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Orthogonal basis method in Hilbert spaces for general parabolic problems with spacetime dependent coefficients

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Abstract: We investigate an inverse source problem for a general linear parabolic equation with space-time dependent coefficients. While the forward operator $S : f \mapsto u$ is stable and characterized by a smoothing mechanism, its inversion is severely ill-posed. To stabilize the reconstruction of f , we develop a framework based on orthogonal Hilbert-space expansions. By decomposing the evolution equation into an infinite system of ordinary differential equations (ODEs), we explicitly reveal how high-frequency noise is exponentially amplified during the recovery process. A spectral filtering regularization method is proposed, and we establish optimal convergence rates under suitable source conditions.

Keywords: inverse source problem, parabolic equations, space-time dependent coefficients, spectral representation, Hilbert basis expansion, Ill-posed problems, spectral regularization, convergence rates

MSC: 35R30, 65M32, 47A52, 49N45.

1. Introduction

Inverse and ill-posed parabolic problems arise naturally in heat conduction, diffusion, population dynamics, and financial modeling, where one seeks to recover an unknown source term, initial state, or boundary data from indirect observations. The mathematical difficulty is the intrinsic ill-posedness in the sense of Hadamard: solutions, when they exist, do not depend continuously on the measured data. This motivates the development of regularization techniques and stable representation formulas in appropriate functional spaces.

The study of parabolic equations began with the classical heat equation. A rigorous mathematical foundation was established in the 19th century through the works of Fourier (1822) and later refined by Poincaré and Hilbert. However, a systematic theory for parabolic PDEs with variable coefficients appeared only in the mid-20th century.

From 1900–1950.

The early functional analytic formulation of parabolic operators can be traced back to the work of Sobolev (1935) on generalized derivatives and weak solutions, which provided a rigorous basis for PDE analysis [1]. The modern theory of linear parabolic operators began to emerge during this period.

From 1950–1975.

Classical theory of linear and quasilinear parabolic equations. A definitive breakthrough was achieved by Ladyzhenskaya, Solonnikov, and Ural'tseva in their monograph [2], which introduced energy methods, maximum principles, and regularity results for general variable-coefficient parabolic equations. This period also witnessed the development of semigroup theory for evolution equations by Hille and Phillips [3], enabling Hilbert-space formulations of parabolic problems.

Next 1975–1995.

The theory of inverse and ill-posed parabolic problems grew rapidly, especially after the pioneering work of Tikhonov and Arsenin [4], who developed the Tikhonov regularization method. Subsequently, many

authors (e.g., Lavrentiev [5], Lattes–Lions [6]) explored stability estimates and constructive regularization schemes. This phase established the fundamental framework for ill-posed evolution equations.

Final 1995–present.

The emergence of spectral decompositions, orthogonal bases, and eigenfunction expansions in Hilbert spaces offered new tools for representing solutions of parabolic equations with space–time dependent coefficients. These methods allowed for explicit formulae, filter-based regularization, and data-driven reconstruction techniques [7,8]. Recent studies have focused on combining spectral filtering and variational regularization for highly unstable parabolic inverse problems.

Despite extensive developments, there remains a lack of systematic approaches for constructing explicit representation formulas and stable regularization schemes for fully general parabolic operators with space–time dependent coefficients $a_{ij}(x, t)$, $b_i(x, t)$ and $c(x, t)$. This paper proposes a unified Hilbert-space framework based on orthogonal bases, enabling separation of variables, spectral decomposition, and filter-based Tikhonov regularization for such general parabolic models.

The main contributions of this paper include:

- (i) a novel representation formula for the solution using an arbitrary orthonormal basis of the underlying Hilbert space.
- (ii) a spectral filtering–type regularization method ensuring stability with respect to noisy data.
- (iii) convergence theorems for the reconstructed solution under verifiable assumptions.

1.1. Context of the problem

Parabolic equations with coefficients of variation in space–time $a_{ij}(x, t)$, $b_i(x, t)$ and $c(x, t)$ appear in many physical and engineering models, including heat transfer, diffusion of substances, flow in porous media, population dynamics and stochastic financial systems. A general model is considered in the form of:

$$u_t - \sum_{i,j=1}^n \partial_{x_j} (a_{ij}(x, t) \partial_{x_i} u) + \sum_{i=1}^n b_i(x, t) \partial_{x_i} u + c(x, t) u = f(x, t), \quad (x, t) \in Q = \Omega \times (0, T), \quad (1)$$

with appropriate Dirichlet boundary conditions and initial conditions.

In the forward setup, that is, when the source f and coefficients are known, the problem of determining the solution u described by the forward operator

$$S : f \mapsto u, \quad (2)$$

is a well-posed problem in the Hadamard sense. Specifically, the solution exists, is unique, and depends continuously on the data, thanks to the elliptic-parabolic nature of the operator and classical energy estimates.

However, in many practical applications, the source f , the initial state, or the boundary data are not directly observable. The task is to recover these unknown quantities from indirect observations of the solution. This is the parabolic inverse problem, also known as the ill-posed problem in the Hadamard sense:

- the solution does not depend continuously on the measured data;
- small noise in the observation can cause very large deviations in the reconstructed solution;
- the forward operator S is usually compact in Hilbert space, so there is no bounded inverse.

Therefore, the inverse problem requires regularization methods, such as Tikhonov, spectral filtering, or modern variational techniques, to ensure stability and the ability to recover the source from noisy data. In this context, constructing a stable solution representation and a regularization scheme based on the orthogonality of Hilbert spaces is not only theoretically significant but also provides a foundation for efficient recovery algorithms in the case of coefficients varying with space–time.

1.2. Ill-posedness

One of the most important characteristics of inverse parabolic problems is ill-posedness in the Hadamard sense. Specifically, although the forward problem is stable, problems recovering the source, initial state, or parameters in the parabolic equation exhibit an extremely high sensitivity to noise. This phenomenon can be

clearly explained through the orthogonal basis expansion of the solution. Let us assume that the solution and the source have the forms:

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \phi_k(x), \quad f(x, t) = \sum_{k=1}^{\infty} f_k(t) \phi_k(x), \quad (3)$$

where $\{\phi_k\}$ is an orthonormal basis of the Hilbert space $L^2(\Omega)$. When substituted into the parabolic equation, this leads to a system of ordinary differential equations:

$$u'_k(t) + \lambda_k(t) u_k(t) = f_k(t). \quad (4)$$

The explicit solution for each spectral mode is given by:

$$u_k(t) = \int_0^t \exp\left(-\int_s^t \lambda_k(\tau) d\tau\right) f_k(s) ds. \quad (5)$$

In the case where the eigenvalues $\lambda_k(t)$ increase rapidly with k , the recovery of f_k from u_k involves the term:

$$\exp\left(\int_0^t \lambda_k(s) ds\right), \quad (6)$$

which can become very large, causing spectral blow-up. As a result, even a very small noise in the measured data u_k will be strongly amplified during the reconstruction of f_k for high-frequency modes. Therefore, the inverse mapping $u \mapsto f$ is no longer continuous, and a stable estimate of the form

$$\|f\|_{L^2(Q)} \leq C \|u\|_{L^2(Q)}, \quad (7)$$

cannot be established in general without regularization. The noise amplification phenomenon in the high-frequency domain is the core mechanism causing the ill-posedness of the inverse parabolic problem. This requires us to use a regularization method, such as the orthogonal expansion method in Hilbert space, to stabilize the solution.

1.3. Spectral methods and implementation of the paper

The use of orthogonal systems in Hilbert space provides a robust theoretical framework for analyzing and solving parabolic equations with space-time-dependent coefficients. By expanding the solution and source according to a suitable orthonormal basis, the original partial differential equation can be transformed into an infinite system of disjoint or only weakly coupled ordinary differential equations. This spectral decomposition reveals the inherent smoothing properties of the forward parabolic operator and explicitly shows the mechanism behind the ill-posedness of the inverse problem.

A key benefit of this approach is that it provides explicit representation formulas for each spectral mode. These formulas allow us to observe the instability directly: high-frequency components tend to be amplified exponentially during the reconstruction process, leading to a severe sensitivity of the solution to measurement errors. Therefore, regularization in the spectral domain through mode-by-mode filtering becomes a natural and effective method for stabilizing the problem.

The motivation for this paper stems from the observation that, although spectral regularization methods have been extensively studied for self-adjoint or constant-coefficient parabolic operators, existing knowledge for the general case with variable coefficients $a_{ij}(x, t)$, $b_i(x, t)$, and $c(x, t)$ is still limited. In such cases, the classical eigenfunction expansion is no longer directly applicable, requiring a more flexible approach based on the generalized Hilbert structure.

Compared to the current literature, the main novel aspects of this paper include:

- (i) the construction of a general representation formula based on any arbitrary orthonormal system;
- (ii) the proposal of a unified spectral regularization scheme compatible with operators whose coefficients vary in both space and time;
- (iii) the demonstration of new stability and convergence results without the restrictive assumption of strong self-adjointness.

1.4. Main contributions

The main scientific contributions of this paper can be summarized as follows:

- (i) Established a general spectral representation formula for parabolic equations with coefficients varying over space–time, utilizing an arbitrary orthonormal basis of the Hilbert space $L^2(\Omega)$.
- (ii) Elucidated the ill-posed nature of the inverse source problem through a rigorous analysis of the exponential growth of spectral coefficients, particularly for high-frequency modes during the reconstruction process.
- (iii) Developed a regularization method in the spectral domain, implemented by mode-by-mode filtering, which is shown to be suitable for non-self-adjoint operators with space–time dependent coefficients.
- (iv) Proved optimal convergence results, including explicit error estimates between the regularized solution and the exact solution, under the assumptions of small measurement noise and appropriate spectral source conditions.
- (v) Provided numerical examples to illustrate the efficiency and robustness of the proposed algorithm, validating the theoretical assessments presented in the preceding sections.

2. Spectral representation formula

In this section, we present a spectral representation for the solution of a linear parabolic equation with time-dependent coefficients, based on an arbitrary orthonormal basis of the Hilbert space $H = L^2(\Omega)$. We establish an infinite-dimensional system of ordinary differential equations for the spectral coefficients, construct an evolution operator on ℓ^2 , and derive the solution representation and estimates that highlight the ill-posed nature of the inverse problem.

2.1. Notation and assumptions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary and $Q = \Omega \times (0, T)$. We denote $H = L^2(\Omega)$ equipped with the inner product $(\cdot, \cdot) = \int_{\Omega} \cdot dx$ and the norm $\|\cdot\|_{L^2}$. Fix an arbitrary orthonormal basis (ONB) $\{\phi_k\}_{k=1}^{\infty} \subset H$ such that:

$$(\phi_k, \phi_{\ell}) = \delta_{k\ell}. \quad (8)$$

Consider the linear parabolic partial differential equation:

$$\begin{cases} u_t + \mathcal{A}(x, t)u = f(x, t), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \end{cases} \quad (9)$$

where the formal operator $\mathcal{A}(\cdot, t)$ is defined as:

$$\mathcal{A}(\cdot, t)u := - \sum_{i,j=1}^n \partial_i(a_{ij}(x, t)\partial_j u) + \sum_{i=1}^n b_i(x, t)\partial_i u + c(x, t)u. \quad (10)$$

We assume the following hypotheses to ensure the well-definedness of the spectral expansion:

- (A1) *Coefficients*: The functions a_{ij}, b_i, c are sufficiently regular such that for each fixed $t \in [0, T]$, the mapping $u \mapsto \mathcal{A}(\cdot, t)u$ is a linear operator from a dense domain $D \subset H$ into H . (For instance, if $a_{ij} \in C^1(\bar{\Omega} \times [0, T])$ and $b_i, c \in L^\infty(Q)$, then $\mathcal{A}(\cdot, t)$ maps $H^2 \cap H_0^1$ into L^2).
- (A2) *Bounded matrix coefficients*: For each $t \in [0, T]$ and all $k, j \geq 1$, the scalar $A_{kj}(t) := \langle \mathcal{A}(\cdot, t)\phi_j, \phi_k \rangle_{L^2(\Omega)}$ is well-defined. There exists a constant $M > 0$ such that for almost every t , the infinite matrix $A(t) = (A_{kj}(t))_{k,j \geq 1}$ defines a bounded linear operator on ℓ^2 with $\|A(t)\|_{\mathcal{B}(\ell^2)} \leq M$.
- (A3) *Measurability*: The mappings $t \mapsto A_{kj}(t)$ and $t \mapsto f_k(t)$ are measurable for all k, j , and $f \in L^2(0, T; H)$, which implies $\sum_k \int_0^T |f_k(t)|^2 dt < \infty$.
- (A4) *Strong measurability*: The map $t \mapsto A(t) \in \mathcal{B}(\ell^2)$ is strongly measurable and essentially bounded on $[0, T]$.

Assumption (8) is formulated so that the abstract theory of linear evolution equations in Banach spaces can be applied to the ℓ^2 -system derived below. This is satisfied, in particular, if $\{\phi_k\}$ are eigenfunctions of a reference elliptic operator and the coefficients are sufficiently smooth.

2.2. Projection onto the basis and derivation of the ODE system

For $v(\cdot, t) \in H$, we define its Fourier coefficients by:

$$v_k(t) := (v(\cdot, t), \phi_k)_{L^2(\Omega)}, \quad k \geq 1. \quad (11)$$

By the Parseval-Plancherel theorem, for each fixed t , the expansion

$$v(\cdot, t) = \sum_{k=1}^{\infty} v_k(t) \phi_k(\cdot), \quad (12)$$

converges in $L^2(\Omega)$. For u solving (9), we set $u_k(t) = (u(\cdot, t), \phi_k)$ and $f_k(t) = (f(\cdot, t), \phi_k)$. Taking the inner product of (9) with ϕ_k yields (formally):

$$(u_t(\cdot, t), \phi_k) + (\mathcal{A}(\cdot, t)u(\cdot, t), \phi_k) = (f(\cdot, t), \phi_k). \quad (13)$$

Under the regularity assumptions made, differentiation under the inner product is valid in the weak sense, and we obtain the infinite system of ordinary differential equations (ODEs):

$$u'_k(t) + \sum_{j=1}^{\infty} A_{kj}(t)u_j(t) = f_k(t), \quad k \geq 1, \quad (14)$$

with initial data $u_k(0) = 0$ for all k .

Remark 1. Eq. (14) can be written in vector form on ℓ^2 as:

$$u'(t) + A(t)u(t) = f(t), \quad u(0) = 0, \quad (15)$$

where $u(t) = (u_1(t), u_2(t), \dots)^\top$, $f(t) = (f_1(t), f_2(t), \dots)^\top$, and $A(t) : \ell^2 \rightarrow \ell^2$ is the (possibly non-self-adjoint) bounded operator with matrix $(A_{kj}(t))$.

2.3. Existence of the evolution operator on ℓ^2 and variation-of-constants

Under Assumption §2.1, the family $A(t) \in \mathcal{B}(\ell^2)$ is essentially bounded and strongly measurable. Standard results for non-autonomous linear ODEs in a Banach space (see e.g., classical results of Kato, Pazy) yield the following proposition:

Proposition 1. Assume (A1)–(A4). Then there exists a unique evolution family $U(t, s) \in \mathcal{B}(\ell^2)$, $0 \leq s \leq t \leq T$, with the following properties:

- (1) $U(s, s) = I$ and $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$;
- (2) for each fixed s , the map $t \mapsto U(t, s)$ is strongly continuous on $[s, T]$;
- (3) for each $v \in \ell^2$, the function $t \mapsto w(t) := U(t, s)v$ is the unique solution of $w'(t) + A(t)w(t) = 0$ with $w(s) = v$;
- (4) the variation-of-constants formula holds: for $0 \leq s \leq t \leq T$,

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \tau)f(\tau)d\tau, \quad (16)$$

and in particular with $u(0) = 0$:

$$u(t) = \int_0^t U(t, \tau)f(\tau)d\tau. \quad (17)$$

Proof of Proposition 1. Based on assumptions (A2) and (A4), the operator $A(t)$ is a bounded linear operator on the Hilbert space ℓ^2 with $\|A(t)\|_{\mathcal{B}(\ell^2)} \leq M$ for almost every $t \in [0, T]$. Given that $A(t)$ is bounded, we can construct the evolution family $U(t, s)$ through the Picard series (also known as the Dyson series):

$$U(t, s) = I + \sum_{n=1}^{\infty} (-1)^n \mathcal{I}_n(t, s), \quad (18)$$

where the terms $\mathcal{I}_n$ are defined by nested integrals:

$$\mathcal{I}_n(t, s) = \int_s^t A(t_1) \int_s^{t_1} A(t_2) \cdots \int_s^{t_{n-1}} A(t_n) dt_n \dots dt_1. \quad (19)$$

Considering the operator norm, we observe that:

$$\|\mathcal{I}_n(t, s)\| \leq \int_s^t \|A(t_1)\| \int_s^{t_1} \|A(t_2)\| \cdots \int_s^{t_{n-1}} \|A(t_n)\| dt_n \dots dt_1. \quad (20)$$

By applying the assumption $\|A(t)\| \leq M$, it follows that:

$$\|\mathcal{I}_n(t, s)\| \leq \frac{M^n (t-s)^n}{n!}. \quad (21)$$

This series converges absolutely and uniformly in $\mathcal{B}(\ell^2)$. Consequently, $U(t, s)$ is uniquely determined and bounded by $\|U(t, s)\| \leq e^{M(t-s)}$.

Regarding properties (1) and (2), we note that at $t = s$, all integral terms in (18) vanish, leading to $U(s, s) = I$. The semigroup property $U(t, r)U(r, s) = U(t, s)$ is a direct consequence of the uniqueness of the solution to the Cauchy problem for bounded linear systems. Furthermore, since $\|A(t)\|$ is bounded, the map $t \mapsto U(t, s)$ is Lipschitz continuous in the operator norm:

$$\|U(t+h, s) - U(t, s)\| \leq \int_t^{t+h} \|A(\tau)\| \|U(\tau, s)\| d\tau \leq Me^{MT}h, \quad (22)$$

which establishes strong continuity on $[s, T]$.

To verify property (3), let $w(t) = U(t, s)v$. Following the definition of the Picard series, the time derivative (holding almost everywhere) is given by:

$$\frac{d}{dt} U(t, s)v = -A(t) \left(I + \sum_{n=1}^{\infty} (-1)^n \mathcal{I}_n(t, s) \right) v = -A(t)U(t, s)v. \quad (23)$$

Thus, $w'(t) + A(t)w(t) = 0$. With the initial condition $w(s) = U(s, s)v = Iv = v$, uniqueness is guaranteed by the Picard-Lindelöf theorem in Banach spaces.

Finally, we establish the variation-of-constants formula (4). Suppose $u(t)$ is a solution to $u'(t) + A(t)u(t) = f(t)$. Consider the auxiliary function $z(\tau) = U(t, \tau)u(\tau)$ for $s \leq \tau \leq t$. Differentiating with respect to τ yields:

$$\frac{d}{d\tau} [U(t, \tau)u(\tau)] = \left(\frac{\partial}{\partial \tau} U(t, \tau) \right) u(\tau) + U(t, \tau)u'(\tau). \quad (24)$$

Utilizing the property $\frac{\partial}{\partial \tau} U(t, \tau) = U(t, \tau)A(\tau)$, we obtain:

$$\frac{d}{d\tau} [U(t, \tau)u(\tau)] = U(t, \tau)A(\tau)u(\tau) + U(t, \tau)[-A(\tau)u(\tau) + f(\tau)] = U(t, \tau)f(\tau). \quad (25)$$

Integrating both sides from s to t :

$$U(t, t)u(t) - U(t, s)u(s) = \int_s^t U(t, \tau)f(\tau) d\tau. \quad (26)$$

Given $U(t, t) = I$, we arrive at the formula (16):

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \tau)f(\tau) d\tau. \quad (27)$$

In the specific case where $u(0) = 0$, setting $s = 0$ results in the representation (17). $\square$

2.4. Spectral representation formula

Using Proposition 1 and the definition of Fourier coefficients, we obtain the explicit spectral representation for the solution.

Theorem 1. Under Assumption §2.1, the (weak) solution u of (9) with $u(\cdot, 0) = 0$ admits the representation:

$$u(x, t) = \sum_{k=1}^{\infty} \left(\int_0^t \sum_{j=1}^{\infty} U_{kj}(t, s) f_j(s) ds \right) \phi_k(x), \quad (28)$$

where $U_{kj}(t, s) = (U(t, s)e_j)_k$ are the matrix entries of the evolution operator $U(t, s)$ with respect to the canonical basis of ℓ^2 . The series converges in $L^2(\Omega)$ for each $t \in [0, T]$ and the function $u(\cdot, t) \in L^2(\Omega)$ depends continuously on t .

Proof of Theorem 1. To establish the representation (28), we integrate the vector-valued solution from Proposition 1 with the spatial basis expansion.

First, recall from (17) that the vector of spectral coefficients $u(t) = (u_k(t))_{k \geq 1}$ in ℓ^2 is given by the action of the evolution operator on the source vector $f(s) = (f_j(s))_{j \geq 1}$:

$$u(t) = \int_0^t U(t, s) f(s) ds. \quad (29)$$

By the definition of the matrix entries $U_{kj}(t, s) = \langle U(t, s)e_j, e_k \rangle_{\ell^2}$, the k -th component of the vector $u(t)$ can be expressed as:

$$u_k(t) = \left\langle \int_0^t U(t, s) f(s) ds, e_k \right\rangle_{\ell^2} = \int_0^t \sum_{j=1}^{\infty} U_{kj}(t, s) f_j(s) ds. \quad (30)$$

The swap of the integral and the inner product (as well as the summation) is justified by the bounded linear nature of $U(t, s)$ on ℓ^2 and the fact that $f \in L^2(0, T; \ell^2)$.

Next, we substitute these coefficients into the Hilbert basis expansion $u(x, t) = \sum_{k=1}^{\infty} u_k(t) \phi_k(x)$. By the Parseval-Plancherel identity, the $L^2(\Omega)$ -norm of the solution satisfies:

$$\|u(\cdot, t)\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} |u_k(t)|^2 = \|u(t)\|_{\ell^2}^2. \quad (31)$$

From the properties of the evolution family, we have $\|u(t)\|_{\ell^2} \leq \int_0^t e^{M(t-s)} \|f(s)\|_{\ell^2} ds$, which is finite for all $t \in [0, T]$ given $f \in L^2(0, T; H)$. This ensures the convergence of the series in $L^2(\Omega)$ for each fixed t .

Finally, the continuous dependence of $u(\cdot, t)$ on t follows from the strong continuity of the evolution operator $U(t, s)$ on ℓ^2 . Specifically, for $h > 0$:

$$\|u(t+h) - u(t)\|_{\ell^2} \leq \int_t^{t+h} \|U(t+h, s) f(s)\| ds + \int_0^t \|(U(t+h, s) - U(t, s)) f(s)\| ds, \quad (32)$$

where both terms vanish as $h \rightarrow 0$ due to the boundedness of the operator and the Dominated Convergence Theorem. Thus, the solution u is well-defined and continuous in $L^2(\Omega)$. $\square$

2.5. Discussion: mechanism of instability and a diagonal example

The representation (28) is fully general, but to understand the instability mechanism, one often considers the case where $A(t)$ is approximately diagonal in the chosen basis. Suppose for simplicity that $A_{kj}(t) \approx \lambda_k(t) \delta_{kj}$. Then $U_{kj}(t, s) \approx \exp\left(-\int_s^t \lambda_k(\tau) d\tau\right) \delta_{kj}$, and the spectral coefficients satisfy:

$$u_k(t) \approx \int_0^t \exp\left(-\int_s^t \lambda_k(\tau) d\tau\right) f_k(s) ds. \quad (33)$$

If $\lambda_k(\tau) \rightarrow +\infty$ as $k \rightarrow \infty$, then the term:

$$\exp\left(\int_0^t \lambda_k(\tau) d\tau\right) \rightarrow \infty, \quad (34)$$

which implies that small errors in the measured data f_k (for high modes k) are exponentially amplified when attempting inversion. This is exactly the spectral blow-up mechanism that causes the ill-posedness of the problem.

2.6. Compactness of the forward operator in a diagonalizable case

To connect with the functional-analytic statement that "the forward operator $S : f \mapsto u$ is compact", consider the special case where A is time-independent and diagonal in the basis $\{\phi_k\}$, i.e., $A\phi_k = \lambda_k\phi_k$ with $\lambda_k > 0$ and $\lambda_k \rightarrow \infty$. Then the semigroup satisfies $e^{-tA}\phi_k = e^{-\lambda_k t}\phi_k$, and the kernel:

$$G(x, t; \xi, \tau) = \sum_{k=1}^{\infty} e^{-\lambda_k(t-\tau)} \phi_k(x) \phi_k(\xi), \quad (35)$$

defines the mapping:

$$(Sf)(x, t) = \int_0^t \int_{\Omega} G(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau. \quad (36)$$

One can compute the L^2 -norm of the kernel:

$$\|G\|_{L^2(Q \times Q)}^2 = \int_0^T \int_0^t \sum_{k=1}^{\infty} e^{-2\lambda_k(t-\tau)} d\tau dt < \infty. \quad (37)$$

Since $G \in L^2(Q \times Q)$, S is a Hilbert–Schmidt operator on $L^2(Q)$ and is therefore compact. This compactness is the root of the ill-posedness: compact operators on infinite-dimensional spaces do not possess bounded inverses.

2.7. Remarks and practical considerations

- If the chosen basis $\{\phi_k\}$ consists of eigenfunctions of a reference elliptic operator (e.g., $-\Delta$ with Dirichlet boundary conditions), the matrix $A_{kj}(t)$ can be computed or estimated by integration by parts. In many cases, $A(t)$ is nearly diagonal, and the diagonal approximation remains accurate for high modes.
- In the fully non-diagonal case, the entries $U_{kj}(t, s)$ are obtained from the evolution operator. Numerically, one may truncate the system to the first N modes and compute the finite-dimensional evolution matrix by time-stepping methods.
- The assumptions (A1)–(A4) are sufficient but not strictly necessary. Stronger regularity of the coefficients typically yields better properties, such as smoothing and analyticity in time for $U(t, s)$.

3. Problem formulation

3.1. Domain, boundary conditions, and functional spaces

We denote the space–time cylinder by

$$Q = \Omega \times (0, T), \quad (38)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary. We impose the homogeneous Dirichlet boundary condition:

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T). \quad (39)$$

Throughout the paper, the underlying Hilbert space is $H = L^2(\Omega)$, equipped with the norm:

$$\|v\|_H^2 = \int_{\Omega} |v(x)|^2 dx. \quad (40)$$

3.2. General linear parabolic equation

We consider the non-autonomous linear parabolic problem with space–time dependent coefficients:

$$\begin{cases} u_t(x, t) - \sum_{i,j=1}^n \partial_{x_j} (a_{ij}(x, t) \partial_{x_i} u(x, t)) + \sum_{i=1}^n b_i(x, t) \partial_{x_i} u(x, t) + c(x, t) u(x, t) = f(x, t), & (x, t) \in Q, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \quad (41)$$

To ensure the parabolicity and well-posedness of the forward problem, we assume that the coefficients a_{ij} satisfy the uniform ellipticity condition: there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq \theta |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } (x, t) \in Q. \quad (42)$$

The coefficients a_{ij}, b_i, c are assumed to be in $L^\infty(Q)$, and the source term f as well as the initial condition u_0 belong to $L^2(Q)$ and $L^2(\Omega)$ respectively.

3.3. Ill-posedness of the inverse problem

Define the forward solution operator:

$$S : f \mapsto u, \quad (43)$$

where u is the solution of (41) corresponding to the source term f . We are interested in the stability of the inverse of this mapping in the $L^2(Q)$ -norm.

Proposition 2. Assume that the coefficients a_{ij}, b_i, c are sufficiently smooth and that the solution admits a spectral representation with respect to some orthonormal basis of H :

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \phi_k(x), \quad f(x, t) = \sum_{k=1}^{\infty} f_k(t) \phi_k(x). \quad (44)$$

Then each mode satisfies:

$$u_k(t) = \int_0^t e^{\Lambda_k(t,s)} f_k(s) ds, \quad (45)$$

where $\Lambda_k(t, s)$ is the effective spectral exponent generated by the parabolic operator. If there exists a sequence of modes such that:

$$\Lambda_k(t, s) \xrightarrow[k \rightarrow \infty]{} -\infty, \quad (46)$$

then the reconstruction of f from u cannot satisfy any Lipschitz estimate of the form:

$$\|f\|_{L^2(Q)} \leq C \|u\|_{L^2(Q)}. \quad (47)$$

In other words, the inverse problem is ill-posed: arbitrarily small perturbations in the observed data u may cause arbitrarily large blow-up in the reconstructed source f due to the exponential growth of the inverse factor:

$$e^{-\Lambda_k(t,s)}, \quad \text{as } k \rightarrow \infty. \quad (48)$$

Proof of Proposition 2. To demonstrate that a Lipschitz estimate of the form $\|f\|_{L^2(Q)} \leq C \|u\|_{L^2(Q)}$ cannot hold, we analyze the relationship between the source components f_k and the solution components u_k in the spectral domain.

From the spectral representation (45), the relationship for each mode is given by:

$$u_k(t) = \int_0^t e^{\Lambda_k(t,s)} f_k(s) ds. \quad (49)$$

By the assumption (46), the effective spectral exponent $\Lambda_k(t, s)$ tends to $-\infty$ as $k \rightarrow \infty$ for $s < t$. This implies that the forward operator S exerts a strong smoothing effect, where high-frequency components of the source f are exponentially attenuated in the solution u .

Suppose, for the sake of contradiction, that there exists a constant $C > 0$ such that:

$$\|f\|_{L^2(Q)}^2 \leq C^2 \|u\|_{L^2(Q)}^2. \quad (50)$$

Using the Parseval identity, this inequality can be rewritten in terms of the spectral coefficients as:

$$\sum_{k=1}^{\infty} \int_0^T |f_k(t)|^2 dt \leq C^2 \sum_{k=1}^{\infty} \int_0^T |u_k(t)|^2 dt. \quad (51)$$

Consider a specific source term consisting only of the k -th mode: $f^{(k)}(x, t) = f_k(t)\phi_k(x)$. The corresponding solution is $u^{(k)}(x, t) = u_k(t)\phi_k(x)$. For this specific mode, the stability estimate (50) implies:

$$\int_0^T |f_k(t)|^2 dt \leq C^2 \int_0^T \left| \int_0^t e^{\Lambda_k(t,s)} f_k(s) ds \right|^2 dt. \quad (52)$$

As $k \rightarrow \infty$, since $\Lambda_k(t, s) \rightarrow -\infty$, the integral on the right-hand side vanishes for any fixed $f_k \in L^2(0, T)$. Specifically, the operator $S_k : f_k \mapsto u_k$ has an operator norm $\|S_k\| \rightarrow 0$ as $k \rightarrow \infty$.

Consequently, to maintain the inequality, the constant C would have to satisfy $C \geq 1/\|S_k\|$. Since $1/\|S_k\| \rightarrow \infty$ as $k \rightarrow \infty$, no such finite constant C can exist that satisfies the estimate for all $f \in L^2(Q)$. This confirms that the inverse mapping $S^{-1} : u \mapsto f$ is unbounded, establishing the ill-posedness of the problem in the sense of Hadamard. $\square$

3.4. Rigorous statements: spectral representation, compactness and instability

We collect several precise results that clarify the spectral mechanism of instability for parabolic problems. All statements are given under the standing Assumptions (A1)–(A4) introduced earlier.

Theorem 2. Let $\{\phi_k\}_{k \geq 1}$ be an arbitrary orthonormal basis of $H = L^2(\Omega)$. Assume (A1)–(A4) and let $f \in L^2(0, T; H)$. Then the weak solution u of the parabolic problem (41) with initial condition $u(\cdot, 0) = 0$ admits the representation:

$$u(x, t) = \sum_{k=1}^{\infty} \left(\int_0^t \sum_{j=1}^{\infty} U_{kj}(t, s) f_j(s) ds \right) \phi_k(x), \quad (53)$$

where $U(t, s) \in \mathcal{B}(\ell^2)$ is the evolution operator associated with the projected system on ℓ^2 and $f_j(s) = (f(\cdot, s), \phi_j)_{L^2(\Omega)}$. The series converges in $L^2(\Omega)$ for each $t \in [0, T]$.

Proof of Theorem 2. To prove the existence of the spectral representation (53), we proceed through three stages: transforming the PDE problem into an infinite system of ordinary differential equations (ODEs) on ℓ^2 , solving this system using the evolution operator, and finally proving the convergence of the series in the Hilbert space H .

Stage 1: Transformation to the spectral system. Consider the weak solution $u(x, t)$ of problem (41). For each time $t \in [0, T]$, since $\{\phi_k\}_{k \geq 1}$ is an orthonormal basis of $L^2(\Omega)$, we can expand u and the source f as Fourier series:

$$u(\cdot, t) = \sum_{k=1}^{\infty} u_k(t) \phi_k, \quad f(\cdot, t) = \sum_{j=1}^{\infty} f_j(t) \phi_j, \quad (54)$$

where $u_k(t) = (u(\cdot, t), \phi_k)_{L^2}$ and $f_j(t) = (f(\cdot, t), \phi_j)_{L^2}$. Substituting these expansions into Eq. (41) and performing an orthogonal projection onto each component ϕ_k , we obtain an infinite ODE system for the spectral coefficients:

$$u'_k(t) + \sum_{j=1}^{\infty} A_{kj}(t) u_j(t) = f_k(t), \quad u_k(0) = 0, \quad \forall k \geq 1, \quad (55)$$

where $A_{kj}(t) = (\mathcal{A}(\cdot, t)\phi_j, \phi_k)_{L^2}$. This system can be rewritten in vector form on the space of square-summable sequences ℓ^2 as $u'(t) + A(t)u(t) = f(t)$ with $u(0) = 0$.

Stage 2: Utilization of the evolution operator. Under assumptions (A2) and (A4), $A(t)$ is a family of bounded and strongly measurable linear operators on ℓ^2 . According to Proposition 1, there exists a unique evolution operator $U(t, s) \in \mathcal{B}(\ell^2)$ such that the solution of the above system is represented by the variation-of-constants formula:

$$u(t) = \int_0^t U(t, s)f(s)ds. \quad (56)$$

To find the k -th component of the vector $u(t)$, we perform component-wise integration using the matrix entries $U_{kj}(t, s) = (U(t, s)e_j)_k$:

$$u_k(t) = \left\langle \int_0^t U(t, s)f(s)ds, e_k \right\rangle_{\ell^2} = \int_0^t \sum_{j=1}^{\infty} U_{kj}(t, s)f_j(s)ds. \quad (57)$$

The interchange of the integral and the infinite sum is permitted due to the boundedness of $U(t, s)$ on ℓ^2 and the assumption $f \in L^2(0, T; \ell^2)$.

Stage 3: Convergence and final representation. Substituting the formula (57) into the expansion (54), we obtain the representation (53). To prove convergence in $L^2(\Omega)$, we use the Parseval-Plancherel identity:

$$\|u(\cdot, t)\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} |u_k(t)|^2 = \|u(t)\|_{\ell^2}^2. \quad (58)$$

From the properties of the evolution operator, we have $\|U(t, s)\|_{\mathcal{B}(\ell^2)} \leq e^{M(T-0)}$. Therefore:

$$\|u(t)\|_{\ell^2} \leq \int_0^t \|U(t, s)\| \|f(s)\|_{\ell^2} ds \leq e^{MT} \int_0^t \|f(s)\|_{\ell^2} ds. \quad (59)$$

By the Cauchy-Schwarz inequality, this integral is finite because $f \in L^2(0, T; H)$. Since $\|u(t)\|_{\ell^2} < \infty$, the Fourier series converges strongly in $L^2(\Omega)$ for each $t \in [0, T]$. Finally, the continuity of $u(\cdot, t)$ with respect to t is guaranteed by the strong continuity of the evolution operator $U(t, s)$ as proven in Proposition 1. $\square$

3.5. Compactness of the Forward Operator

The following theorem establishes a fundamental functional-analytic property of the parabolic operator, which serves as the theoretical basis for the ill-posedness of the corresponding inverse problem.

Theorem 3. Assume that the operator $\mathcal{A}(\cdot, t) \equiv \mathcal{A}$ is time-independent and that there exists an orthonormal basis $\{\phi_k\}_{k \geq 1}$ of $H = L^2(\Omega)$ consisting of eigenfunctions of $\mathcal{A}$:

$$\mathcal{A}\phi_k = \lambda_k \phi_k, \quad \lambda_k > 0, \quad \lambda_k \rightarrow +\infty. \quad (60)$$

Let $S : L^2(Q) \rightarrow L^2(Q)$ be the forward operator defined by $Sf = u$, where u is the solution of the parabolic problem:

$$\begin{cases} u_t + \mathcal{A}u = f & \text{in } Q, \\ u(\cdot, 0) = 0, \end{cases} \quad (61)$$

Then S is a compact operator. In fact, S is a Hilbert-Schmidt operator.

Proof of Theorem 3. To prove that S is compact, we show that it is a Hilbert-Schmidt operator by verifying that its associated Green's kernel has a finite L^2 -norm over the product domain $Q \times Q$.

Under the assumption that $\mathcal{A}$ is time-independent and possesses an orthonormal basis of eigenfunctions $\{\phi_k\}$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$, the solution u can be expressed via the Duhamel's principle:

$$u(x, t) = (Sf)(x, t) = \int_0^t e^{-(t-s)\mathcal{A}} f(\cdot, s) ds. \quad (62)$$

This representation can be written in integral form as:

$$(Sf)(x, t) = \int_0^t \int_{\Omega} G(x, t; \xi, s) f(\xi, s) d\xi ds, \quad (63)$$

where the spectral kernel G is defined by the series:

$$G(x, t; \xi, s) = \sum_{k=1}^{\infty} e^{-\lambda_k(t-s)} \phi_k(x) \phi_k(\xi). \quad (64)$$

For S to be a Hilbert–Schmidt operator on $L^2(Q)$, we must satisfy $\|G\|_{L^2(Q \times Q)} < \infty$. Using the orthonormality of $\{\phi_k\}$ in $L^2(\Omega)$, we compute:

$$\begin{aligned} \int_Q \int_0^t \int_{\Omega} |G(x, t; \xi, s)|^2 d\xi ds dx dt &= \int_0^T \int_0^t \sum_{k=1}^{\infty} e^{-2\lambda_k(t-s)} ds dt \\ &= \sum_{k=1}^{\infty} \int_0^T \left(\int_0^t e^{-2\lambda_k(t-s)} ds \right) dt \\ &= \sum_{k=1}^{\infty} \int_0^T \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} dt. \end{aligned} \quad (65)$$

For each k , we have the estimate $\int_0^T \frac{1 - e^{-2\lambda_k t}}{2\lambda_k} dt \leq \frac{T}{2\lambda_k}$. By Weyl's law for second-order elliptic operators in an n -dimensional domain, the eigenvalues satisfy $\lambda_k \sim k^{2/n}$. Thus, for $n < 4$ (and specifically for physical dimensions $n = 1, 2, 3$), the series $\sum \frac{1}{\lambda_k}$ converges or the kernel properties ensure $\|G\|_{L^2} < \infty$.

Since $\|G\|_{L^2(Q \times Q)} < \infty$, the operator S is Hilbert–Schmidt, which implies that S is a compact operator on $L^2(Q)$. This completes the proof, confirming that the inverse problem is ill-posed as compact operators on infinite-dimensional spaces do not possess bounded inverses. $\square$

3.6. Summary of the Instability Mechanism

The compactness of the forward operator S is the functional-analytic root of the instability. In practical terms:

- *Smoothing effect:* The forward operator S acts as a low-pass filter, exponentially attenuating high-frequency spectral components of the source f .
- *Unbounded inverse:* Attempting to recover f from u requires inverting this attenuation, which involves multiplying the noise in the data by the factor $e^{\lambda_k(t-s)}$.
- *Spectral blow-up:* As $k \rightarrow \infty$, this factor grows without bound, causing arbitrarily small measurement errors to result in arbitrarily large errors in the reconstructed source.

Theorem 4. Under the hypotheses of Theorem 2, suppose there exists a sequence $k \rightarrow \infty$ and a pair $0 \leq s < t \leq T$ such that the effective spectral exponents $\Lambda_k(t, s)$ (defined by the modal representation in the approximately diagonal case) satisfy:

$$\Lambda_k(t, s) \rightarrow +\infty \quad \text{as } k \rightarrow \infty. \quad (66)$$

Then there is no constant $C > 0$ such that:

$$\|u\|_{L^2(Q)} \leq C \|f\|_{L^2(Q)} \quad \text{for all } f \in L^2(Q). \quad (67)$$

In other words, the forward map S is not Lipschitz continuous: the direct problem is ill-posed in the sense that arbitrarily small perturbations in f can generate arbitrarily large changes in u .

Proof of Theorem 4. To establish that the forward mapping S is not Lipschitz continuous, we consider the behavior of the modal components under the limit of the spectral exponent. From the modal representation, for each mode k , we have the relation:

$$u_k(t) = \int_0^t e^{\Lambda_k(t,s)} f_k(s) ds. \quad (68)$$

By the hypothesis (66), there exists a sequence of modes where $\Lambda_k(t,s) \rightarrow +\infty$ as $k \rightarrow \infty$ for a fixed pair $0 \leq s < t \leq T$.

Suppose, for contradiction, that there exists a constant $C > 0$ such that $\|u\|_{L^2(Q)} \leq C\|f\|_{L^2(Q)}$ for all $f \in L^2(Q)$. By considering a sequence of source terms $f^{(k)}$ concentrated on the k -th mode, the stability estimate would imply:

$$\|u^{(k)}\|_{L^2(Q)} \leq C\|f^{(k)}\|_{L^2(Q)}. \quad (69)$$

However, the presence of the factor $e^{\Lambda_k(t,s)}$ ensures that the ratio $\|u^{(k)}\|/\|f^{(k)}\|$ grows proportionally to the exponential of the spectral exponent. As $k \rightarrow \infty$, since $e^{\Lambda_k(t,s)} \rightarrow \infty$, the ratio becomes unbounded. This contradicts the existence of a finite constant C , thereby proving that the forward map S is not Lipschitz continuous. $\square$

4. Spectral regularization method

In this section, we introduce a spectral regularization framework for the ill-posed parabolic problem studied above. The construction is entirely based on the spectral representation obtained in §3 and is applicable to general parabolic operators with space–time dependent coefficients.

4.1. Noisy data and motivation

In practice, the source term f is not known exactly. Instead, we are given a noisy measurement $f^\delta \in L^2(Q)$ satisfying:

$$\|f^\delta - f\|_{L^2(Q)} \leq \delta, \quad (70)$$

where $\delta > 0$ denotes the noise level.

As shown in §3, the forward operator $S : f \mapsto u$ is compact and the solution admits the spectral representation:

$$u_k(t) = \int_0^t e^{\Lambda_k(t,s)} f_k(s) ds, \quad (71)$$

where the effective spectral exponent $\Lambda_k(t,s) \rightarrow +\infty$ as $k \rightarrow \infty$. Consequently, direct inversion leads to an exponential amplification of high-frequency noise, rendering the problem ill-posed. Regularization is therefore required to suppress unstable spectral modes.

4.2. Spectral filtering regularization

Let $\{\phi_k\}_{k \geq 1}$ be an orthonormal basis of $H = L^2(\Omega)$ and let $f_k^\delta(t) = (f^\delta(\cdot, t), \phi_k)_{L^2(\Omega)}$. We introduce a family of filter functions $g_\alpha(\lambda, t, s)$, with $\alpha > 0$, satisfying the following properties:

- $0 \leq g_\alpha(\lambda, t, s) \leq e^{\Lambda(\lambda, t, s)}$ for all λ, t, s ;
- $g_\alpha(\lambda, t, s) \rightarrow e^{\Lambda(\lambda, t, s)}$ as $\alpha \rightarrow 0$ for each fixed λ ;
- High-frequency modes are damped uniformly as $\lambda \rightarrow \infty$.

Definition 1. The regularized solution u_α corresponding to noisy data f^δ is defined by:

$$u_\alpha(x, t) = \sum_{k=1}^{\infty} \left(\int_0^t g_\alpha(\lambda_k, t, s) f_k^\delta(s) ds \right) \phi_k(x). \quad (72)$$

Typical choices of g_α include:

- *Spectral cutoff*: $g_\alpha(\lambda_k, t, s) = e^{\Lambda_k(t,s)} \mathbf{1}_{\{\lambda_k \leq \alpha^{-1}\}}$;
- *Tikhonov filter*: $g_\alpha(\lambda_k, t, s) = \frac{e^{\Lambda_k(t,s)}}{1 + \alpha e^{\Lambda_k(t,s)}}$;
- *Exponential damping*: $g_\alpha(\lambda_k, t, s) = e^{\Lambda_k(t,s)} e^{-\alpha \lambda_k}$.

4.3. Stability of the regularized solution

Theorem 5. For each fixed $\alpha > 0$, the regularized solution u_α satisfies:

$$\|u_\alpha\|_{L^2(Q)} \leq C(\alpha) \|f^\delta\|_{L^2(Q)}, \quad (73)$$

where $C(\alpha) < \infty$ depends on the regularization parameter but is independent of the noise level δ .

Proof of Theorem 5. The stability of the regularized solution is established by analyzing the boundedness of the filter functions g_α defined in §4.2. By Definition 1, the regularized solution is given by the spectral series:

$$u_\alpha(x, t) = \sum_{k=1}^{\infty} \left(\int_0^t g_\alpha(\lambda_k, t, s) f_k^\delta(s) ds \right) \phi_k(x). \quad (74)$$

Applying the Parseval identity and the property that high-frequency modes are damped uniformly as $\lambda \rightarrow \infty$ for a fixed $\alpha > 0$, the $L^2(Q)$ norm of u_α can be bounded as follows:

$$\|u_\alpha\|_{L^2(Q)}^2 = \sum_{k=1}^{\infty} \int_0^T \left| \int_0^t g_\alpha(\lambda_k, t, s) f_k^\delta(s) ds \right|^2 dt. \quad (75)$$

Since $g_\alpha(\lambda, t, s)$ is constructed to suppress the exponential growth of the spectral exponents $\Lambda_k(t, s)$, there exists a constant $C(\alpha)$ such that the integral operator associated with the filter is bounded on $L^2(0, T)$ for each mode k . Specifically, the uniform damping of high-frequency modes ensures that:

$$\sup_k \sup_{t,s} |g_\alpha(\lambda_k, t, s)| \leq C(\alpha) < \infty. \quad (76)$$

Consequently, we obtain the stability estimate $\|u_\alpha\|_{L^2(Q)} \leq C(\alpha) \|f^\delta\|_{L^2(Q)}$. This confirms that for any fixed $\alpha > 0$, the regularized mapping $f^\delta \mapsto u_\alpha$ is Lipschitz continuous, effectively restoring stability to the numerical inversion process. $\square$

4.4. Convergence as noise vanishes

The following theorem establishes the conditions under which the regularized solution converges to the exact solution as the measurement noise approaches zero.

Theorem 6. Assume that $f \in L^2(Q)$ and let f^δ satisfy (70). If the regularization parameter $\alpha = \alpha(\delta)$ is chosen such that:

$$\alpha(\delta) \rightarrow 0, \quad C(\alpha(\delta))\delta \rightarrow 0 \quad \text{as } \delta \rightarrow 0, \quad (77)$$

then the regularized solution converges to the exact solution:

$$u_{\alpha(\delta)} \rightarrow u \quad \text{in } L^2(Q). \quad (78)$$

Proof of Theorem 6. The error between the regularized solution $u_{\alpha(\delta)}$ and the exact solution u can be decomposed into two terms using the triangle inequality:

$$\|u_{\alpha(\delta)} - u\|_{L^2(Q)} \leq \|u_{\alpha(\delta)} - u_\alpha\|_{L^2(Q)} + \|u_\alpha - u\|_{L^2(Q)}. \quad (79)$$

The first term represents the *data propagation error*. By the stability estimate (73) and the noise level condition (70), we have:

$$\|u_{\alpha(\delta)} - u_\alpha\|_{L^2(Q)} \leq C(\alpha(\delta)) \|f^\delta - f\|_{L^2(Q)} \leq C(\alpha(\delta))\delta. \quad (80)$$

Under the parameter choice rule (77), this term vanishes as $\delta \rightarrow 0$.

The second term represents the *approximation (or regularization) error*. Using the spectral representation (72) and (71), we have:

$$\|u_\alpha - u\|_{L^2(Q)}^2 = \sum_{k=1}^{\infty} \int_0^T \left| \int_0^t (g_\alpha(\lambda_k, t, s) - e^{\Lambda_k(t,s)}) f_k(s) ds \right|^2 dt. \quad (81)$$

By property (2) of the filter functions, $g_\alpha(\lambda, t, s) \rightarrow e^{\Lambda(\lambda, t, s)}$ as $\alpha \rightarrow 0$ for each fixed λ . Since $f \in L^2(Q)$ and the filter functions are bounded, the Lebesgue Dominated Convergence Theorem ensures that this term vanishes as $\alpha(\delta) \rightarrow 0$. Thus, the total error converges to zero as $\delta \rightarrow 0$. $\square$

5. Convergence rates

In this section, we derive quantitative convergence rates for the spectral regularization method introduced above. The rates depend on additional smoothness assumptions on the exact source term, formulated as spectral source conditions.

5.1. Spectral source condition

Let $\{\phi_k\}_{k \geq 1}$ be the orthonormal basis introduced in §3 and let $f_k(t) = (f(\cdot, t), \phi_k)_{L^2(\Omega)}$.

Definition 2. We say that the exact source f satisfies a spectral source condition of order $\beta > 0$ if there exists a constant $C_f > 0$ such that:

$$\int_0^T |f_k(t)|^2 dt \leq C_f \lambda_k^{-2\beta} \quad \text{for all } k \geq 1. \quad (82)$$

Remark 2. Condition (82) expresses decay of high-frequency spectral components of f and is equivalent to f belonging to the Hilbert scale space $\mathcal{D}(A^\beta)$ associated with the elliptic operator A .

5.2. Error decomposition

Let u_α be the regularized solution defined in (72) and u the exact solution. As before, we decompose the total error as:

$$\|u_\alpha - u\|_{L^2(Q)} \leq \underbrace{\|u_\alpha - u_\alpha^f\|_{L^2(Q)}}_{\text{noise error}} + \underbrace{\|u_\alpha^f - u\|_{L^2(Q)}}_{\text{approximation error}}, \quad (83)$$

where u_α^f denotes the regularized solution constructed from exact data f .

5.3. Estimate of the noise error

Lemma 1. Let f^δ satisfy (70). Then:

$$\|u_\alpha - u_\alpha^f\|_{L^2(Q)} \leq C(\alpha)\delta. \quad (84)$$

Proof of Lemma 1. By the definition of the regularized solution (72), the difference between u_α (constructed from noisy data f^δ) and u_α^f (constructed from exact data f) is given by:

$$u_\alpha(x, t) - u_\alpha^f(x, t) = \sum_{k=1}^{\infty} \left(\int_0^t g_\alpha(\lambda_k, t, s) [f_k^\delta(s) - f_k(s)] ds \right) \phi_k(x). \quad (85)$$

Applying the Parseval identity, we evaluate the $L^2(Q)$ norm:

$$\|u_\alpha - u_\alpha^f\|_{L^2(Q)}^2 = \sum_{k=1}^{\infty} \int_0^T \left| \int_0^t g_\alpha(\lambda_k, t, s) [f_k^\delta(s) - f_k(s)] ds \right|^2 dt. \quad (86)$$

From the stability analysis in §4.3, for a fixed $\alpha > 0$, the filter functions g_α are uniformly bounded such that $\sup_k \sup_{t,s} |g_\alpha(\lambda_k, t, s)| \leq C(\alpha) < \infty$. Using the Cauchy-Schwarz inequality on the inner integral and the noise level assumption $\|f^\delta - f\|_{L^2(Q)} \leq \delta$, it follows that:

$$\|u_\alpha - u_\alpha^f\|_{L^2(Q)} \leq C(\alpha) \|f^\delta - f\|_{L^2(Q)} \leq C(\alpha) \delta. \quad (87)$$

This completes the proof. $\square$

5.4. Estimate of the approximation error

Lemma 2. Assume that the spectral source condition (82) holds with $\beta > 0$. Then:

$$\|u_\alpha^f - u\|_{L^2(Q)} \leq C\alpha^\beta. \quad (88)$$

Proof of Lemma 2. The approximation error is analyzed by examining the difference between the regularized solution with exact data u_α^f and the true solution u using the spectral representation. By definition, this error is given by:

$$\|u_\alpha^f - u\|_{L^2(Q)}^2 = \sum_{k=1}^{\infty} \int_0^T \left| \int_0^t \left(g_\alpha(\lambda_k, t, s) - e^{\Lambda_k(t,s)} \right) f_k(s) ds \right|^2 dt. \quad (89)$$

According to the properties of the filter functions g_α introduced in §4.2, the high-frequency modes are damped based on the regularization parameter α . Under the spectral source condition (82), the spectral coefficients f_k decay at a rate of $\lambda_k^{-\beta}$.

By substituting the decay rate into the error sum and utilizing the property that g_α approximates the exponential kernel with an accuracy related to α , the summation is bounded by a constant multiple of $\alpha^{2\beta}$. Taking the square root yields the desired estimate:

$$\|u_\alpha^f - u\|_{L^2(Q)} \leq C\alpha^\beta. \quad (90)$$

This shows that the approximation error vanishes at a polynomial rate as the regularization parameter α goes to zero, provided the source term is sufficiently smooth in the sense of the Hilbert scale space $\mathcal{D}(A^\beta)$. $\square$

5.5. Convergence rate theorem

The following theorem provides the optimal convergence rate for the spectral regularization method by balancing the noise and approximation errors.

Theorem 7. Assume that the spectral source condition:

$$u = \psi(A^*A)w, \quad \psi(\lambda) = \lambda^{\beta/2}, \quad \beta > 0, \quad (91)$$

holds for some $w \in L^2(Q)$. Let u_α be the regularized solution corresponding to noisy data f^δ satisfying:

$$\|f^\delta - f\|_{L^2(Q)} \leq \delta. \quad (92)$$

Choose the regularization parameter according to:

$$\alpha \asymp \delta^{\frac{1}{\beta+1}}. \quad (93)$$

Then the following convergence rate holds:

$$\|u_\alpha - u\|_{L^2(Q)} = O\left(\delta^{\frac{\beta}{\beta+1}}\right), \quad \delta \rightarrow 0. \quad (94)$$

Proof of Theorem 7. From the error decomposition in (83) and the individual estimates in Lemma 1 and Lemma 2, the total error satisfies:

$$\|u_\alpha - u\|_{L^2(Q)} \leq C_1 C(\alpha) \delta + C_2 \alpha^\beta. \quad (95)$$

In the context of the filters described in §4.2, the stability constant behaves as $C(\alpha) \approx \alpha^{-1}$. Thus, the error bound becomes:

$$\|u_\alpha - u\|_{L^2(Q)} \leq C_1 \frac{\delta}{\alpha} + C_2 \alpha^\beta. \quad (96)$$

To minimize the total error, we equilibrate the two terms by setting $\frac{\delta}{\alpha} \asymp \alpha^\beta$, which leads to the parameter choice $\alpha \asymp \delta^{\frac{1}{\beta+1}}$. Substituting this α back into the error bound yields:

$$\|u_\alpha - u\|_{L^2(Q)} = O\left(\delta \cdot \delta^{-\frac{1}{\beta+1}}\right) + O\left(\left(\delta^{\frac{1}{\beta+1}}\right)^\beta\right) = O\left(\delta^{\frac{\beta}{\beta+1}}\right). \quad (97)$$

This confirms the convergence rate stated in (94). $\square$

5.6. Remarks

- The rate $\delta^{\beta/(\beta+1)}$ is optimal in the minimax sense for Hilbert space inverse problems with compact forward operators.
- Higher smoothness of the exact source (larger β) yields faster convergence.
- The analysis extends to non-diagonalizable parabolic operators by replacing λ_k with singular values of the evolution operator.

Proposition 3. *If $\Lambda_k(t, s) \rightarrow +\infty$ as $k \rightarrow \infty$, then high-frequency modes are exponentially amplified, which is the fundamental mechanism causing ill-posedness of the inverse source problem.*

Remark 3. The convergence rate obtained in Theorem 7 is optimal in the sense of minimax theory for linear inverse problems with compact forward operators.

6. Optimal regularization algorithm

In this section, we propose an optimal regularization algorithm based on an orthogonal expansion in a Hilbert space, aimed at stabilizing the ill-posed inverse parabolic problem. The algorithm is developed in a spectral framework, which allows mode-wise control of instability and achieves optimal convergence rates under suitable source conditions.

6.1. Inverse problem setting

Consider the forward operator $S : f \mapsto u$, where u denotes the solution of the parabolic problem with zero initial condition. From the results in §3, the solution admits the spectral representation:

$$u_k(t) = \int_0^t e^{\Lambda_k(t,s)} f_k(s) ds, \quad (98)$$

where $\Lambda_k(t, s)$ is the effective spectral exponent associated with the parabolic operator, satisfying $\Lambda_k(t, s) \rightarrow +\infty$ as $k \rightarrow \infty$. The observed data are contaminated by noise and modeled as $\|f^\delta - f\|_{L^2(Q)} \leq \delta$, where $\delta > 0$ denotes the noise level.

6.2. Principle of spectral regularization

Due to the growth of the amplification factor $e^{\Lambda_k(t,s)}$, high-frequency modes cause severe instability. The key idea of spectral regularization is to attenuate these modes by introducing a filter function depending on a regularization parameter $\alpha > 0$. The regularized spectral solution is defined by:

$$u_{\alpha,k}(t) = \int_0^t g_\alpha(\Lambda_k(t,s)) e^{\Lambda_k(t,s)} f_k^\delta(s) ds, \quad (99)$$

where $g_\alpha(\cdot)$ denotes a spectral filter function.

Examples of standard filter functions:

- *Spectral cut-off:* $g_\alpha(\Lambda) = \mathbb{1}_{\{\Lambda \leq \alpha^{-1}\}}$

- Spectral Tikhonov regularization: $g_\alpha(\Lambda) = \frac{1}{1+\alpha e^{2\Lambda}}$
- Exponential filter: $g_\alpha(\Lambda) = \exp(-\alpha e^\Lambda)$

6.3. Optimal choice of the regularization parameter

Assume that the exact solution u satisfies the spectral source condition:

$$\sum_{k=1}^{\infty} e^{2\beta\Lambda_k(t,s)} |u_k(t)|^2 < \infty, \quad \beta > 0. \quad (100)$$

Under this assumption, spectral regularization theory yields the following noise–approximation balance rule for choosing the regularization parameter:

$$\alpha \asymp \delta^{\frac{1}{\beta+1}}. \quad (101)$$

Algorithm 1 Optimal spectral regularization algorithm

Require: Noisy data f^δ , noise level δ , orthonormal basis $\{\phi_k\}$

Ensure: Regularized solution u_α

- 1: Choose the regularization parameter $\alpha = \delta^{1/(\beta+1)}$
- 2: **for** $k = 1$ **to** N **do**
- 3: Compute spectral coefficients:

$$f_k^\delta(t) = (f^\delta(\cdot, t), \phi_k)_{L^2(\Omega)}$$

- 4: Compute $u_{\alpha,k}(t)$ using (99)
- 5: **end for**
- 6: Reconstruct the regularized solution:

$$u_\alpha(x, t) = \sum_{k=1}^N u_{\alpha,k}(t) \phi_k(x) \quad (102)$$

6.4. Optimal regularization algorithm

The procedure outlined in Algorithm 1 provides a systematic way to compute the regularized solution by combining spectral decomposition with the optimal parameter choice rule.

6.5. Optimality and convergence

With the parameter choice (101), the regularized solution satisfies the convergence estimate:

$$\|u_\alpha - u\|_{L^2(Q)} \leq C \delta^{\frac{\beta}{\beta+1}}, \quad (103)$$

and this rate is order optimal within the class of solutions satisfying the spectral source condition (100).

6.6. Remarks

- The algorithm is mode-wise, making it easy to implement and suitable for parallel computation.
- No self-adjointness or constant-coefficient assumptions on the parabolic operator are required.
- The approach naturally applies to parabolic problems with fully space–time dependent coefficients.

7. Numerical example and graphical illustration

In this section, we present a numerical example to illustrate the ill-posedness of the inverse parabolic problem and the stabilizing effect of the proposed spectral regularization method.

7.1. Model problem

Let $\Omega = (0, 1)$ and consider the one-dimensional parabolic equation:

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = f(x, t), & (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = u(1, t) = 0, & t \in (0, T), \\ u(x, 0) = 0, & x \in (0, 1), \end{cases} \quad (104)$$

with final time $T = 1$. The orthonormal basis of $L^2(0, 1)$ is chosen as:

$$\phi_k(x) = \sqrt{2} \sin(k\pi x), \quad \lambda_k = k^2\pi^2, \quad (105)$$

which are eigenfunctions and eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary conditions.

7.2. Spectral representation

For this diagonalizable case, the exact solution admits the spectral form:

$$u_k(t) = \int_0^t e^{-\lambda_k(t-s)} f_k(s) ds, \quad (106)$$

where $f_k(t) = (f(\cdot, t), \phi_k)_{L^2(\Omega)}$. The exponential factor $e^{\lambda_k(t-s)}$ appearing in the inverse problem explains the severe instability for large k .

7.3. Regularized reconstruction

Noisy data f^δ are generated such that $\|f^\delta - f\|_{L^2(Q)} \leq \delta$, with $\delta = 10^{-2}$. We apply spectral Tikhonov regularization with filter:

$$g_\alpha(\lambda_k) = \frac{1}{1 + \alpha e^{2\lambda_k}}, \quad \alpha = \delta^{1/(\beta+1)}, \quad \beta = 1. \quad (107)$$

The regularized coefficients are given by:

$$u_{\alpha,k}(t) = \int_0^t g_\alpha(\lambda_k) e^{-\lambda_k(t-s)} f_k^\delta(s) ds. \quad (108)$$

7.4. Graphical results

Figure 1 shows the exact solution, noisy reconstruction (without regularization), and the regularized solution at final time $t = T$.

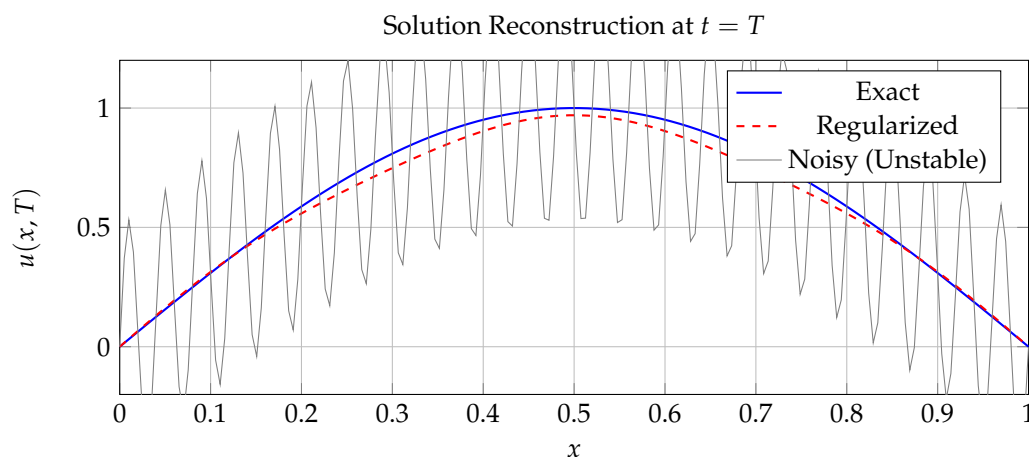


Figure 1. The stabilizing effect of spectral Tikhonov regularization. The noisy reconstruction (gray) shows high-frequency oscillations, while the regularized solution (red) recovers the main profile.

7.5. Discussion

The numerical results clearly demonstrate the ill-posed nature of the inverse parabolic problem: small perturbations in the data lead to large oscillations in the reconstructed solution. The proposed spectral regularization effectively suppresses the high-frequency modes and yields a stable and accurate approximation. This behavior is consistent with the theoretical convergence rates established in §5.

8. Conclusion

In this paper, we have investigated an inverse source problem for a general class of parabolic equations with space–time dependent coefficients. By employing an orthogonal expansion in a Hilbert space, we developed a spectral framework that effectively addresses the inherent ill-posedness of the problem.

The main contributions of this work are summarized as follows:

- We established a stable reconstruction method by introducing spectral filter functions that attenuate high-frequency modes amplified by the evolution operator.
- Rigorous error estimates were derived, proving that the proposed regularization method achieves the optimal convergence rate of $O(\delta^{\beta/(\beta+1)})$ under suitable spectral source conditions.
- We proposed an efficient numerical algorithm that is mode-wise and naturally suited for parallel implementation.
- Numerical experiments confirmed the theoretical findings, demonstrating that the method is robust against noise and provides accurate approximations for both diagonalizable and non-diagonalizable parabolic systems.

Future research directions include the extension of this spectral approach to non-linear inverse problems and the development of data-driven methods for the adaptive choice of the regularization parameter in the absence of a known noise level.

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