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Path-connected components of the space of generalized integration operators on Fock spaces

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Abstract: We characterize the boundedness and compactness of generalized integration operators acting between Fock spaces and apply these characterizations to investigate the path-connected components and the singleton of path-connected components of the space of bounded generalized integration operators. Moreover, we describe the essential norm of these operators.

Keywords: Fock spaces, generalized integration operator, bounded, compact, essential norm, path-connected components

MSC: 46E, 47B, 30H20.

1. Introduction

Let $\mathcal{H}(\mathbb{C})$ be space entire functions on $\mathbb{C}$. For $f, g \in \mathcal{H}(\mathbb{C})$, the Volterra type integral operator $\mathcal{V}_g$, induced by g, is

$$\mathcal{V}_g f(z) := \int_0^z f(w)g'(w)dw.$$

In recent decades, the study of Volterra type integral operators acting between different Fock-type spaces has developed significantly. Inspired by earlier contributions of Aleman and Siskakis [1,2] on the Hardy and Bergman spaces, Constantin [3] investigated various properties of the operator, particularly its boundedness and compactness on Fock spaces. Subsequently, in joint work with Peláez [4], they extended this study to weighted Fock spaces whose weight functions grow more rapidly than the standard Gaussian weight in Fock spaces. Related problems on Fock–Sobolev spaces were later addressed by Mengestie in [5]. Despite the progress on boundedness and compactness, there remains considerable interest in understanding the dynamical and topological behavior of these operators on Fock spaces. The first author of this paper, in collaboration with Bonet and Mengestie [6], studied several dynamical properties on Fock spaces endowed with weight functions of the form $|z|^l$, for $l > 0$. Moreover, topological properties such as path-connectedness, connected components, and isolated points of the space of bounded Volterra-type integral operators were studied in [7,8].

In [9], Chalmoukis introduced a new generalization of the Volterra type integral operator, referred to as the generalized integration operator $\mathcal{T}_g^{n,m}$, defined as follows. For a nonnegative integers m and n with $0 \le m < n$, and $f, g \in \mathcal{H}(\mathbb{C})$, the generalized integration operator, $\mathcal{T}_g^{n,m}$, is

$$\mathcal{T}_g^{n,m} f(z) := \int_0^z \int_0^{\eta_1} \cdots \int_0^{\eta_{n-1}} f^{(m)}(\eta_n) g^{(n-m)}(\eta_n) d\eta_n \cdots d\eta_1.$$

That is, $\mathcal{T}_g^{n,m}$ is the n-th iterate of the integral operator

$$\mathcal{V}_g^{n,m} f(z) := \int_0^z f^{(m)}(w) g^{(n-m)}(w) dw,$$

where $f^{(m)}$ denotes the m-th derivative of f and $f^{(0)} = f$. In particular, for $m = 0$ and $n = 1$, the operator $\mathcal{T}_g^{n,m}$ reduces to the Volterra type integral operator. To guarantee the linearity and well-definedness of the generalized integration operator $\mathcal{T}_g^{n,m}$, the constants of integration are fixed to be zero, namely $\mathcal{V}_g^{n,m} f(0) = 0$. Since $f^{(m)}$ and $g^{(n-m)}$ are entire functions, the integrals involved in $\mathcal{T}_g^{n,m}$ are independent of the choice of integration path.

Chalmoukis investigated certain properties of the generalized integration operator on Hardy spaces, and his work motivated further interest in studying the operator on various other function spaces [10,11]. The main goal of this paper is to characterize topological properties of $\mathcal{T}_g^{n,m}$ on Fock spaces $\mathcal{F}_p^\alpha$, including boundedness, compactness, essential norm, as well as path-connected and the singleton of path-connected components of space of bounded generalized integration operators.

For a real number $\alpha > 0$ and $0 < p < \infty$, Fock space $\mathcal{F}_p^\alpha$ is the space of all functions f in $\mathcal{H}(\mathbb{C})$ such that $f(z)e^{-\frac{\alpha}{2}|z|^2}$ is in $L^p(\mathbb{C}, dA)$, that is,

$$\|f\|_{(p,\alpha)} := \left(\int_{\mathbb{C}} |f(z)|^p \, e^{-\frac{p\alpha}{2}|z|^2} dA(z) \right)^{\frac{1}{p}} < \infty,$$

where dA is Euclidean area measure on the complex plane. And, for $\alpha > 0$ and $p = \infty$, Fock space $\mathcal{F}_\infty^\alpha$ is the space of all functions f in $\mathcal{H}(\mathbb{C})$ such that $f(z)e^{-\frac{\alpha}{2}|z|^2}$ is in $L^\infty(\mathbb{C})$, that is,

$$\|f\|_{(\infty,\alpha)} := \sup_{z \in \mathbb{C}} |f(z)| \, e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

For $f \in \mathcal{H}(\mathbb{C})$ and $0 < p \le \infty$, $f \in \mathcal{F}_p^\alpha$ if and only if the function $f^{(m)}(z)(1+|z|)^{-m} e^{-\frac{\alpha}{2}|z|^2}$ belongs to $L^p(\mathbb{C}, dA)$ (see, Theorem 2.1 of [12], Theorem 1 of [13], and Lemma 2.10 of [14]). Moreover, $\|f\|_{(p,\alpha)}$ can be estimated in terms of m-th derivative of f as follows;

$$\|f\|_{(p,\alpha)} \asymp \begin{cases} \sum_{j=0}^{m-1} \left| f^{(j)}(0) \right| + \left(\int_{\mathbb{C}} \frac{|f^{(m)}(z)|^p}{(1+|z|)^{mp}} e^{-\frac{p\alpha}{2}|z|^2} dA(z) \right)^{\frac{1}{p}}, & 0 < p < \infty \\ \sum_{j=0}^{m-1} \left| f^{(j)}(0) \right| + \sup_{z \in \mathbb{C}} \frac{|f^{(m)}(z)| e^{-\frac{\alpha}{2}|z|^2}}{(1+|z|)^m}, & p = \infty, \end{cases} \tag{1}$$

The notation " $\asymp$ " in (1) above is to mean $S(z) \preceq T(z)$ and $T(z) \preceq S(z)$, where $S(z) \preceq T(z)$ (or equivalently $T(z) \succeq S(z)$) means that there is a constant C such that $S(z) \le CT(z)$, for each $z \in \mathbb{C}$.

In particular, for $p = 2$, $\mathcal{F}_2^\alpha$ is a reproducing kernel Hilbert space with kernel function $K_{(w,\alpha)}(z) := e^{\alpha z \overline{w}}$, and normalized kernel function $k_{(w,\alpha)}(z) := e^{\alpha z \overline{w} - \frac{\alpha}{2}|w|^2}$. The function $k_{(w,\alpha)}$ belongs to every $\mathcal{F}_p^\alpha$, for $0 < p \le \infty$, with $\|k_{(w,\alpha)}\|_{(p,\alpha)} = 1$. Furthermore, Fock spaces satisfy a natural inclusion property: $\mathcal{F}_p^\alpha \subseteq \mathcal{F}_q^\alpha$ whenever $p \le q \le \infty$. For more information about Fock spaces, we refer to the book in [15].

2. Bounded and compact $\mathcal{T}_g^{n,m}$

In this section, we characterize the boundedness and compactness properties of the generalized integration operator $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ for later use. We first give a characterization of these properties in terms of the function property of

$$M_{(\alpha\beta,g,nm)}(z) := \frac{\left| g^{(n-m)}(z) \right| e^{\frac{\alpha-\beta}{2}|z|^2}}{(1+|z|)^{n-m}},$$

and then the simplified characterization will be given in the subsequent Corollaries.

Theorem 1.

(i) *Let $0 < p \le q \le \infty$ and $g \in \mathcal{H}(\mathbb{C})$. Then $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded (respectively, compact) if and only if $M_{(\alpha\beta,g,nm)} \in L^\infty(\mathbb{C})$ (respectively, $\lim_{|w| \to \infty} M_{(\alpha\beta,g,nm)}(w) = 0$).*

(ii) *Let* $0 < q < p \leq \infty$ *and* $g \in \mathcal{H}(\mathbb{C})$. *Then* $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ *is bounded or compact if and only if* $M_{(\alpha\beta,g,nm)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ *for* $p < \infty$ *and* $M_{(\alpha\beta,g,nm)} \in L^q(\mathbb{C}, dA)$ *for* $p = \infty$.

Proof. The proof follows arguments similar to those used in the proof of Theorem 1.3 in [16]; for the sake of completeness, we include the details here.

(i) If the operator $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded, then using the inclusion $\mathcal{F}_q^\beta \subseteq \mathcal{F}_\infty^\beta$ for $q \leq \infty$, and the estimate (1),

$$\begin{aligned}
\infty > \left\|\mathcal{T}_g^{n,m}\right\| \geq \left\|\mathcal{T}_g^{n,m} k_{(w,\alpha)}\right\|_{(q,\beta)} &\succeq \sup_{z\in\mathbb{C}} \left|\mathcal{T}_g^{n,m} k_{(w,\alpha)}(z)\right| e^{-\frac{\beta}{2}|z|^2} \\
&\succeq \frac{|\alpha \overline{w}|^m \left|g^{(n-m)}(z)\right|}{(1+|z|)^n} \left|e^{\alpha z\overline{w} - \frac{\alpha|w|^2}{2}}\right| e^{-\frac{\beta}{2}|z|^2} \\
&\succeq \frac{(1+|w|)^m \left|g^{(n-m)}(z)\right|}{(1+|z|)^n} \left|e^{\alpha z\overline{w} - \frac{\alpha|w|^2}{2}}\right| e^{-\frac{\beta}{2}|z|^2},
\end{aligned}$$

for all $w \in \mathbb{C}$. Putting $w = z$, we obtain $M_{(\alpha\beta,g,nm)} \in L^\infty(\mathbb{C})$. On the other hand, if $M_{(\alpha\beta,g,nm)}$ is bounded over $\mathbb{C}$, for $q < \infty$,

$$\begin{aligned}
\left\|\mathcal{T}_g^{n,m} f\right\|_{(q,\beta)}^q &\asymp \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q \left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \\
&\leq \left(\sup_{z\in\mathbb{C}} M^q_{(\alpha\beta,g,nm)}(z)\right) \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} dA(z) \\
&\preceq \left(\sup_{z\in\mathbb{C}} M^q_{(\alpha\beta,g,nm)}(z)\right) \|f\|_{(q,\alpha)}^q \preceq \left(\sup_{z\in\mathbb{C}} M^q_{(\alpha\beta,g,nm)}(z)\right) \|f\|_{(p,\alpha)}^q.
\end{aligned}$$

Thus, $\left\|\mathcal{T}_g^{n,m}\right\| \preceq \sup_{z\in\mathbb{C}} M_{(\alpha\beta,g,nm)}(z)$ and hence $\mathcal{T}_g^{n,m}$ is bounded. The case $q = \infty$ is similar to the above, which only needs to replace the integral in the above estimate by supremum, we omit it.

Next, if $\mathcal{T}_g^{n,m}$ is compact, then using the fact that $k_{(w,\alpha)}$ is uniformly bounded on $\mathcal{F}_p^\alpha$ and converges to zero on a compact subsets of $\mathbb{C}$ as $|w|$ goes to ∞,

$$\lim_{|w|\to\infty} M_{(\alpha\beta,g,nm)}(w) \preceq \lim_{|w|\to\infty} \left\|\mathcal{T}_g^{n,m} k_{(w,\alpha)}\right\|_{(q,\beta)} = 0.$$

Therefore $\lim_{|w|\to\infty} M_{(\alpha\beta,g,nm)}(w) = 0$. For the other direction, we let f_l to be bounded sequence in $\mathcal{F}_p^\alpha$ that converges to 0 uniformly on a compact subsets of $\mathbb{C}$ as $l \to \infty$. Then, for $R > 0$ and $q < \infty$, using (1),

$$\begin{aligned}
\left\|\mathcal{T}_g^{n,m} f_l\right\|_{(q,\beta)}^q &\asymp \int_{\mathbb{C}} \frac{\left|f_l^{(m)}(z)\right|^q \left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \\
&= \left(\int_{|z|\leq R} + \int_{|z|>R}\right) \frac{\left|f_l^{(m)}(z)\right|^q \left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \\
&\preceq \max_{|z|\leq R} \left|f_l^{(m)}(z)\right|^q \int_{|z|\leq R} \frac{\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \\
&\quad + \left(\sup_{|z|>R} M^q_{(\alpha\beta,g,nm)}(z)\right) \int_{|z|>R} \frac{\left|f_l^{(m)}(z)\right|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} dA(z) \\
&\preceq \|\mathcal{T}_g^{n,m} z^m\|_{(q,\beta)}^q \max_{|z|\leq R} \left|f_l^{(m)}(z)\right|^q + \|f_l\|_{(q,\alpha)}^q \left(\sup_{|z|>R} M^q_{(\alpha\beta,g,nm)}(z)\right) \\
&\preceq \max_{|z|\leq R} \left|f_l^{(m)}(z)\right|^q + \|f_l\|_{(p,\alpha)}^q \left(\sup_{|z|>R} M^q_{(\alpha\beta,g,nm)}(z)\right)
\end{aligned}$$

$$\preceq \max_{|z|\le R}\left|f_l^{(m)}(z)\right|^q+\sup_{|z|>R} M^q_{(\alpha\beta,g,nm)}(z).$$

Letting $l\to\infty$ and then $R\to\infty$, we get

$$\left\|\mathcal{T}_g^{n,m} f_l\right\|_{(q,\beta)}\to 0,$$

and hence $\mathcal{T}_g^{n,m}$ is compact. The case $q=\infty$ is very similar to the above. Thus, we omit it.

(ii) We first show that the integral conditions imply compactness of the operator. Let (f_l) be a uniformly bounded sequence in $\mathcal{F}_p^\alpha$ and $f_l\to 0$ uniformly on compact subsets of $\mathbb{C}$ as $l\to\infty$. Then, for $R>0$

$$\begin{aligned}
\left\|\mathcal{T}_g^{n,m} f_l\right\|^q_{(q,\beta)} &\asymp \int_{\mathbb{C}} \frac{\left|f_l^{(m)}(z)\right|^q\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2}dA(z)\\
&=\left(\int_{|z|\le R}+\int_{|z|>R}\right)\frac{\left|f_l^{(m)}(z)\right|^q\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2}dA(z)\\
&\preceq \max_{|z|\le R}\left|f_l^{(m)}(z)\right|^q\int_{|z|\le R}\frac{\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2}dA(z)\\
&\quad+\int_{|z|>R}\left(M^q_{(\alpha\beta,g,nm)}(z)\right)\left(\frac{\left|f_l^{(m)}(z)\right|^q}{(1+|z|)^{mq}}e^{-\frac{q\alpha}{2}|z|^2}\right)dA(z)\\
&\preceq \max_{|z|\le R}\left|f_l^{(m)}(z)\right|^q+\int_{|z|>R}\left(M^q_{(\alpha\beta,g,nm)}(z)\right)\left(\frac{\left|f_l^{(m)}(z)\right|^q}{(1+|z|)^{mq}}e^{-\frac{q\alpha}{2}|z|^2}\right)dA(z).
\end{aligned}\tag{2}$$

Now, for $p<\infty$, applying Hölder's inequality, and then using (1), the integral in (2) is further estimated as follows;

$$\begin{aligned}
&\int_{|z|>R}\left(M^q_{(\alpha\beta,g,nm)}(z)\right)\left(\frac{\left|f_l^{(m)}(z)\right|^q}{(1+|z|)^{mq}}e^{-\frac{q\alpha}{2}|z|^2}\right)dA(z)\\
&\le\left(\int_{|z|>R}\frac{\left|f_l^{(m)}(z)\right|^p}{(1+|z|)^{mp}}e^{-\frac{p\alpha}{2}|z|^2}dA(z)\right)^{\frac{q}{p}}\left(\int_{|z|>R}M^{\frac{pq}{p-q}}_{(\alpha\beta,g,nm)}(z)dA(z)\right)^{\frac{p-q}{p}}\\
&\preceq \|f_l\|^q_{(p,\alpha)}\left(\int_{|z|>R}M^{\frac{pq}{p-q}}_{(\alpha\beta,g,nm)}(z)dA(z)\right)^{\frac{p-q}{p}}\preceq\left(\int_{|z|>R}M^{\frac{pq}{p-q}}_{(\alpha\beta,g,nm)}(z)dA(z)\right)^{\frac{p-q}{p}},
\end{aligned}$$

and hence

$$\left\|\mathcal{T}_g^{n,m} f_l\right\|^q_{(q,\beta)}\preceq \max_{|z|\le R}\left|f_l^{(m)}(z)\right|^q+\left(\int_{|z|>R}M^{\frac{pq}{p-q}}_{(\alpha\beta,g,nm)}(z)dA(z)\right)^{\frac{p-q}{p}}.\tag{3}$$

Similarly, for $p=\infty$, from (2) we have

$$\begin{aligned}
\left\|\mathcal{T}_g^{n,m} f_l\right\|^q_{(q,\beta)} &\preceq \max_{|z|\le R}\left|f_l^{(m)}(z)\right|^q+\int_{|z|>R}\left(M^q_{(\alpha\beta,g,nm)}(z)\right)\left(\frac{\left|f_l^{(m)}(z)\right|^q}{(1+|z|)^{mq}}e^{-\frac{q\alpha}{2}|z|^2}\right)dA(z)\\
&\le \max_{|z|\le R}\left|f_l^{(m)}(z)\right|^q+\left(\sup_{|z|>R}\frac{\left|f_l^{(m)}(z)\right|^q}{(1+|z|)^{mq}}e^{-\frac{q\alpha}{2}|z|^2}\right)\int_{|z|>R}M^q_{(\alpha\beta,g,nm)}(z)dA(z)\\
&\preceq \max_{|z|\le R}\left|f_l^{(m)}(z)\right|^q+\|f_l\|^q_{(\infty,\alpha)}\int_{|z|>R}M^q_{(\alpha\beta,g,nm)}(z)dA(z)
\end{aligned}$$

$$\preceq \max_{|z|\le R} \left| f_l^{(m)}(z) \right|^q + \int_{|z|>R} M^q_{(\alpha\beta,g,nm)}(z)dA(z). \tag{4}$$

Letting $l \to \infty$, and then $R \to \infty$, in (3) and (4), $\|\mathcal{T}_g^{n,m} f_l\|_{(q,\beta)} \to 0$ as $l \to \infty$, which implies that $\mathcal{T}_g^{n,m}$ is compact.

Next, we show that boundedness of the operator implies the integral conditions. For this, using (1),

$$\begin{aligned}
\left\|\mathcal{T}_g^{n,m}\right\|^q \|f\|^p_{(p,\alpha)} &\ge \left\|\mathcal{T}_g^{n,m} f\right\|^q_{(q,\beta)} \\
&\asymp \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q \left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \\
&= \int_{\mathbb{C}} \left(\left|f^{(m)}(z)\right|^q e^{-\frac{q\alpha}{2}|z|^2}\right) \frac{\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{\frac{(\alpha-\beta)q}{2}|z|^2} dA(z) \\
&= \int_{\mathbb{C}} \left|f^{(m)}(z)\right|^q e^{-\frac{q\alpha}{2}|z|^2} d\mu_{(\alpha\beta,g,nm,q)}(z),
\end{aligned}$$

where

$$d\mu_{(\alpha\beta,g,nm,q)}(z) := \frac{\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{\frac{(\alpha-\beta)q}{2}|z|^2} dA(z).$$

Thus, if the operator $\mathcal{T}_g^{n,m}$ is bounded, then the differential operator $D^m f = f^{(m)}$,

$$D^m : \mathcal{F}_p^\alpha \to L^q_w(\mathbb{C}, d\mu_{(\alpha\beta,g,nm,q)}),$$

is bounded, where $L^q_w(\mathbb{C}, d\mu_{(\alpha\beta,g,nm,q)})$ denote the space of all measure function f such that $|f(z)|\, e^{-\frac{q\alpha}{2}|z|^2} \in L^p(\mathbb{C}, d\mu_{(\alpha\beta,g,nm,q)})$. By Theorem 3.3 and 3.4 of [14], D^m is bounded if and only if

$$\tilde{\mu}(w) := \int_{\mathbb{C}} (1+|z|)^{mq} e^{-\frac{\alpha q}{2}|z-w|^2} d\mu_{(\alpha\beta,g,nm,q)}(z),$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$ for $p < \infty$, and belongs to $L^1(\mathbb{C}, dA)$ for $p = \infty$. But,

$$\begin{aligned}
\tilde{\mu}(w) &= \int_{\mathbb{C}} (1+|z|)^{mq} e^{-\frac{\alpha q}{2}|z-w|^2} d\mu_{(\alpha\beta,g,nm,q)}(z) \\
&= \int_{\mathbb{C}} (1+|z|)^{mq} e^{-\frac{\alpha q}{2}|z-w|^2} \frac{\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}} e^{\frac{(\alpha-\beta)q}{2}|z|^2} dA(z) \\
&\ge \int_{D(w,1)} e^{-\frac{\alpha q}{2}|z-w|^2} \frac{\left|g^{(n-m)}(z)\right|^q}{(1+|z|)^{(n-m)q}} e^{\frac{(\alpha-\beta)q}{2}|z|^2} dA(z) \\
&\succeq \frac{\left|g^{(n-m)}(w)\right|^q}{(1+|w|)^{(n-m)q}} e^{\frac{(\alpha-\beta)q}{2}|w|^2} = M^q_{(\alpha\beta,g,nm)}(w).
\end{aligned}$$

Therefore, $M_{(\alpha\beta,g,nm)}$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ for $p < \infty$, and belongs to $L^q(\mathbb{C}, dA)$ for $p = \infty$. □

Next, we investigate how the modulating constants α and β in the Fock spaces affect the boundedness and compactness of $\mathcal{T}_g^{n,m}$. Accordingly, we consider different cases for α and β to obtain the following consequences of the above theorem.

Corollary 1. *Let $0 < p, q \le \infty$, $\alpha > \beta$ and $g \in \mathcal{H}(\mathbb{C})$. Then $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded or compact if and only if g is a polynomial of degree at most $(n-m)-1$. i.e $\mathcal{T}_g^{n,m}$ is a zero operator.*

Proof. If $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$ is bounded, then, by Theorem 1, $M_{(\alpha\beta,g,nm)}$ is uniformly bounded over $\mathbb{C}$, and hence there exists a positive real number N such that

$$M_{(\alpha\beta,g,nm)}(z) = \frac{\left|g^{(n-m)}(z)\right| e^{\frac{\alpha-\beta}{2}|z|^2}}{(1+|z|)^{n-m}} \leq N,$$

which implies that

$$\left|g^{(n-m)}(z)\right| \leq \frac{N(1+|z|)^{n-m}}{e^{\frac{\alpha-\beta}{2}|z|^2}}.$$

The right hand side of the above inequality goes to zero as $|z| \to \infty$. Thus, $g^{(n-m)}$ is identically zero function, and hence g is a polynomial of degree at most $(n-m)-1$. On the other hand, if g is a polynomial of degree at most $(n-m)-1$, then clearly $M_{(\alpha\beta,g,nm)}(z)$ goes to zero, and by Theorem 1, $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$ is compact. Since compactness of the operator implies boundedness, we have the desired equivalence. □

The above corollary shows that when $\alpha > \beta$, the operator $\mathcal{T}_g^{n,m}$ is bounded or compact if and only if it is the zero operator. As the zero operator trivially satisfies many topological and dynamical properties, we restrict our attention to the case $\alpha \leq \beta$, and in the special case when $\alpha = \beta$, we have the following.

Corollary 2. (i) *Let* $0 < p \leq q \leq \infty$, $\alpha = \beta$ *and* $g \in \mathcal{H}(\mathbb{C})$. *Then* $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$ *is bounded (respectively, compact) if and only if* g *is a polynomial of degree at most* $2(n-m)$ *(respectively,* g *is a polynomial of degree at most* $2(n-m)-1$*).*

(ii) *Let* $0 < q < p \leq \infty$, $\alpha = \beta$ *and* $g \in \mathcal{H}(\mathbb{C})$. *Then* $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$ *is bounded or compact if and only if* $q > \begin{cases} \frac{2p}{p+2}, & p < \infty \\ 2, & p = \infty \end{cases}$ *and* g *is a polynomial of degree at most* $2(n-m)-1$.

Proof. (i) Suppose $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$ is bounded. Then, by Theorem 1, there exists a positive real number N such that

$$\left|g^{(n-m)}(z)\right| \leq N(1+|z|)^{n-m},$$

which by Liouville's Theorem gives that $g^{(n-m)}$ is a polynomial of degree at most $n-m$, and hence g is a polynomial of degree at most $2(n-m)$. On the other hand, if g is a polynomial of degree at most $2(n-m)$, then $M_{(\alpha\beta,g,nm)}$ is clearly uniformly bounded over $\mathbb{C}$, and by Theorem 1, $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$ is bounded. The compactness case also follows from similar arguments.

(ii) By Theorem 1, boundedness or compactness of the operator is equivalent with $M_{(\alpha\beta,g,nm)} \in L^r(\mathbb{C}, dA)$, where $r = \frac{pq}{p-q}$ for $p < \infty$ and $r = q$ for $p = \infty$. That is, the operator is bounded or compact if and only if there exists a real number N such that

$$\int_{\mathbb{C}} \left(\frac{\left|g^{(n-m)}(z)\right|}{(1+|z|)^{n-m}} \right)^r dA(z) \leq N,$$

from which the restriction on exponents p and q, and g follows. □

The corollary shows that, in the case $\alpha = \beta$, the form of the function g that generates a bounded or compact operator $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$ is independent of the modulating constants α and β of the spaces. We next consider the case $\alpha < \beta$, and $\mathcal{T}_g^{n,m} : \mathcal{F}_p^{\alpha} \to \mathcal{F}_q^{\beta}$, $0 < p, q \leq \infty$. In this case, we characterize the boundedness and compactness of the operator in terms of the order and type of the inducing function g associated with $\mathcal{T}_g^{n,m}$. Recall that, the order $\rho(f)$ of an entire function f is defined as

$$\rho(f) := \lim_{r\to\infty} \sup \frac{\log(\log M_f(r))}{\log r},$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$, and if f is of finite order ρ, its type $\sigma(f)$ is defined to be

$$\sigma(f) := \lim_{r\to\infty} \sup \frac{\log M_f(r)}{r^{\rho(f)}}.$$

Proposition 1.

(i) *Let* $0 < p \leq q \leq \infty$, $\alpha < \beta$ *and* $g \in \mathcal{H}(\mathbb{C})$. *Then*

(a) $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ *is bounded if and only if* $\rho(g) < 2$ *or* $\rho(g) = 2$ *and* $\sigma(g) \leq \frac{\beta-\alpha}{2}$.

(b) $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ *is compact if and only if* $\rho(g) < 2$ *or* $\rho(g) = 2$ *and* $\sigma(g) < \frac{\beta-\alpha}{2}$.

(ii) *Let* $0 < q < p \leq \infty$, $\alpha < \beta$ *and* $g \in \mathcal{H}(\mathbb{C})$. *Then* $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ *is bounded or compact if and only if* $\rho(g) < 2$ *or* $\rho(g) = 2$ *and* $\sigma(g) < \frac{\beta-\alpha}{2}$.

Proof. The proof follows arguments similar to those used in the proof of Proposition 1.5 of [16], with the only modification being the replacement of the function g' there by $g^{(n-m)}$. □

In particular, if we assume that the symbol function g is zero-free, we obtain the following corollary, which can be proved by following arguments similar to those used in the proof of Theorem 1.6 in [16]. We include the proof here for completeness.

Corollary 3. (i) *Let* $0 < p \leq q \leq \infty$, $\alpha < \beta$ *and* g *be non vanishing function in* $\mathcal{H}(\mathbb{C})$. *Then* $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ *is*

(a) *bounded if and only if* g *has the form*

$$g(z) = e^{bz+az^2}, \tag{5}$$

for some $a, b \in \mathbb{C}$ *with* $|a| < \frac{\beta-\alpha}{2}$, *or* $|a| = \frac{\beta-\alpha}{2}$ *and either* $b = 0$ *or* $a = -\frac{(\beta-\alpha)b^2}{2|b|^2}$.

(b) *compact if and only if* g *has the form in* (5) *and* $|a| < \frac{\beta-\alpha}{2}$.

(ii) *Let* $0 < q < p \leq \infty$, $\alpha < \beta$ *and* g *be non vanishing function in* $\mathcal{H}(\mathbb{C})$. *Then* $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ *is bounded or compact if and only if* g *has the form in* (5) *and* $|a| < \frac{\beta-\alpha}{2}$.

Proof. (i) (a) First we assume the operator is bounded. Since by Proposition 1, $\rho(g) \leq 2$ and g is zero-free on $\mathbb{C}$, Hadamard's product formula gives $g(z) = e^{bz+az^2}$ for some $a, b \in \mathbb{C}$. Thus, we only show that the restrictions on a and b hold. If $a = 0$, then clearly $|a| < \frac{\beta-\alpha}{2}$. Assume $a \neq 0$, then $\rho(g) = 2$ and by Proposition 1, $\sigma(g) = |a| \leq \frac{\beta-\alpha}{2}$. In particular, if $|a| = \frac{\beta-\alpha}{2}$, then using Faá di Bruno's formula,

$$g^{(n-m)}(z) = e^{bz+az^2} \sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}, \tag{6}$$

where m_1 and m_2 are nonnegative integers such that $1m_1 + 2m_2 = n - m$ and the sum is over all such a partition (m_1, m_2) of $n - m$, we obtain

$$\begin{aligned} M_{(\alpha\beta,g,nm)}(z) &= \frac{\left|g^{(n-m)}(z)\right| e^{\frac{\alpha-\beta}{2}|z|^2}}{(1+|z|)^{n-m}} \\ &= \frac{\left|e^{bz+az^2} \sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}\right|}{(1+|z|)^{n-m}} e^{-\frac{(\beta-\alpha)}{2}|z|^2} \\ &= \frac{\left|\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}\right|}{(1+|z|)^{n-m}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2}, \end{aligned}$$

for all $z \neq 0$. setting $a = |a|e^{-2i\theta} = \frac{(\beta-\alpha)}{2}e^{-2i\theta}$, $0 \leq \theta < \pi$, and replacing z by $e^{i\theta}w$ in the above equation, we obtain

$$M_{(\alpha\beta,g,nm)}(we^{i\theta}) = \frac{\left|\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2awe^{i\theta}+b)^{m_1}a^{m_2}\right|}{(1+|w|)^{n-m}} e^{\Re(bwe^{i\theta})+\frac{(\beta-\alpha)}{2}(\Re(w^2)-|w|^2)},$$

for all $w \in \mathbb{C}$. Particularly, for a real number w, we get

$$M_{(\alpha\beta,g,nm)}(we^{i\theta}) = \frac{\left|\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2awe^{i\theta}+b)^{m_1}a^{m_2}\right|}{(1+|w|)^{n-m}} e^{w\Re(be^{i\theta})}. \tag{7}$$

Since $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded, by Theorem 1, the function $M_{(\alpha\beta,g,nm)}(we^{i\theta})$ above is uniformly bounded, which implies that $\Re(be^{i\theta}) = 0$. Thus, either $b = 0$ or $e^{-i\theta} = \pm\frac{ib}{|b|}$. Therefore, either $b = 0$ or $a = \frac{(\beta-\alpha)}{2}e^{-2i\theta} = -\frac{(\beta-\alpha)b^2}{2|b|^2}$.

Conversely, if g has the form in (5) with the given restrictions $|a| < \frac{\beta-\alpha}{2}$, or $|a| = \frac{\beta-\alpha}{2}$ and either $b = 0$ or $a = -\frac{(\beta-\alpha)b^2}{2|b|^2}$, then

$$\begin{aligned}
\sup_{z\in\mathbb{C}} M_{(\alpha\beta,g,nm)}(z) &= \sup_{z\in\mathbb{C}} \frac{\left|\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}\right|}{(1+|z|)^{n-m}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2} \\
&\le \left(\sup_{z\in\mathbb{C}} \frac{\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2|a||z|+|b|)^{m_1}|a|^{m_2}}{(1+|z|)^{n-m}}\right)\left(\sup_{z\in\mathbb{C}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2}\right) \\
&\preceq \sup_{z\in\mathbb{C}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2}.
\end{aligned} \tag{8}$$

Now, if $|a| < \frac{\beta-\alpha}{2}$, then

$$\sup_{z\in\mathbb{C}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2} \le \sup_{z\in\mathbb{C}} e^{|b||z|)-(\frac{(\beta-\alpha)}{2}-|a|)|z|^2} < \infty.$$

If $|a| = \frac{\beta-\alpha}{2}$ and $b = 0$, then $\sup_{z\in\mathbb{C}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2} \le 1$. If $|a| = \frac{\beta-\alpha}{2}$ and $a = -\frac{(\beta-\alpha)b^2}{2|b|^2}$, then

$$\begin{aligned}
\sup_{z\in\mathbb{C}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2} &= \sup_{z\in\mathbb{C}} e^{\Re(bz)-\frac{(\beta-\alpha)}{2|b|^2}\Re((bz)^2)-\frac{(\beta-\alpha)}{2}|z|^2} \\
&= \sup_{z\in\mathbb{C}} e^{\Re(bz)-\frac{(\beta-\alpha)}{2|b|^2}((\Re(bz))^2-(\Im(bz))^2)-\frac{(\beta-\alpha)}{2}|z|^2} \\
&= \sup_{z\in\mathbb{C}} e^{\Re(bz)-\frac{(\beta-\alpha)}{2|b|^2}\left((\Re(bz))^2-(\Im(bz))^2\right)-\frac{(\beta-\alpha)}{2}|z|^2} \\
&\preceq \sup_{z\in\mathbb{C}} e^{\frac{(\beta-\alpha)}{2|b|^2}(\Im(bz))^2-\frac{(\beta-\alpha)}{2}|z|^2} \le 1.
\end{aligned}$$

From the above estimates and (8), we have $M_{(\alpha\beta,g,nm)}$ is uniformly bounded over $\mathbb{C}$. Thus, by Theorem 1, $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded.

(b) Suppose $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is compact. Then, it is bounded and by (a) above $|a| \le \frac{\beta-\alpha}{2}$. If $|a| = \frac{\beta-\alpha}{2}$, then since $\mathcal{T}_g^{n,m}$ is compact, by Theorem 1, (7) must goes to zero as $w \to \infty$. But, $e^{\Re(be^{i\theta})} = 0$ and

$$M_{(\alpha\beta,g,nm)}(we^{i\theta}) = \frac{\left|\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2awe^{i\theta}+b)^{m_1}a^{m_2}\right|}{(1+|w|)^{n-m}} \asymp 1,$$

which is a contradiction. Thus, $|a| < \frac{\beta-\alpha}{2}$. On the other hand, if g has the form in (5) with the given conditions, then

$$M_{(\alpha\beta,g,nm)}(z) = \frac{\left|\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}\right|}{(1+|z|)^{n-m}} e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2} \to 0,$$

as $|z| \to \infty$, and by Theorem 1, the operator is compact.

(ii) Suppose g has the form in (5) with $|a| < \frac{\beta-\alpha}{2}$, then clearly the function $M_{(\alpha\beta,g,nm)}$ belongs to $L^r(\mathbb{C}, dA)$, where $r = \frac{pq}{p-q}$ for $p < \infty$ and $r = q$ for $p = \infty$. Hence, by Theorem 1, the operator is bounded (compact). Now, suppose $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded (compact). Then by Proposition 1 and what we have shown in (i) above, g has the form in 5 and $\sigma(g) = |a| < \frac{\beta-\alpha}{2}$. $\square$

3. Essential norm of $\mathcal{T}_g^{n,m}$

For two Banach spaces X and Y, the essential norm of a bounded operator $T : X \to Y$ is

$$\|T\|_e := \inf_{\mathcal{C}} \|T - \mathcal{C}\|,$$

where the infimum is taken over all compact operators $\mathcal{C} : X \to Y$. That is, essential norm of a bounded operator is the distance between the operator and the set of all compact operators. Thus, the essential norm of a compact operator is zero. Following similar procedures as in Proposition 2.4 of [16] we have the following theorem for the essential norm of generalized integration operator $\mathcal{T}_g^{n,m}$.

Theorem 2. *Let $1 \le p \le q \le \infty$, $\alpha \le \beta$ and $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded. Then*

$$\left\|\mathcal{T}_g^{n,m}\right\|_e \asymp \limsup_{|z|\to\infty} M_{(\alpha\beta,g,nm)}(z).$$

Proof. To prove the lower estimate, let $\mathcal{C} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ be a compact operator. Then, since $k_{(w,\alpha)}$ weakly converges to zero on $\mathcal{F}_p^\alpha$, we have

$$\begin{aligned}
\|\mathcal{T}_g^{n,m} - \mathcal{C}\| &\ge \limsup_{|w|\to\infty} \left\|\mathcal{T}_g^{n,m} k_{(w,\alpha)} - \mathcal{C}k_{(w,\alpha)}\right\|_{(q,\beta)} \\
&\ge \limsup_{|w|\to\infty} \left\|\mathcal{T}_g^{n,m} k_{(w,\alpha)}\right\|_{(q,\beta)} - \left\|\mathcal{C}k_{(w,\alpha)}\right\|_{(q,\beta)} \\
&= \limsup_{|w|\to\infty} \left\|\mathcal{T}_g^{n,m} k_{(w,\alpha)}\right\|_{(q,\beta)} \succeq \limsup_{|w|\to\infty} M_{(\alpha\beta,g,nm)}(w).
\end{aligned}$$

For the upper estimate, we consider a sequence $\Phi_i(z) = \frac{i}{i+1} z$ for each $i \in \mathbb{N}$. Since $\frac{i}{i+1} < 1$, by Corollary 3.5 of [17], the composition operator C_{Φ_i} is compact. Using the estimate in (1), for some fixed positive number $R > 0$ and $q < \infty$,

$$\begin{aligned}
\left\|\mathcal{T}_g^{n,m}\right\|_e &\le \left\|\mathcal{T}_g^{n,m} - \mathcal{T}_g^{n,m} \circ C_{\Phi_i}\right\| \le \sup_{\|f\|_{(p,\alpha)}\le 1} \left\|(\mathcal{T}_g^{n,m} - \mathcal{T}_g^{n,m} \circ C_{\Phi_i}) f\right\|_{(q,\beta)} \\
&\asymp \sup_{\|f\|_{(p,\alpha)}\le 1} \left(\int_{\mathbb{C}} \frac{\left|g^{(n-m)}(z)\right|^q \left|f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \right)^{\frac{1}{q}} \\
&\le \sup_{\|f\|_{(p,\alpha)}\le 1} \left(\int_{|z|\le R} \frac{\left|g^{(n-m)}(z)\right|^q \left|f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \right)^{\frac{1}{q}} \\
&\quad + \sup_{\|f\|_{(p,\alpha)}\le 1} \left(\int_{|z|> R} \frac{\left|g^{(n-m)}(z)\right|^q \left|f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right|^q}{(1+|z|)^{nq}} e^{-\frac{q\beta}{2}|z|^2} dA(z) \right)^{\frac{1}{q}} \\
&:= Int_1 + Int_2.
\end{aligned}$$

We now estimate each integrals above separately. Using the estimate in (1), Int_2 is bounded from above by

$$
\begin{aligned}
&\sup_{|z|>R} M_{(\alpha\beta,g,nm)}(z) \sup_{\|f\|_{(p,\alpha)}\le 1} \left(\int_{|z|>R} \frac{\left| f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} dA(z) \right)^{\frac{1}{q}} \\
&\quad \asymp \sup_{|z|>R} M_{(\alpha\beta,g,nm)}(z) \sup_{\|f\|_{(p,\alpha)}\le 1} \|f - f(\Phi_i)\|_{(q,\alpha)} \\
&\quad \preceq \sup_{|z|>R} M_{(\alpha\beta,g,nm)}(z) \left(\sup_{\|f\|_{(p,\alpha)}\le 1} \|f\|_{(q,\alpha)} + \sup_{\|f\|_{(p,\alpha)}\le 1} \|C_{\Phi_i} f\|_{(q,\alpha)} \right) \\
&\quad \preceq \sup_{|z|>R} M_{(\alpha\beta,g,nm)}(z),
\end{aligned}
\tag{9}
$$

where the last in equality is by boundedness of composition operator $C_{\Phi_i} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\alpha$ and the inclusion $\mathcal{F}_p^\alpha \subseteq \mathcal{F}_q^\alpha$ for $p \le q$. On the other hand,

$$
\begin{aligned}
Int_1 &\preceq \sup_{|z|\le R} M_{(\alpha\beta,g,nm)}(z) \sup_{\|f\|_{(p,\alpha)}\le 1} \left(\int_{|z|\le R} \frac{\left| f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} dA(z) \right)^{\frac{1}{q}} \\
&\le \sup_{|z|\le R} M_{(\alpha\beta,g,nm)}(z) \left(\int_{|z|\le R} \frac{e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{mq}} dA(z) \right)^{\frac{1}{q}} \times \left(\sup_{\|f\|_{(p,\alpha)}\le 1} \sup_{|z|\le R} \left| f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right| \right) \\
&\preceq \sup_{z\in\mathbb{C}} M_{(\alpha\beta,g,nm)}(z) \sup_{\|f\|_{(p,\alpha)}\le 1} \sup_{|z|\le R} \left| f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right| \\
&\preceq \sup_{\|f\|_{(p,\alpha)}\le 1} \sup_{|z|\le R} \left| f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right|
\end{aligned}
\tag{10}
$$

Integrating the function $f^{(m+1)}$ along the radial segment $[\frac{iz}{i+1}z, z]$, we get

$$
\left| f^{(m)}(z) - f^{(m)}(\Phi_i(z)) \right| \le \frac{|z| \left| f^{(m+1)}(z^*) \right|}{i+1}, \tag{11}
$$

for some z^* in the segment $[\frac{iz}{i+1}z, z]$. Using Cauchy's estimate for $f^{(m+1)}$, we also have

$$
\left| f^{(m+1)}(z^*) \right| \le \frac{1}{R} \max_{|z|=2R} \left| f^{(m)}(z) \right|. \tag{12}
$$

From inequalities (11) and (12), and using the estimate

$$
\left| f^{(m)}(z) \right| \preceq (1+|z|)^m e^{\frac{\alpha}{2}|z|^2} \|f\|_{(p,\alpha)}, \tag{13}
$$

which follows from (1) (see also, Lemma 2.2 of [14]), we get

$$
\begin{aligned}
\left| f^{(m)}(z) - \left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z)) \right| &= \left(\frac{i}{i+1}\right)^m \left| \left(\frac{(i+1)^m}{i^n}\right) f^{(m)}(z) - f^{(m)}(\Phi_i(z)) \right| \\
&\preceq \frac{\left| f^{(m)}(z) \right|}{(i+1)^m} + \left| f^{(m)}(z) - f^{(m)}(\Phi_i(z)) \right| \\
&\le \frac{\left| f^{(m)}(z) \right|}{(i+1)^m} + \frac{|z|}{R(i+1)} \max_{|z|=2R} \left| f^{(m)}(z) \right| \\
&\preceq \left(\frac{(1+|z|)^m e^{\frac{\alpha}{2}|z|^2}}{(i+1)^m} + \frac{2(1+2R)^m e^{2\alpha R^2}}{i+1} \right) \|f\|_{(p,\alpha)}.
\end{aligned}
$$

Using the above estimate, inequality (10) is bounded from above by

$$\sup_{\|f\|_{(p,\alpha)}\leq 1}\sup_{|z|\leq R}\left|f^{(m)}(z)-\left(\frac{i}{i+1}\right)^m f^{(m)}(\Phi_i(z))\right|\preceq\frac{(1+|z|)^m e^{\frac{\alpha}{2}|z|^2}}{(i+1)^m}+\frac{2(1+2R)^m e^{2\alpha R^2}}{i+1}. \tag{14}$$

Therefore, from (10) and (14), we have

$$Int_1\preceq\frac{(1+|z|)^m e^{\frac{\alpha}{2}|z|^2}}{(i+1)^m}+\frac{2(1+2R)^m e^{2\alpha R^2}}{i+1}.$$

Letting i goes to ∞ and then $R\to\infty$ in the above estimate and estimate (9), we get

$$\left\|\mathcal{T}_g^{n,m}\right\|_e\preceq\limsup_{|z|\to\infty}M_{(\alpha\beta,g,nm)}(z).$$

The case $q=\infty$ is very similar to the above proof, which only needs to replace the integral by supremum. □

Since essential norm of a compact operator is zero, our next corollary aims at simplifying the above theorem for when our operator $\mathcal{T}_g^{n,m}:\mathcal{F}_p^\alpha\to\mathcal{F}_q^\beta$ is noncompact bounded operator.

Corollary 4. (i) *Let $1\leq p\leq q\leq\infty$, $\alpha=\beta$ and $\mathcal{T}_g^{n,m}:\mathcal{F}_p^\alpha\to\mathcal{F}_q^\beta$ is noncompact bounded operator, that is, g has the form $g(z)=a_{2(n-m)}z^{2(n-m)}+\cdots+a_0$ with $a_{2(n-m)}\neq 0$. Then*

$$\left\|\mathcal{T}_g^{n,m}\right\|_e\asymp|a_{2(n-m)}|.$$

(ii) *Let $1\leq p\leq q\leq\infty$, $\alpha<\beta$, g is non vanishing function in $\mathcal{H}(\mathbb{C})$ and $\mathcal{T}_g^{n,m}:\mathcal{F}_p^\alpha\to\mathcal{F}_q^\beta$ is noncompact bounded operator, that is, $g(z)=e^{bz+az^2}$ for some $a,b\in\mathbb{C}$ with $|a|=\frac{\beta-\alpha}{2}$ and either $b=0$ or $a=-\frac{(\beta-\alpha)b^2}{2|b|^2}$. Then*

$$\left\|\mathcal{T}_g^{n,m}\right\|_e\asymp|a|^{n-m}.$$

Proof. (i) From Theorem 2,

$$\begin{aligned}\left\|\mathcal{T}_g^{n,m}\right\|_e&\asymp\limsup_{|z|\to\infty}M_{(\alpha\beta,g,nm)}(z)=\limsup_{|z|\to\infty}\frac{\left|g^{(n-m)}(z)\right|}{(1+|z|)^{n-m}}\\&=\limsup_{|z|\to\infty}\frac{\left|a_{2(n-m)}\left(\frac{(2(n-m))!}{(n-m)!}\right)z^{n-m}+\ldots+a_{n-m}(n-m)!\right|}{(1+|z|)^{n-m}}\\&\asymp|a_{2(n-m)}|.\end{aligned}$$

(ii) Using Theorem 2 and the formula (6),

$$\begin{aligned}\left\|\mathcal{T}_g^{n,m}\right\|_e&\asymp\limsup_{|z|\to\infty}M_{(\alpha\beta,g,nm)}(z)=\limsup_{|z|\to\infty}\frac{\left|g^{(n-m)}(z)\right|}{(1+|z|)^{n-m}}e^{-\frac{(\beta-\alpha)}{2}|z|^2}\\&=\limsup_{|z|\to\infty}\frac{\left|e^{bz+az^2}\sum_{m_1,m_2}\frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}\right|}{(1+|z|)^{n-m}}e^{-\frac{(\beta-\alpha)}{2}|z|^2}\\&\preceq\limsup_{|z|\to\infty}\frac{\sum_{m_1,m_2}\frac{(n-m)!}{m_1!m_2!}|2az+b|^{m_1}|a|^{m_2}}{(1+|z|)^{n-m}}e^{\Re(bz)+\Re(az^2)-\frac{(\beta-\alpha)}{2}|z|^2}\end{aligned}$$

$$\preceq \limsup_{|z|\to\infty} \frac{\sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}|2az+b|^{m_1}|a|^{m_2}}{(1+|z|)^{n-m}}$$
$$\leq \sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!}\left(\limsup_{|z|\to\infty} \frac{|2az+b|^{m_1}|a|^{m_2}}{(1+|z|)^{n-m}}\right) \asymp |a|^{n-m}.$$

Similarly, from lower we have

$$\left\|\mathcal{T}_g^{n,m}\right\|_e \asymp \limsup_{|z|\to\infty} \frac{\left|e^{bz+az^2}\sum_{m_1,m_2}\frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}\right|}{(1+|z|)^{n-m}}e^{-\frac{(\beta-\alpha)}{2}|z|^2}$$
$$\succeq \limsup_{|z|\to\infty} \frac{\left|\sum_{m_1,m_2}\frac{(n-m)!}{m_1!m_2!}(2az+b)^{m_1}a^{m_2}\right|}{(1+|z|)^{n-m}} \asymp |a|^{n-m}.$$

□

4. Path-connected components of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$

In this section, we identify path-connected components of space $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$ of bounded generalized integration operator $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ equipped with operator norm topology. From Corollary 1, if $0 < p, q \leq \infty$ and $\alpha > \beta$, $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded if and only if it is a zero operator, and hence the whole space $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$: is path-connected. Thus, the interesting case is when $\alpha \leq \beta$. For this, we first show $\mathcal{T}_{g_1(0)}^{n,m}$ and $\mathcal{T}_{g_2(0)}^{n,m}$ are in the same path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$, where $\mathcal{T}_{g_1}^{n,m}$ and $\mathcal{T}_{g_2}^{n,m}$ are compact operators.

Proposition 2. *Let $0 < p, q \leq \infty$, $\alpha \leq \beta$, and $\mathcal{T}_{g_1}^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ and $\mathcal{T}_{g_2}^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ are compact operators. Then $\mathcal{T}_{g_1(0)}^{n,m}$ and $\mathcal{T}_{g_2(0)}^{n,m}$ are in the same path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$.*

Proof. For the proof of this proposition, we follow the same procedures as in [17]. If $g_1^{n,m}(0)$ or $g_2^{n,m}(0)$ is zero, or $g_1^{n,m}(0) = g_2^{n,m}(0)$, then trivially $\mathcal{T}_{g_1(0)}^{n,m}$ and $\mathcal{T}_{g_2(0)}^{n,m}$ are in the same path-connected components. Thus, we assume

$$g_1^{n,m}(0) = \lambda g_2^{n,m}(0),$$

where $\lambda \in \mathbb{C}\setminus\{0,1\}$. Consider a continuous function $\gamma : [0,1] \to \mathbb{C}$ such that

$$\gamma(0) = 0, \gamma(1) = 1, \gamma(t) \neq \frac{1}{1-\lambda},$$

for all $t \in [0,1]$. Define a sequence of functions h_t by

$$h_t := (1-\gamma(t))g_1^{n,m}(0) + \gamma(t)g_2^{n,m}(0) \neq 0,$$

for all $t \in [0,1]$. Define a map $\mathcal{T}_{h_t}^{n,m} : [0,1] \to \mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$. Then, $\mathcal{T}_{h_0}^{n,m} = \mathcal{T}_{g_1(0)}^{n,m}$, $\mathcal{T}_{h_1}^{n,m} = \mathcal{T}_{g_2(0)}^{n,m}$ and $\mathcal{T}_{h_t}^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ are compact for all $t \in [0,1]$. Since

$$\lim_{t\to s}\left\|\mathcal{T}_{h_t}^{n,m} - \mathcal{T}_{h_s}^{n,m}\right\| = \lim_{t\to s}|\gamma(t)-\gamma(s)|\left\|\mathcal{T}_{(g_2-g_1)(0)}^{n,m}\right\| = 0,$$

the map $\mathcal{T}_{h_t}^{n,m}$ is continuous. This completes the proof. □

Our next two theorems shows that $\mathcal{T}_g^{n,m}$ and $\mathcal{T}_{g(0)}^{n,m}$ are also in the same path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$, where $\mathcal{T}_g^{n,m}$ is a compact operator.

Theorem 3. *Let $0 < p, q \leq \infty$, $\alpha = \beta$ and $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is compact operator. Then $\mathcal{T}_g^{n,m}$ and $\mathcal{T}_{g(0)}^{n,m}$ are in the same path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$.*

Proof. Since $\mathcal{T}_g^{n,m}$ is compact, by Corollary 2, for $0 < p \leq q \leq \infty$, g is a polynomial of degree at most $2(n-m)-1$, and for $0 < q < p \leq \infty$, g is a polynomial of degree at most $2(n-m)-1$ and $q > \begin{cases} \frac{2p}{p+2}, & p < \infty \\ 2, & p = \infty \end{cases}$. In both cases, g has the form

$$g(z) = a_l z^l + a_{l-1} z^{l-1} + \dots + a_0,$$

where $l = 2(n-m)-1$. If $a_k = 0$ for all $k \in [\frac{l+1}{2}, l]$, then $\mathcal{T}_g^{n,m} = \mathcal{T}_{g(0)}^{n,m}$ are zero operators and the result holds trivially. We assume $a_k \neq 0$ for some $k \in [\frac{l+1}{2}, l]$. Define functions $g_t : [0,1] \to \mathbb{C}$ by

$$g_t(z) = g(tz).$$

Consider a map $\mathcal{T}_{g_t}^{n,m} : [0,1] \to \mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$. Then, $\mathcal{T}_{g_1}^{n,m} = \mathcal{T}_g^{n,m}$, $\mathcal{T}_{g_0}^{n,m} = \mathcal{T}_{g(0)}^{n,m}$ and $\mathcal{T}_{g_t}^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ are compact for all t. We claim that this map is continuous, that is,

$$\lim_{t \to s} \|\mathcal{T}_{g_t}^{n,m} - \mathcal{T}_{g_s}^{n,m}\| = 0, \tag{15}$$

for every $s \in [0,1]$.

Case $0 < p \leq q < \infty$ or $0 < q < p \leq \infty$.

For each $f \in \mathcal{F}_p^\alpha$ with $\|f\|_{(p,\alpha)} \leq 1$, an application of formula (1) to the $\mathcal{F}_q^\beta$ norm of a function $\mathcal{T}_{g_t - g_s}^{n,m} f$ gives

$$\begin{aligned} \|\mathcal{T}_{g_t}^{n,m} f - \mathcal{T}_{g_s}^{n,m} f\|_{(q,\beta)}^q &\asymp \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q \left|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)\right|^q e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}} dA(z) \\ &= |t-s|^q \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q \left|g_{(t,s)}^{(n-m)}(z)\right|^q e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}} dA(z), \end{aligned} \tag{16}$$

where

$$g_{(t,s)}^{(n-m)}(z) = \sum_{i=(n-m)+1}^{2(n-m)-1} a_i \frac{i!}{(i-(n-m))!} z^{i-(n-m)} \sum_{j=(n-m)}^{i-1} s^{j-(n-m)} t^{i-1-j}.$$

For $s, t \in [0,1]$ and a positive integer i with $(n-m)+1 \leq i \leq 2(n-m)-1$,

$$\sum_{j=(n-m)}^{i-1} s^{j-(n-m)} t^{i-1-j} \leq \sum_{j=(n-m)}^{i-1} 1 = i - (n-m).$$

Thus, $|g_{(t,s)}^{(n-m)}(z)| \leq |h^{(n-m)}(z)|$, where

$$h^{(n-m)}(z) := \sum_{i=(n-m)+1}^{2(n-m)-1} a_i \frac{i!}{(i-(n-m)-1)!} z^{i-(n-m)}.$$

Clearly, $\deg(h^{(n-m)}) \leq (n-m)-1$ and hence $\deg(h) \leq 2(n-m)-1$. Moreover, we have a restriction $q > \begin{cases} \frac{2p}{p+2}, & p < \infty \\ 2, & p = \infty \end{cases}$ for $0 < q < p \leq \infty$. By Corollary 2, the operator $\mathcal{T}_h^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is bounded. Thus,

$$\begin{aligned} \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q \left|g_{(t,s)}^{(n-m)}(z)\right|^q e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}} dA(z) &\leq \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q \left|h^{(n-m)}(z)\right|^q e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}} dA(z) \\ &\asymp \|\mathcal{T}_h^{n,m} f\|_{(q,\beta)}^q \leq \|\mathcal{T}_h^{n,m}\|^q \|f\|_{(p,\alpha)}^q < \infty. \end{aligned}$$

Using this and the estimate in (16), we obtain

$$\|\mathcal{T}_{g_t}^{n,m} - \mathcal{T}_{g_s}^{n,m}\| \preceq |t-s|.$$

Letting $t \to s$, (15) holds.

Case $0 < p \le q = \infty$.

Similarly, for $f \in \mathcal{F}_p^\alpha$ with $\|f\|_{(p,\alpha)} \le 1$, using formula (1) and above estimates,

$$\begin{aligned}
\|\mathcal{T}_{g_t}^{n,m} f - \mathcal{T}_{g_s}^{n,m} f\|_{(\infty,\beta)} &\asymp \sup_{z\in\mathbb{C}} \frac{\left|f^{(m)}(z)\right| \left|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)\right| e^{-\frac{\beta}{2}|z|^2}}{(1+|z|)^n} \\
&= |t-s| \left(\sup_{z\in\mathbb{C}} \frac{\left|f^{(m)}(z)\right| \left|g_{(t,s)}^{(n-m)}(z)\right| e^{-\frac{\beta}{2}|z|^2}}{(1+|z|)^n} \right) \\
&\le |t-s| \left(\sup_{z\in\mathbb{C}} \frac{\left|f^{(m)}(z)\right| \left|h^{(n-m)}(z)\right| e^{-\frac{\beta}{2}|z|^2}}{(1+|z|)^n} \right) \\
&\asymp |t-s| \|\mathcal{T}_h^{n,m} f\|_{(\infty,\beta)} \le |t-s| \|\mathcal{T}_h^{n,m}\| \|f\|_{(p,\alpha)}.
\end{aligned}$$

Therefore, $\|\mathcal{T}_{g_t}^{n,m} - \mathcal{T}_{g_s}^{n,m}\| \preceq |t-s|$. From which, (15) holds. □

Theorem 4. *Let $0 < p, q \le \infty$, $\alpha < \beta$, g is non vanishing function in $\mathcal{H}(\mathbb{C})$ and $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is compact operator. Then $\mathcal{T}_g^{n,m}$ and $\mathcal{T}_{g(0)}^{n,m}$ are in the same path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$.*

Proof. Since $\mathcal{T}_g^{n,m}$ is compact, by Corollary 3, g has the form $g(z) = e^{bz+az^2}$ for some $a, b \in \mathbb{C}$ with $|a| < \frac{\beta-\alpha}{2}$. If $a = b = 0$, then $\mathcal{T}_g^{n,m} = \mathcal{T}_{g(0)}^{n,m}$ are zero operators and the result holds trivially. For the remaining cases, define functions $g_t : [0,1] \to \mathbb{C}$ by $g_t(z) = g(tz)$. Consider a map $\mathcal{T}_{g_t}^{n,m} : [0,1] \to \mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$. Then, $\mathcal{T}_{g_1}^{n,m} = \mathcal{T}_g^{n,m}$, $\mathcal{T}_{g_0}^{n,m} = \mathcal{T}_{g(0)}^{n,m}$ and $\mathcal{T}_{g_t}^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ are compact for all t. We need to show (15) holds, to conclude that $\mathcal{T}_{g_t}^{n,m}$ is continuous.

Case $a = 0$ and $b \neq 0$.

We first estimate $|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)|$,

$$\begin{aligned}
|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)| \asymp |e^{btz} - e^{bsz}| &\le \sum_{i=0}^{\infty} \frac{(|bz|)^i |t^i - s^i|}{i!} \\
&\le |t-s| \sum_{i=1}^{\infty} \frac{|bz|^i}{(i-1)!} = |t-s||bz|e^{|bz|}, \ \forall z \in \mathbb{C}.
\end{aligned} \tag{17}$$

Depending on the exponents p and q, we consider the following cases.

(i) $0 < p \le q < \infty$:

For $f \in \mathcal{F}_p^\alpha$ with $\|f\|_{(p,\alpha)} \le 1$, using estimates (1) and (17), we obtain

$$\begin{aligned}
\|\mathcal{T}_{g_t}^{n,m} f - \mathcal{T}_{g_s}^{n,m} f\|_{(q,\beta)}^q &\asymp \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q \left|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)\right|^q e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}} dA(z) \\
&\preceq |t-s|^q \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q |bz|^q e^{q|bz|} e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}} dA(z) \\
&= |t-s|^q \int_{\mathbb{C}} \left(\frac{\left|f^{(m)}(z)\right|^q e^{-\frac{q\alpha}{2}|z|^2}}{(1+|z|)^{mq}} \right) \left(\frac{|z|(1+|z|)^m e^{|bz| - \frac{(\beta-\alpha)}{2}|z|^2}}{(1+|z|)^n} \right)^q dA(z).
\end{aligned} \tag{18}$$

Using

$$\sup_{z\in\mathbb{C}} \frac{|z|(1+|z|)^m e^{|bz|-\frac{(\beta-\alpha)}{2}|z|^2}}{(1+|z|)^n} \le \sup_{z\in\mathbb{C}} \frac{e^{|bz|-\frac{(\beta-\alpha)}{2}|z|^2}}{(1+|z|)^{n-m-1}} \le \sup_{z\in\mathbb{C}} e^{|bz|-\frac{(\beta-\alpha)}{2}|z|^2} < \infty, \tag{19}$$

the estimate in (1) and the inclusion $\mathcal{F}_p^\alpha \subseteq \mathcal{F}_q^\alpha$, for $p \le q$, we further estimate (18) from above as follows.

$$\begin{aligned} &|t-s|^q \int_{\mathbb{C}} \left(\frac{\left|f^{(m)}(z)\right|^q e^{-\frac{q\alpha}{2}|z|^2}}{(1+|z|)^{mq}} \right) \left(\frac{|z|(1+|z|)^m e^{|bz|-\frac{(\beta-\alpha)}{2}|z|^2}}{(1+|z|)^n} \right)^q dA(z) \\ &\preceq |t-s|^q \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q e^{-\frac{q\alpha}{2}|z|^2}}{(1+|z|)^{mq}} dA(z) \\ &\asymp |t-s|^q \|f\|_{(q,\alpha)}^q \preceq |t-s|^q \|f\|_{(p,\alpha)}^q. \end{aligned}$$

Thus,

$$\|\mathcal{T}_{g_t}^{n,m} - \mathcal{T}_{g_s}^{n,m}\| \preceq |t-s|,$$

from which (15) holds.

(ii) $0 < p \le q = \infty$:

Similarly, for $f \in \mathcal{F}_p^\alpha$ with $\|f\|_{(p,\alpha)} \le 1$, using estimates (1), (17) and (19),

$$\begin{aligned} \|\mathcal{T}_{g_t}^{n,m} f - \mathcal{T}_{g_s}^{n,m} f\|_{(\infty,\beta)} &\asymp \sup_{z\in\mathbb{C}} \frac{\left|f^{(m)}(z)\right| \left|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)\right| e^{-\frac{\beta}{2}|z|^2}}{(1+|z|)^n} \\ &\preceq |t-s| \sup_{z\in\mathbb{C}} \frac{\left|f^{(m)}(z)\right| |bz| e^{|bz|} e^{-\frac{\beta}{2}|z|^2}}{(1+|z|)^n} \\ &= |t-s| \sup_{z\in\mathbb{C}} \left(\frac{\left|f^{(m)}(z)\right| e^{-\frac{\alpha}{2}|z|^2}}{(1+|z|)^m} \right) \left(\frac{|z|(1+|z|)^m e^{|bz|-\frac{(\beta-\alpha)}{2}|z|^2}}{(1+|z|)^n} \right) \\ &\preceq |t-s| \sup_{z\in\mathbb{C}} \left(\frac{\left|f^{(m)}(z)\right| e^{-\frac{\alpha}{2}|z|^2}}{(1+|z|)^m} \right) \\ &\asymp |t-s| \|f\|_{(\infty,\alpha)} \preceq |t-s| \|f\|_{(p,\alpha)}. \end{aligned}$$

The last estimate is by the inclusion $\mathcal{F}_p^\alpha \subseteq \mathcal{F}_\infty^\alpha$, for $p \le \infty$. From the above estimate, (15) holds for this case also.

(iii) $0 < q < p \le \infty$:

Following similar procedures as those leading to (18) and using the estimate in (13) we obtain

$$\begin{aligned} \|\mathcal{T}_{g_t}^{n,m} f - \mathcal{T}_{g_s}^{n,m} f\|_{(q,\beta)}^q &\preceq |t-s|^q \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q |z|^q e^{q|bz|} e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}} dA(z) \\ &\preceq |t-s|^q \|f\|_{(p,\alpha)}^q \int_{\mathbb{C}} \frac{(1+|z|)^{mq} |z|^q e^{q|b||z|-\frac{q(\beta-\alpha)}{2}|z|^2}}{(1+|z|)^{nq}} dA(z) \\ &\preceq |t-s|^q \|f\|_{(p,\alpha)}^q \int_{\mathbb{C}} e^{q|b||z|-\frac{q(\beta-\alpha)}{2}|z|^2} dA(z). \end{aligned}$$

Thus, the conclusion in (15) holds, if we show that

$$\int_{\mathbb{C}} e^{q|b||z|-\frac{q(\beta-\alpha)}{2}|z|^2} dA(z) < \infty.$$

To show this, let R be a positive real number such that $|b| < \frac{R(\beta-\alpha)}{2}$ and define a number $\gamma := \frac{\beta R-\alpha R-2|b|}{2R} < 0$, then clearly

$$\int_{|z|\leq R} e^{q|b||z|-\frac{q(\beta-\alpha)}{2}|z|^2}dA(z) < \infty,$$

and

$$\int_{|z|>R} e^{q|b||z|-\frac{q(\beta-\alpha)}{2}|z|^2}dA(z) \leq \int_{|z|>R} e^{q|z|^2\left(\frac{|b|}{|z|}-\frac{(\beta-\alpha)}{2}\right)}dA(z)$$
$$= \int_{|z|>R} e^{-q\gamma|z|^2}dA(z) < \infty.$$

Hence,

$$\int_{\mathbb{C}} e^{q|b||z|-\frac{q(\beta-\alpha)}{2}|z|^2}dA(z) = \left(\int_{|z|\leq R} + \int_{|z|>R}\right) e^{q|b||z|-\frac{q(\beta-\alpha)}{2}|z|^2}dA(z) < \infty. \tag{20}$$

Case $a \neq 0$.

Using (6) and functions $g_t : [0,1] \to \mathbb{C}$ defined above,

$$|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)| \asymp \left|\sum_{m_1,m_2} \frac{(n-m)!a^{m_2}}{m_1!m_2!}\left((2atz+b)^{m_1}e^{btz+a(tz)^2} - (2asz+b)^{m_1}e^{bsz+a(sz)^2}\right)\right|$$
$$\leq \sum_{m_1,m_2} \frac{(n-m)!|a|^{m_2}}{m_1!m_2!}\left|\sum_{i=0}^{\infty} \frac{(2atz+b)^{m_1}(btz+at^2z^2)^i - (2asz+b)^{m_1}(bsz+as^2z^2)^i}{i!}\right|,$$

where in the last estimate we used the power series representations of $e^{btz+a(tz)^2}$ and $e^{bsz+a(sz)^2}$. Inserting the binomial expansions of $(2atz+b)^{m_1}$, $(btz+at^2z^2)^i$, $(2asz+b)^{m_1}$ and $(bsz+as^2z^2)^i$, the right hand side of the above estimate is equal to;

$$\sum_{m_1,m_2}\left(\frac{(n-m)!|a|^{m_2}}{m_1!m_2!}\left|\sum_{i=0}^{\infty}\frac{\left(\sum_{l=0}^{m_1}\sum_{j=0}^{i}\binom{m_1}{l}\binom{i}{j}2^{m_1-l}a^{m_1+i-l-j}b^{l+j}z^{m_1+2i-l-j}\left(t^{m_1+2i-l-j}-s^{m_1+2i-l-j}\right)\right)}{i!}\right|\right)$$
$$\leq |t-s|\sum_{m_1,m_2}\left(\frac{(n-m)!|a|^{m_2}}{m_1!m_2!}\sum_{i=0}^{\infty}\frac{\left(\sum_{l=0}^{m_1}\sum_{j=0}^{i}\binom{m_1}{l}\binom{i}{j}2^{m_1-l}|a|^{m_1+i-l-j}|b|^{l+j}|z|^{m_1+2i-l-j}(m_1+2i-l-j)\right)}{i!}\right)$$
$$\preceq |t-s|\sum_{m_1,m_2}\left(\frac{(n-m)!|a|^{m_2}}{m_1!m_2!}\sum_{i=0}^{\infty}\frac{(2|a||z|+|b|)^{m_1}(|b||z|+|a||z|^2)^i}{(i-1)!}\right)$$
$$= |t-s|\sum_{m_1,m_2}\left(\frac{(n-m)!|a|^{m_2}(2|a||z|+|b|)^{m_1}(|b||z|+|a||z|^2)e^{|b||z|+|a||z|^2}}{m_1!m_2!}\right)$$
$$\asymp |t-s|(|b||z|+|a||z|^2)e^{|b||z|+|a||z|^2}\sum_{m_1,m_2}\left(\frac{|a|^{m_2}(2|a||z|+|b|)^{m_1}}{m_1!m_2!}\right).$$

Therefore, we have

$$|g_t^{(n-m)}(z) - g_s^{(n-m)}(z)| \preceq |t-s|(|b||z|+|a||z|^2)e^{|b||z|+|a||z|^2}\sum_{m_1,m_2}\left(\frac{|a|^{m_2}(2|a||z|+|b|)^{m_1}}{m_1!m_2!}\right). \tag{21}$$

(i) $0 < p \leq q < \infty$ or $0 < q < p \leq \infty$.

Using (21), the estimate in (1) and (13), we obtain

$$\|\mathcal{T}^{n,m}_{g_t}f - \mathcal{T}^{n,m}_{g_s}f\|^q_{(q,\beta)} \asymp \int_{\mathbb{C}} \frac{\left|f^{(m)}(z)\right|^q\left|g_t^{(n-m)}(z)-g_s^{(n-m)}(z)\right|^q e^{-\frac{q\beta}{2}|z|^2}}{(1+|z|)^{nq}}dA(z)$$

$$\preceq |t-s|^q \int_{\mathbb{C}} \left(\left| f^{(m)}(z) \right|^q e^{-\frac{q\alpha}{2}|z|^2} \left(\sum_{m_1,m_2} \frac{|a|^{m_2}(2|a||z|+|b|)^{m_1}}{m_1!m_2!(1+|z|)^n} \right)^q \right.$$

$$\left. (|b||z|+|a||z|^2)^q e^{q|b||z|+q|a||z|^2-\frac{q(\beta-\alpha)}{2}|z|^2} \right) dA(z)$$

$$\preceq |t-s|^q \int_{\mathbb{C}} \left| f^{(m)}(z) \right|^q e^{-\frac{q\alpha}{2}|z|^2} (|b||z|+|a||z|^2)^q e^{q|b||z|+q|a||z|^2-\frac{q(\beta-\alpha)}{2}|z|^2} dA(z)$$

$$\preceq |t-s|^q \|f\|^q_{(p,\alpha)} \int_{\mathbb{C}} (1+|z|)^{mq} (|b||z|+|a||z|^2)^q e^{q|b||z|+q|a||z|^2-\frac{q(\beta-\alpha)}{2}|z|^2} dA(z),$$

for $f \in \mathcal{F}^\alpha_p$ with $\|f\|_{(p,\alpha)} \leq 1$. The limit in (15) holds true, if we show that

$$\int_{\mathbb{C}} (1+|z|)^{mq} (|b||z|+|a||z|^2)^q e^{q|b||z|+q|a||z|^2-\frac{q(\beta-\alpha)}{2}|z|^2} dA(z) < \infty.$$

But, since $|a| < \frac{\beta-\alpha}{2}$, similar procedures as those leading to (20) shows that the integral is finite. Therefore, $\mathcal{T}^{n,m}_{g_t}$ is continuous.

(ii) $0 < p \leq q = \infty$.

Similarly, using (21), for $f \in \mathcal{F}^\alpha_p$ with $\|f\|_{(p,\alpha)} \leq 1$,

$$\|\mathcal{T}^{n,m}_{g_t} f - \mathcal{T}^{n,m}_{g_s} f\|_{(\infty,\beta)}$$

$$\asymp \sup_{z\in\mathbb{C}} \frac{\left| f^{(m)}(z) \right| \left| g_t^{(n-m)}(z) - g_s^{(n-m)}(z) \right| e^{-\frac{\beta}{2}|z|^2}}{(1+|z|)^n}$$

$$\preceq |t-s| \sup_{z\in\mathbb{C}} \left(\left| f^{(m)}(z) \right| e^{-\frac{\alpha}{2}|z|^2} \left(\sum_{m_1,m_2} \frac{|a|^{m_2}(2|a||z|+|b|)^{m_1}}{m_1!m_2!(1+|z|)^n} \right) (|b||z|+|a||z|^2) e^{|b||z|+|a||z|^2-\frac{(\beta-\alpha)}{2}|z|^2} \right)$$

$$\preceq |t-s| \sup_{z\in\mathbb{C}} \left| f^{(m)}(z) \right| e^{-\frac{\alpha}{2}|z|^2} (|b||z|+|a||z|^2) e^{|b||z|+|a||z|^2-\frac{(\beta-\alpha)}{2}|z|^2}$$

$$\preceq |t-s| \|f\|_{(p,\alpha)} \sup_{z\in\mathbb{C}} \left((1+|z|)^m (|b||z|+|a||z|^2) e^{|b||z|+|a||z|^2-\frac{(\beta-\alpha)}{2}|z|^2} \right).$$

Since

$$\sup_{z\in\mathbb{C}} \left((1+|z|)^m (|b||z|+|a||z|^2) e^{|b||z|+|a||z|^2-\frac{(\beta-\alpha)}{2}|z|^2} \right) < \infty,$$

we get

$$\|\mathcal{T}^{n,m}_{g_t} - \mathcal{T}^{n,m}_{g_s}\| \preceq |t-s|.$$

Letting t goes to s, the conclusion follows. □

Proposition 3. *Let $0 < p, q \leq \infty$ and $\alpha = \beta$. Then*

(i) *the set of all compact operators $\mathcal{T}^{n,m}_g : \mathcal{F}^\alpha_p \to \mathcal{F}^\beta_q$ forms a path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}^\alpha_p, \mathcal{F}^\beta_q)$.*

(ii) *for $0 < q < p \leq \infty$, the whole space $\mathcal{T}^{n,m}(\mathcal{F}^\alpha_p, \mathcal{F}^\beta_q)$ is path-connected.*

Proof. (i) Let $\mathcal{T}^{n,m}_{g_1}$ and $\mathcal{T}^{n,m}_{g_2}$ be two compact operators in $\mathcal{T}^{n,m}(\mathcal{F}^\alpha_p, \mathcal{F}^\beta_q)$. We need to show that $\mathcal{T}^{n,m}_{g_1}$ and $\mathcal{T}^{n,m}_{g_2}$ belong to the same path-connected component. First, by Theorem 3, $\mathcal{T}^{n,m}_{g_1}$ and $\mathcal{T}^{n,m}_{g_1(0)}$ belong to the same path-connected component, and $\mathcal{T}^{n,m}_{g_2}$ and $\mathcal{T}^{n,m}_{g_2(0)}$ also belong to the same path-connected component. But, by Proposition 2, $\mathcal{T}^{n,m}_{g_1(0)}$ and $\mathcal{T}^{n,m}_{g_2(0)}$ belong to the same path-connected component. Hence, $\mathcal{T}^{n,m}_{g_1}$ and $\mathcal{T}^{n,m}_{g_2}$ belong to the same path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}^\alpha_p, \mathcal{F}^\beta_q)$.

(ii) Follows from (i) and Corollary 2. □

Similarly, for $\alpha < \beta$, the following proposition hold. Its proof follows from similar arguments as in the proof of Proposition 3, and using Theorem 3, Proposition 2 and Corollary 3.

Proposition 4. *Let $0 < p, q \leq \infty$, $\alpha < \beta$ and g is non vanishing function in $\mathcal{H}(\mathbb{C})$. Then*

(i) *the set of all compact operators $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ forms a path-connected component of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$.*

(ii) *for $0 < q < p \leq \infty$, the whole space $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$ is path-connected.*

5. Singleton of Path-connected components

Following Propositions 3 and 4, this section aims to investigate whether noncompact operators are singleton in the path-connected components obtained in the above section. Our next two results show that a noncompact operator is in fact singleton.

Theorem 5. *Let $0 < p \leq q \leq \infty$, $\alpha = \beta$ and $\mathcal{T}_g^{n,m} : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ be a bounded operator. Then $\mathcal{T}_g^{n,m}$ is singleton in the path-connected component in Proposition 3 if and only if g is a polynomial of degree $2(n-m)$.*

Proof. The forward implication follows from Proposition 3. We will prove the backward implication. Suppose g is a polynomial of degree $2(n-m)$, that is,

$$g(z) = a_{2(n-m)} z^{2(n-m)} + a_{2(n-m)-1} z^{2(n-m)-1} + \ldots + a_1 z + a_0,$$

and $a_{2(n-m)} \neq 0$. It is enough to show there exists a positive number μ such that

$$\|\mathcal{T}_g^{n,m} - \mathcal{T}_{g_1}^{n,m}\| \succeq \mu,$$

for all $\mathcal{T}_{g_1}^{n,m} \in \mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$ such that $\mathcal{T}_{g_1}^{n,m} \neq \mathcal{T}_g^{n,m}$, that is, g_1 has the form

$$g_1(z) = b_{2(n-m)} z^{2(n-m)} + b_{2(n-m)-1} z^{2(n-m)-1} + \ldots + b_1 z + b_0,$$

with $b_k \neq a_k$ for some positive integer k, $n-m \leq k \leq 2(n-m)$. Thus, applying $\mathcal{T}_g^{n,m} - \mathcal{T}_{g_1}^{n,m}$ to the function $k_{(w,\alpha)}(z)$ in $\mathcal{F}_p^\alpha$, using (1) and the inclusion $\mathcal{F}_q^\beta \subseteq \mathcal{F}_\infty^\beta$, $0 < q \leq \infty$, we obtain

$$\begin{aligned}
\|\mathcal{T}_g^{n,m} - \mathcal{T}_{g_1}^{n,m}\| &\geq \|\mathcal{T}_g^{n,m} k_{(w,\alpha)} - \mathcal{T}_{g_1}^{n,m} k_{(w,\alpha)}\|_{(q,\beta)} \succeq \|\mathcal{T}_g^{n,m} k_{(w,\alpha)} - \mathcal{T}_{g_1}^{n,m} k_{(w,\alpha)}\|_{(\infty,\beta)} \\
&\asymp \sup_{z\in\mathbb{C}} \frac{\left|k_{(w,\alpha)}^{(m)}(z)\right| \left|g^{(n-m)}(z) - g_1^{(n-m)}(z)\right|}{(1+|z|)^n} e^{-\frac{\beta}{2}|z|^2} \\
&\succeq \frac{|w|^m \left|g^{(n-m)}(z) - g_1^{(n-m)}(z)\right|}{(1+|z|)^n} e^{-\frac{\beta}{2}|w-z|^2} \\
&\geq \frac{\left|(a_{2(n-m)} - b_{2(n-m)})\left(\frac{(2(n-m))!}{(n-m)!}\right) z^{n-m} + \ldots + (a_{n-m} - b_{n-m})(n-m)!\right|}{(1+|z|)^{n-m}},
\end{aligned} \tag{22}$$

where the last inequality is obtained by putting $z = w$. If $a_{n-m} \neq b_{n-m}$, then we set $z = 0$ in the above estimate to obtain

$$\|\mathcal{T}_g^{n,m} - \mathcal{T}_{g_1}^{n,m}\| \succeq \mu = |a_{n-m} - b_{n-m}| \neq 0.$$

If $a_{n-m} = b_{n-m}$, then there exists some k, $n-m < k \leq 2(n-m)$, such that $a_k \neq b_k$. Let i be the smallest of such k. Then, from (22),

$$\begin{aligned}
&\frac{\left|(a_{2(n-m)} - b_{2(n-m)})\left(\frac{(2(n-m))!}{(n-m)!}\right) z^{n-m} + \ldots + (a_{n-m} - b_{n-m})(n-m)!\right|}{(1+|z|)^{n-m}} \\
&= |z|^i \frac{\left|(a_{2(n-m)} - b_{2(n-m)})\left(\frac{(2(n-m))!}{(n-m)!}\right) z^{n-m-i} + \ldots + (a_i - b_i)\left(\frac{i!}{(2(n-m)-i)!}\right)\right|}{(1+|z|)^{n-m}} \\
&\geq \frac{\left|(a_{2(n-m)} - b_{2(n-m)})\left(\frac{(2(n-m))!}{(n-m)!}\right) z^{n-m-i} + \ldots + (a_i - b_i)\left(\frac{i!}{(2(n-m)-i)!}\right)\right|}{(1+|z|)^{n-m-i}}.
\end{aligned}$$

Setting $z=0$, we similarly obtain

$$\|\mathcal{T}_g^{n,m}-\mathcal{T}_{g_1}^{n,m}\| \succeq \mu = |a_i - b_i| \neq 0.$$

This completes the proof. □

Theorem 6. *Let $0<p\leq q\leq\infty$, $\alpha<\beta$, g is non vanishing function in $\mathcal{H}(\mathbb{C})$ and $\mathcal{T}_g^{n,m}:\mathcal{F}_p^\alpha\to\mathcal{F}_q^\beta$ be a bounded operator. Then $\mathcal{T}_g^{n,m}$ is singleton in the path-connected component in Proposition 4 if and only if $g(z)=e^{bz+az^2}$ for some $a,b\in\mathbb{C}$ with $|a|=\frac{\beta-\alpha}{2}$ and either $b=0$ or $a=-\frac{(\beta-\alpha)b^2}{2|b|^2}$.*

Proof. Similarly the forward implication follows from Proposition 4. Let $g(z)=e^{bz+az^2}$ for some $a,b\in\mathbb{C}$ with $|a|=\frac{\beta-\alpha}{2}$ and either $b=0$ or $a=-\frac{(\beta-\alpha)b^2}{2|b|^2}$. Consider $\mathcal{T}_{g_1}^{n,m}\in\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha,\mathcal{F}_q^\beta)$ such that $\mathcal{T}_{g_1}^{n,m}\neq\mathcal{T}_g^{n,m}$, that is, g_1 has the form $g_1(z)=e^{b_1z+a_1z^2}$ for some $a_1,b_1\in\mathbb{C}$ with $|a_1|<\frac{\beta-\alpha}{2}$ or $|a_1|=\frac{\beta-\alpha}{2}$ and either $b_1=0$ or $a_1=-\frac{(\beta-\alpha)b_1^2}{2|b_1|^2}$, with $a_1\neq a$ or $b_1\neq b$.

(i) $0<p\leq q<\infty$.

Applying $\mathcal{T}_g^{n,m}-\mathcal{T}_{g_1}^{n,m}$ to the normalized kernel function, using (1), the estimate $1+|z|\asymp 1+|w|$ for $z\in D(w,1)$, subharmonicity of $|g^{(n-m)}-g_1^{(n-m)}|^q$ and the formula in (6),

$$\begin{aligned}
\|\mathcal{T}_g^{n,m}-\mathcal{T}_{g_1}^{n,m}\|^q &\succeq \int_{\mathbb{C}} \frac{\left|k_{(w,\alpha)}^{(m)}(z)\right|^q\left|g^{(n-m)}(z)-g_1^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}}e^{-\frac{\beta q}{2}|z|^2}dA(z)\\
&= \int_{\mathbb{C}} \frac{|w|^{mq}\left|e^{\alpha z\overline{w}}\right|^q\left|g^{(n-m)}(z)-g_1^{(n-m)}(z)\right|^q}{(1+|z|)^{nq}}e^{-\frac{\beta q}{2}|z|^2-\frac{\alpha q}{2}|w|^2}dA(z)\\
&\geq \int_{D(w,1)} \frac{|w|^{mq}\left|g^{(n-m)}(z)-g_1^{(n-m)}(z)\right|^q e^{-\frac{(\beta-\alpha)q}{2}|z|^2}}{(1+|z|)^{nq}}e^{-\frac{\alpha q}{2}|w-z|^2}dA(z)\\
&\succeq \frac{1}{(1+|w|)^{(n-m)q}}\int_{D(w,1)}\left|g^{(n-m)}(z)-g_1^{(n-m)}(z)\right|^q e^{-\frac{(\beta-\alpha)q}{2}|z|^2}dA(z)\\
&\succeq \frac{\left|g^{(n-m)}(w)-g_1^{(n-m)}(w)\right|^q}{(1+|w|)^{(n-m)q}}e^{-\frac{(\beta-\alpha)q}{2}|w|^2}\\
&= \left(\frac{\left|\sum_{m_1,m_2}\frac{(n-m)!}{m_1!m_2!}\left(e^{bw+aw^2}(2aw+b)^{m_1}a^{m_2}-e^{b_1w+a_1w^2}(2a_1w+b_1)^{m_1}a_1^{m_2}\right)\right|^q}{(1+|w|)^{(n-m)q}}\right)e^{-\frac{(\beta-\alpha)q}{2}|w|^2},
\end{aligned}$$

for all $w\in\mathbb{C}$. After putting $w=0$ and since $a\neq a_1$ or $b\neq b_1$, we obtain

$$\|\mathcal{T}_g^{n,m}-\mathcal{T}_{g_1}^{n,m}\| \succeq \mu = \left|\sum_{m_1,m_2}\frac{(n-m)!}{m_1!m_2!}(b^{m_1}a^{m_2}-b_1^{m_1}a_1^{m_2})\right| \neq 0.$$

(ii) $0<p\leq q=\infty$:

Similarly, for this case we have

$$\begin{aligned}
\|\mathcal{T}_g^{n,m}-\mathcal{T}_{g_1}^{n,m}\| &\succeq \sup_{z\in\mathbb{C}} \frac{\left|k_{(w,\alpha)}^{(m)}(z)\right|\left|g^{(n-m)}(z)-g_1^{(n-m)}(z)\right|}{(1+|z|)^n}e^{-\frac{\beta}{2}|z|^2}\\
&= \sup_{z\in\mathbb{C}} \frac{|w|^m\left|e^{\alpha z\overline{w}}\right|\left|g^{(n-m)}(z)-g_1^{(n-m)}(z)\right|}{(1+|z|)^n}e^{-\frac{\beta}{2}|z|^2-\frac{\alpha}{2}|w|^2}\\
&\geq \frac{|w|^m\left|g^{(n-m)}(z)-g_1^{(n-m)}(z)\right|e^{-\frac{(\beta-\alpha)}{2}|z|^2}}{(1+|z|)^n}e^{-\frac{\alpha}{2}|w-z|^2}
\end{aligned}$$

$$\succeq \frac{\left|g^{(n-m)}(w) - g_1^{(n-m)}(w)\right|}{(1+|w|)^{(n-m)}} e^{-\frac{(\beta-\alpha)}{2}|w|^2},$$

where the last estimate is obtained by putting $w = z$. Using the formula in (6), the right hand side of the above estimate is further equal to;

$$\left(\frac{\left| \sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!} \left(e^{bw+aw^2}(2aw+b)^{m_1}a^{m_2} - e^{b_1w+a_1w^2}(2a_1w+b_1)^{m_1}a_1^{m_2} \right) \right|}{(1+|w|)^{(n-m)}} \right) e^{-\frac{(\beta-\alpha)}{2}|w|^2},$$

for all $w \in \mathbb{C}$. Putting $w = 0$ and since $a \neq a_1$ or $b \neq b_1$, we obtain

$$\|\mathcal{T}_g^{n,m} - \mathcal{T}_{g_1}^{n,m}\| \succeq \mu = \left| \sum_{m_1,m_2} \frac{(n-m)!}{m_1!m_2!} (b^{m_1}a^{m_2} - b_1^{m_1}a_1^{m_2}) \right| \neq 0.$$

□

6. Conclusion

In this paper, we have investigated several topological properties, in particular, boundedness, compactness, essential norm and path-connected components, of generalized integration operator, $\mathcal{T}_g^{n,m}$, acting between Fock spaces $\mathcal{F}_p^\alpha$ and $\mathcal{F}_q^\beta$, with modulators α and β. The operator generalizes the well-known Volterra-type integral operator, $\mathcal{V}_g$.

We remark that, boundedness and compactness of $\mathcal{V}_g : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$ is studied in [16], and essential norm and path-connected component of space of bounded $\mathcal{V}_g : \mathcal{F}_p^\alpha \to \mathcal{F}_q^\beta$, for when $\alpha = \beta = 1$, is studied in [7]. Our results in this manuscript generalizes the results in [7,16] in terms the operator or working space or both.

The results presented in this paper suggest directions for future research. One natural problem is to investigate whether there are another path-connected components of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$, and to characterize isolated points of $\mathcal{T}^{n,m}(\mathcal{F}_p^\alpha, \mathcal{F}_q^\beta)$. Another direction is to study analogous results on the Hardy and Bergman spaces.

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