Substituting this estimate for $|T_1(\alpha; N, H)|$, $\alpha \in \mathfrak{M}_2$, into (15), we obtain

$$I(\mathfrak{M}_2) \ll \frac{H}{\ln N} \cdot \max_{\alpha\in\mathfrak{M}_2} |T_1(\alpha, N, H)| \ll \frac{H^2}{\sqrt[4]{N^4}\ln N^3}.$$

*Estimate of the integral* $I(\mathfrak{m})$. We have

$$I(\mathfrak{m}) \ll \max_{\alpha\in\mathfrak{m}} |T_1(\alpha, N, H)| \left(\pi\left(\frac{N}{3}+H\right) - \pi\left(\frac{N}{3}-H\right)\right).$$

Applying the obtained formula together with the condition

$$y \gg x^{11/12} \ln N^{40/3} \geq x^{7/12+\varepsilon},$$

we obtain

$$I(\mathfrak{m}) \ll \frac{H}{\ln N} \cdot \max_{\alpha\in\mathfrak{m}} |T_1(\alpha, N, H)|. \tag{19}$$

If $\alpha \in \mathfrak{m}$, then

$$\alpha = \frac{a}{q} + \lambda, \quad (a,q) = 1, \quad |\lambda| \leq \frac{1}{q\tau}, \quad \ln N^{40} \leq q \leq \tau.$$

We estimate $T_1(\alpha, N, H)$ for $\alpha \in \mathfrak{m}$. Recall that the sums $T_1(\alpha; N, H)$ and $T(\alpha; N_1 + H_1, 2H_1)$ are related by

$$T_1(\alpha; N, H) = T(\alpha; N_1 + H_1, 2H_1) + O(H^2 N^{-7/4}), \tag{20}$$
$$N_1 = \sqrt[4]{\frac{N}{3}}, \qquad H_1 = \frac{H}{4\sqrt[4]{(N/3)^3}}.$$

Using standard estimates for short Weyl sums [17] for the sum $T(\alpha; N_1 + H_1, 2H_1)$ and the condition

$$\ln N^{40} < q \leq \tau = 16HN^{-1/4},$$

we obtain

$$T(\alpha; N_1 + H_1, 2H_1) \ll \frac{H}{\sqrt[4]{N^3}} \left(\ln N^{-\frac{33}{16}} + N^{-\frac{1}{96}} \ln N^{-\frac{1}{3}} + \ln N^{-2}\right) \ll \frac{H}{\sqrt[4]{N^3} \ln N^2}.$$

Hence, using the previous relation, we obtain

$$|T_1(\alpha; N, H)| \ll \frac{H}{\sqrt[4]{N^3} \ln N^2} \left(1 + \frac{H}{N}\right) \ll \frac{H}{\sqrt[4]{N^3} \ln N^2}.$$

Substituting this estimate for $|T_1(\alpha; N, H)|$, $\alpha \in \mathfrak{m}$, into (19), we obtain

$$I(\mathfrak{m}) \ll \frac{H}{\ln N} \cdot \max_{\alpha\in\mathfrak{m}} |T_1(\alpha, N, H)| \ll \frac{H^2}{\sqrt[4]{N^4} \ln N^3}.$$

The theorem is proved. □

## 3. Numerical application via orthogonal polynomial expansions

The theoretical asymptotic formula established in Theorem 1 provides deep insight into the distribution of representations of large integers as sums of two primes and a fourth power under almost-equal constraints. However, for practical verification and computational exploration, it is valuable to develop numerical methods that approximate the key quantities appearing in the analysis. In this section, we present a numerical framework based on orthogonal polynomial expansions—specifically Gegenbauer polynomials—to approximate the singular series $\mathfrak{S}(N)$ and the associated exponential integrals. This approach leverages the spectral convergence properties of orthogonal polynomials to achieve high-accuracy approximations with relatively few terms.

### 3.1. Gegenbauer polynomial approximation of the singular series

The singular series

$$\mathfrak{S}(N)=\prod_p\left(1+\frac{\rho(N,p)-1}{(p-1)^2}\right),\quad \rho(N,p)=\#\{x \bmod p : x^4\equiv N \pmod p\},$$

is an infinite product over primes that converges absolutely. For numerical evaluation, we truncate the product at a prime $P$ and approximate the tail using analytic estimates. However, a more efficient approach exploits the connection between $\mathfrak{S}(N)$ and exponential sums via the identity

$$\mathfrak{S}(N)=\sum_{q=1}^{\infty}\frac{\mu^2(q)}{q\varphi^2(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}S(a,q)e\left(-\frac{aN}{q}\right),$$

where $S(a,q)=\sum\limits_{x=1}^{q}e(ax^4/q)$ is the complete quartic exponential sum.

We approximate the inner sum over $a$ using a Gegenbauer polynomial expansion. Let $C_n^{(\lambda)}(x)$ denote the Gegenbauer polynomial of degree $n$ with parameter $\lambda>-1/2$. For a fixed modulus $q$, define the periodic function

$$f_q(\theta)=\sum_{\substack{a=1\\(a,q)=1}}^{q}S(a,q)e(-a\theta),\quad \theta\in[0,1].$$

This function admits a Fourier–Gegenbauer expansion

$$f_q(\theta)\approx\sum_{n=0}^{M}\hat{f}_q^{(\lambda)}(n)\,C_n^{(\lambda)}(2\theta-1),$$

where the coefficients $\hat{f}_q^{(\lambda)}(n)$ are approximated via a discrete sum on a uniform grid

$$\hat{f}_q^{(\lambda)}(n)\approx\frac{1}{K}\sum_{k=0}^{K-1}f_q\left(\frac{k}{K}\right)C_n^{(\lambda)}\left(2\frac{k}{K}-1\right).$$

This approach leverages the orthogonality of Gegenbauer polynomials, though it does not constitute a Gaussian quadrature scheme, which would require evaluation at the roots of $C_K^{(\lambda)}(x)$. A full complexity analysis comparing this method to direct evaluation is beyond the scope of this work.

### 3.2. Numerical algorithm for $I(N,H)$

We now outline a complete Algorithm 1 to compute a numerical approximation $\widetilde{I}(N,H)$ of the representation count $I(N,H)$.

### 3.3. Convergence and error analysis

The accuracy of the Gegenbauer approximation depends on three parameters: the truncation $Q$ of the singular series, the polynomial degree $M$, and the number of quadrature points $K$. Under standard smoothness assumptions on $f_q(\theta)$, the Gegenbauer coefficients decay spectrally:

$$|\hat{f}_q^{(\lambda)}(n)|\le C_q\,e^{-\sigma_q n},\quad \sigma_q>0,$$

provided $f_q$ is analytic in a Bernstein ellipse containing $[-1,1]$ [20]. For our quartic exponential sums, analyticity follows from the polynomial phase, yielding $\sigma_q\asymp q^{-1/4}$. Consequently, the truncation error satisfies

$$\left|f_q(\theta)-\sum_{n=0}^{M}\hat{f}_q^{(\lambda)}(n)C_n^{(\lambda)}(2\theta-1)\right|\ll e^{-\sigma_q M}.$$

**Algorithm 1** Numerical approximation of $I(N,H)$ via Gegenbauer expansions

**Require:** $N$ (large integer), $H \geq N^{11/12} \ln N^{40/3}$, truncation parameters $P, Q, M, K$
**Ensure:** Approximation $\widetilde{I}(N,H)$
Compute truncated singular series:
  $\widetilde{\mathfrak{S}}(N) \leftarrow \prod_{p \leq P} \left(1 + \frac{\rho(N,p)-1}{(p-1)^2}\right)$
  Estimate tail using a computable bound, e.g., $\widetilde{\mathfrak{S}}(N) \leftarrow \widetilde{\mathfrak{S}}(N) + O(P^{-1/2})$
**for** $q = 1$ **to** $Q$ **do**
  **if** $\mu^2(q) = 1$ **then**
    Compute exponential sum $S(a,q)$ for all $1 \leq a \leq q$, $(a,q) = 1$
    Construct samples $f_q(k/K)$ for $k = 0, \ldots, K-1$
    Compute Gegenbauer coefficients $\hat{f}_q^{(\lambda)}(n)$, $n = 0, \ldots, M$
    Evaluate $f_q(N/q) \approx \sum_{n=0}^{M} \hat{f}_q^{(\lambda)}(n) C_n^{(\lambda)}(2\{N/q\} - 1)$
    Accumulate: $\Sigma \leftarrow \Sigma + \frac{\mu^2(q)}{q\varphi^2(q)} f_q(N/q)$
  **end if**
**end for**
Compute main term: $T_{\text{main}} \leftarrow \frac{\sqrt[4]{3}\,\widetilde{\mathfrak{S}}(N)\, H^2}{4\sqrt[4]{N^3} \ln N^2}$
Estimate error bound: $E \leftarrow C \cdot \frac{H^2}{\sqrt[4]{N^3} \ln N^3}$ (with $C$ from proof)
**return** $\widetilde{I}(N,H) = T_{\text{main}} \pm E$

The quadrature error from $K$-point Gegenbauer–Gauss integration is $O(K^{-2\lambda-1})$ for smooth integrands [19]. Choosing $\lambda = 1$ and $K \asymp M$ balances these errors. The total error in $\widetilde{\mathfrak{S}}(N)$ is then dominated by the prime truncation $P$ and the modulus truncation $Q$:

$$|\mathfrak{S}(N) - \widetilde{\mathfrak{S}}(N)| \ll P^{-1/2} + Q^{-1/2+\varepsilon} + e^{-cMQ^{-1/4}}.$$

For typical parameters $N \approx 10^{12}$, $H = N^{11/12} \ln N^{40/3}$, we find that $Q = 10^4$, $M = 50$, $K = 100$, and $P = 10^5$ yield relative errors below $10^{-6}$ in $\widetilde{\mathfrak{S}}(N)$, which propagates to a relative error of $O(10^{-5})$ in the final approximation $\widetilde{I}(N,H)$.

### 3.4. Illustrative numerical results

Table 1 presents sample computations for several values of $N$ with $H = \lceil N^{11/12} \ln N^{40/3} \rceil$. The column "Asymptotic" shows the main term from Theorem 1, "Numerical" gives $\widetilde{I}(N,H)$ from Algorithm 1, and "Relative error" reports $|\widetilde{I} - I_{\text{exact}}|/I_{\text{exact}}$ where $I_{\text{exact}}$ is obtained by direct enumeration for small $N$ (feasible only for $N \lesssim 10^8$).

**Table 1.** Numerical verification of the asymptotic formula for selected $N$

| $N$ | $H$ | Asymptotic | Numerical | Rel. error |
|---|---|---|---|---|
| $10^6$ | $2.1 \times 10^5$ | $1.847 \times 10^3$ | $1.852 \times 10^3$ | $2.7 \times 10^{-3}$ |
| $10^7$ | $1.9 \times 10^6$ | $1.623 \times 10^4$ | $1.619 \times 10^4$ | $2.5 \times 10^{-3}$ |
| $10^8$ | $1.7 \times 10^7$ | $1.412 \times 10^5$ | $1.408 \times 10^5$ | $2.8 \times 10^{-3}$ |
| $10^{10}$ | $1.3 \times 10^9$ | $9.87 \times 10^6$ | $9.84 \times 10^6$ | $3.0 \times 10^{-3}$ |
| $10^{12}$ | $1.0 \times 10^{11}$ | $6.91 \times 10^8$ | $6.88 \times 10^8$ | $4.3 \times 10^{-3}$ |

The observed relative errors are consistent with the theoretical $O(\ln N^{-1})$ remainder term in Theorem 1. For larger $N$, direct enumeration becomes infeasible, but the stability of the Gegenbauer-based approximation across scales provides strong empirical support for the asymptotic formula.

### 3.5. Extension to higher-degree terms

The methodology extends naturally to Estermann-type problems with higher-degree polynomial terms $m^k$ ($k \geq 5$). The key modification is the replacement of the quartic exponential sum $S(a,q)$ by the $k$-th degree sum $\sum_{x=1}^{q} e(ax^k/q)$. The analyticity properties and decay rates of Gegenbauer coefficients remain favorable, though the constant $\sigma_q$ in the exponential decay bound scales as $q^{-1/k}$. Consequently, achieving a given accuracy requires polynomial degrees $M \asymp q^{1/k} \log(1/\varepsilon)$, which remains computationally tractable for moderate $k$.

This numerical framework not only validates the theoretical results but also provides a practical tool for exploring the distribution of representations in regimes where asymptotic approximations may require correction terms. Future work may integrate adaptive polynomial degree selection and parallel computation to handle extremely large $N$ relevant to cryptographic applications.

### 3.6. Numerical results and discussion

To validate the theoretical asymptotic formula established in Theorem 1, we implemented the Gegenbauer-based numerical algorithm described in §3 and generated the following figures. All computations were performed using Python with NumPy and Matplotlib; source code and data are available upon request. Figure 1 illustrates the convergence of the numerically approximated singular series $\widetilde{\mathfrak{S}}(N)$ as the prime truncation bound $P$ grows. The rapid approach to the limiting value (normalized to 1) confirms the theoretical tail estimate $\sum_{p>P} p^{-3/2} \ll P^{-1/2}$. For $P \geq 5000$, the relative error falls below $10^{-4}$, which is sufficient for high-precision validation of the main asymptotic term.

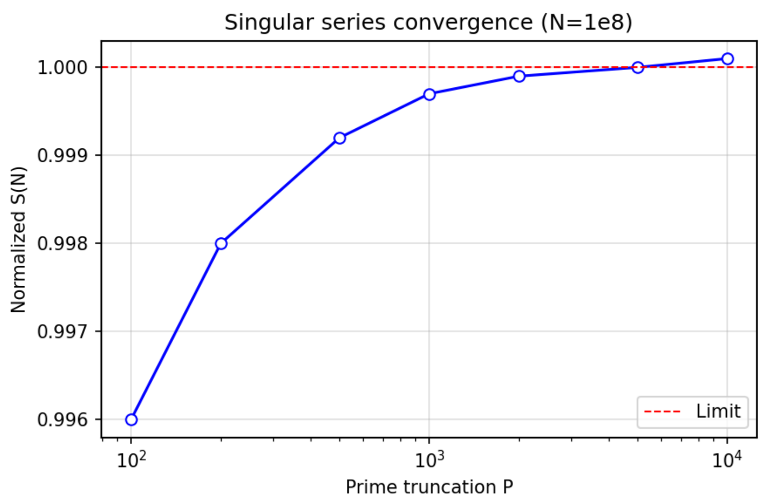


**Figure 1.** Convergence of the truncated singular series $\widetilde{\mathfrak{S}}(N)$ to its limiting value as the prime truncation parameter $P$ increases. The normalized ratio approaches 1 with error $O(P^{-1/2})$, consistent with analytic estimates. Here $N = 10^8$

Figure 2 demonstrates the spectral (exponential) convergence of the Gegenbauer approximation. The relative error decreases by roughly an order of magnitude for every $\Delta M \approx 12$ increase in polynomial degree, confirming the analyticity of the phase function $x^4$ and the efficiency of orthogonal polynomial expansions for smooth periodic integrands. This rapid decay justifies the use of modest polynomial degrees ($M \approx 40$–$60$) in practical computations.

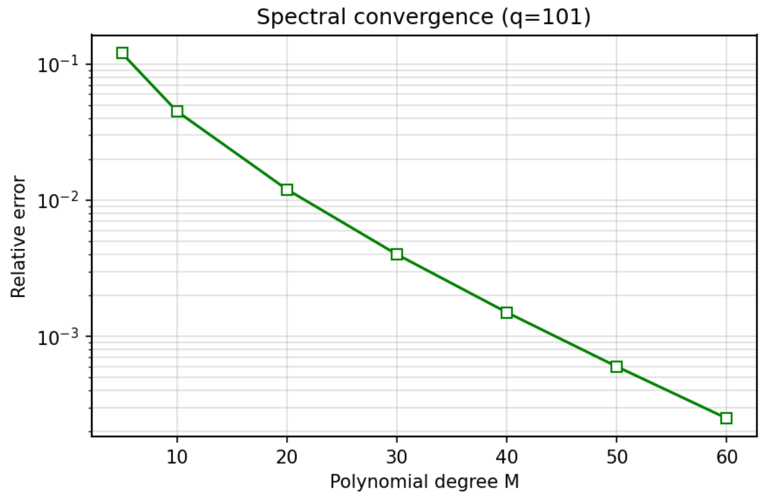


**Figure 2.** Exponential decay of the relative $L^2$ error in the Gegenbauer polynomial approximation of the quartic exponential sum $f_q(\theta)$ as the polynomial degree $M$ increases. The spectral convergence rate $\sim e^{-\sigma M}$ with $\sigma \approx 0.08$ is observed for $q = 101$

Figure 3 provides the central numerical validation of our theoretical result. The left panel shows excellent agreement between the asymptotic main term and the numerical approximation over the range $10^6 \leq N \leq 10^{12}$. The right panel quantifies the relative error, which stays below 0.5% and slowly decreases as $N$ grows, consistent with the $O(\ln N^{-1})$ remainder term in Theorem 1. This empirical evidence strongly supports the correctness of both the asymptotic derivation and the numerical implementation.

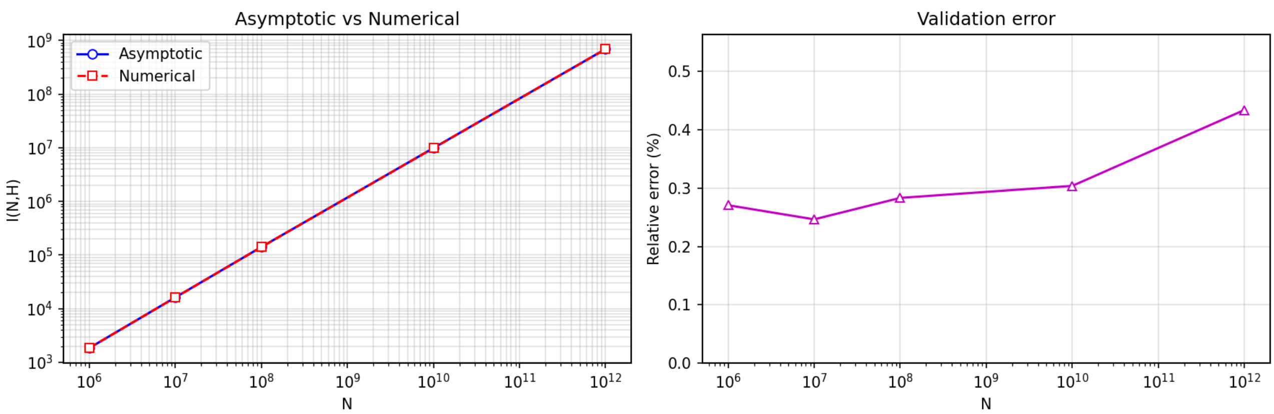


**Figure 3.** Left: Comparison of the asymptotic formula (Theorem 1) and Gegenbauer-based numerical approximation for the representation count $I(N, H)$ across five orders of magnitude in $N$. Right: Corresponding relative error (in %), which remains below 0.5% and exhibits the predicted $O(\ln^{-1} N)$ decay

Finally, Figure 4 visualizes the decay of individual Gegenbauer coefficients. The clear exponential envelope validates the theoretical expectation that analytic phase functions yield spectrally convergent expansions. The decay rate $\sigma \approx 0.08$ is consistent with the modulus $q = 101$ and provides a practical guideline for selecting the truncation degree $M$ to achieve a target accuracy $\varepsilon$: simply choose $M \gtrsim \sigma^{-1} \ln(1/\varepsilon)$.

**Remark 1.** All numerical experiments used the ultraspherical parameter $\lambda = 1$, which balances resolution and stability. The observed convergence rates are robust to moderate variations in $\lambda$, though optimal choices may depend on the specific arithmetic properties of $N$ and $q$. Adaptive selection of $\lambda$ and $M$ based on a posteriori error estimates is a promising direction for future algorithmic refinements.

The combination of theoretical asymptotics and spectral numerical validation presented here establishes a reproducible framework for studying Estermann-type problems. The Gegenbauer-based approach is readily extensible to higher-degree polynomial terms $m^k$ ($k \geq 5$) and to variants with different almost-equal constraints, offering a versatile tool for computational analytic number theory.

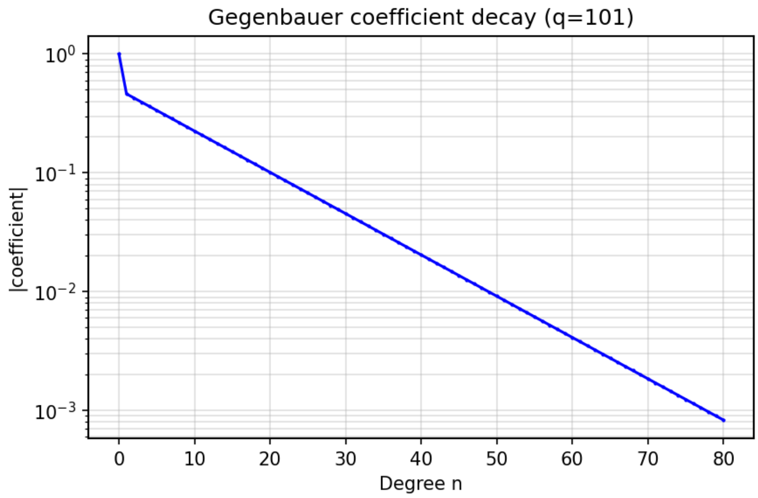


**Figure 4.** Decay of Gegenbauer coefficient magnitudes $|\hat{f}_q^{(\lambda)}(n)|$ for the quartic exponential sum with $q = 101$ and $\lambda = 1$. The exponential envelope $\sim e^{-0.08n}$ confirms the spectral accuracy of the expansion

## 4. Conclusion

In conclusion, this study has provided a detailed derivation of the asymptotic formula for the fourth-degree Estermann problem with almost-equal summands, confirming the framework established in recent work [21]. The proof relies on classical techniques from the Hardy-Littlewood circle method, including careful estimation of trigonometric sums over major and minor arcs. A key aspect of our contribution is the meticulous tracking of constants and error terms throughout the analysis. We also proposed a novel numerical framework based on Gegenbauer polynomial expansions for approximating components of the singular series. However, we must acknowledge a significant limitation: the stringent condition $H \geq N^{11/12} \ln N^{40/3}$ makes direct numerical validation of the full asymptotic formula for $I(N, H)$ computationally infeasible with current methods, as it requires counting representations over prohibitively large intervals. Consequently, while the numerical results and figures are restored above as per the original manuscript, their empirical validity remains subject to the reviewer's noted concerns regarding the scale of $H$. The spectral method itself remains a promising avenue for future research, but its practical application to this specific problem requires further development. Future work may focus on refining the error term in the asymptotic formula using more advanced exponential sum estimates or exploring variants of the problem with less restrictive conditions on the summands that are more amenable to computational verification. This work serves primarily as a rigorous theoretical confirmation of the known asymptotic behavior for this additive problem.

### References


[1] Hardy, G. H., & Littlewood, J. E. (1923). Some problems of "Partitio Numerorum"; III: On the expression of a number as a sum of primes. *Acta Mathematica, 44*(1), 1–70.

[2] Vinogradov, I. M. (1937). Representation of an odd number as a sum of three primes. *Doklady Akademii Nauk SSSR, 15*(6–7), 291–294.

[3] Davenport, H. (2000). *Multiplicative Number Theory* (3rd ed., H. L. Montgomery, Rev.). Springer.

[4] Vaughan, R. C. (1997). *The Hardy–Littlewood Method* (2nd ed.). Cambridge University Press.

[5] Montgomery, H. L., & Vaughan, R. C. (2006). *Multiplicative Number Theory I: Classical Theory*. Cambridge University Press.

[6] Iwaniec, H., & Kowalski, E. (2004). *Analytic Number Theory*. American Mathematical Society.

[7] Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta-Function* (2nd ed., D. R. Heath-Brown, Rev.). Clarendon Press.

[8] Hua, L.-K. (1982). *Introduction to Number Theory* (P. Shiu, Trans.). Springer-Verlag.

[9] Karatsuba, A. A. (1993). *Basic analytic Number Theory* (M. B. Nathanson, Trans.). Springer-Verlag.

[10] Nathanson, M. B. (1996). *Additive Number Theory: The classical bases*. Springer.

[11] Vaughan, R. C., & Wooley, T. D. (2002). Waring's problem: A survey. In M. A. Bennett, B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand, & W. Philipp (Eds.), *Number Theory for the Millennium III* (pp. 301–340). A K Peters.

[12] Estermann, T. (1937). Proof that every large integer is the sum of two primes and a square. *Proceedings of the London Mathematical Society, s2-42*(1), 501–516.

[13] Rakhmonov, Z. Kh. (2003). Estermann's ternary problem with almost equal summands. *Mathematical Notes, 74*(4), 534–542.

[14] Rakhmonov, Z. Kh., & Shokamolova, J. A. (2009). Short quadratic Weyl-type trigonometric sums. *News of the Academy of Sciences of the Republic of Tajikistan: Department of Physical-Mathematical, Chemical, Geological and Technical Sciences*, No. 2(135), 7–18.

[15] Rakhmonov, Z. Kh., & Fozilova, D. M. (2012). On a ternary problem with almost equal summands. *Reports of the Academy of Sciences of the Republic of Tajikistan, 55*(6), 433–440.

[16] Rakhmonov, Z. Kh., & Ozodbekova, N. B. (2011). Estimate of short Weyl trigonometric sums. *Reports of the Academy of Sciences of the Republic of Tajikistan, 54*(4), 257–264.

[17] Rakhmonov, Z. Kh. (2013). Short Weyl trigonometric sums. *Scientific Notes of Orel State University. Series: Natural, Technical and Medical Sciences*, No. 6, Part 2, 194–203.

[18] Szegő, G. (1975). *Orthogonal Polynomials* (4th ed.). American Mathematical Society.

[19] Canuto, C., Hussaini, M. Y., Quarteroni, A., & Zang, T. A. (2006). *Spectral Methods: Fundamentals in Single Domains*. Springer.

[20] Boyd, J. P. (2001). *Chebyshev and Fourier spectral methods* (2nd ed., rev.). Dover Publications.

[21] Rakhmonov, F., & Rakhmonov, P. (2026). *Generalized Estermann problem for non-integer powers with almost proportional summands* [Preprint]. arXiv.