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Picone type-identities for variable exponent $p(\cdot)$-biharmonic operator and applications on stratified Lie groups

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Abstract: In this paper we establish a new nonlinear variable exponent Picone-type identities for $p(x)$-biharmonic operator on a general stratified Lie group. As applications, eigenvalue properties, domain monotonicity, Barta-type estimate are proved for $p(x)$-sub-biharmonic operator. Furthermore, a Díaz-Saa-type inequality is proved and applied to study results on uniqueness of positive solutions of quasilinear elliptic equations involving variable exponent $p(x)$-sub-biharmonic operator.

Keywords: Biharmonic operator, Picone's identity, variable Lebesgue spaces, eigenvalue problems, principal frequency, Hardy-Rellich inequalities

MSC: 35B05 35H20, 35P30, 47J10.

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1. Introduction

The aim of this paper is to find a more direct and simpler approach to prove certain properties of solutions to problems involving operators with non-standard growth conditions (variable exponents) comparable to their constant exponent counterparts. Precisely, we study the $p(x)$-biharmonic operator ($p(x)$-biLaplacian) for $u : \mathbb{G} \to \mathbb{R}$

$$\Delta^2_{\mathbb{G},p(x)} u := \Delta_{\mathbb{G}}(|\Delta_{\mathbb{G}} u|^{p(x)-2}\Delta_{\mathbb{G}} u),$$

on stratified Lie groups with varible exponent variable exponent $p(x) > 1$ and $\Delta_{\mathbb{G}}$ being the usual sub-Laplacian. The variable exponent operators are not homogeneous and thus possess more complicated nonlinearities which make several known results that have been applied in the constant exponent cases breakdown. To this end, and motivated by the sophistication of Picone-type identities for higher order partial differential operators we hereby investigate if there exist such identities for variable exponents $p(x)$-biLaplacian, and if problems involving this operator can be studied via Picone-type identities.

Italian mathematician, Mauro Picone (1885 - 1977), originally derived Picone identity in 1910 for differentiable functions $u = u(x), v = v(x) \geq 0$ with $v \neq 0$ in the form (where $' = \frac{d}{dx}$)

$$\left[\frac{u}{v}(f_1 u' v - f_2 u v')\right]' = (f_1 - f_2)(u')^2 + f_2\left(u' - \frac{u}{v}v'\right)^2 + (g_2 - g_1)u^2, \tag{1}$$

during his study of Sturmian comparison and oscillation principles for the following system of homogeneous linear second order differential equations

$$\begin{cases} [f_1(x)u'(x)]' + g_1(x)u(x) = 0, \\ [f_2(x)v'(x)]' + g_2(x)v(x) = 0. \end{cases} \tag{2}$$

The Sturmian comparison principle for (2) was established using (1) assuming that $f_1(x) \geq f_2(x) > 0$ and $g_1(x) \leq g_2(x)$. Several years later, Picone [1] generalised identity (1) to the second order partial differential operator (Laplacian) under the name Picone identity: For non-negative differentiable functions u and v, $v > 0$

$$\left|\nabla u - \frac{u}{v}\nabla v\right|^2 = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0. \tag{3}$$

Here, $|\cdot|$ and ∇ denote the length of a vector and gradient of a function in Euclidean space $\mathbb{R}^n$, respectively. The identity (3) has been applied extensively to the study of second order elliptic equations and systems involving Laplace operator (see [2] for instance) in which case, among other applications, many interesting results such as existence and non-existence of positive solutions, Sturmian comparison principle, domain monotonicity, Hardy's inequality, Barta's inequality and Caccioppoli inequality have been established. Several extensions and generalisation of (3) are now available for studying more general elliptic operators. For instance, in order to study p-Laplace equations and eigenvalue problems involving p-Laplacian, Allegretto and Huang [3] extended (3) to the case of general $p > 1$ as follows: for $u \geq 0$, $v > 0$, then

$$|\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v \geq 0. \tag{4}$$

Tyagi [4] and Bal [5] established nonlinear versions of (3) and (4), respectively, with several applications (see also [6,7]). For other interesting extension of Picone-type identities for p-Laplacian and related results see [8–19].

Picone identity for p-biharmonic operator is a generalisation of the identity to a class of fourth order elliptic operator as established by Dwivedi and Tyagi [20]. Jaroš [21] (see also Jaroš [22]) obtained a generalised nonlinear analogue of this and he highlighted its applications to comparison results for a class of half linear differential equations of fourth order. Dwivedi [23] introduced the Picone identity for p-biharmonic operator as follows: Let $u \geq 0$ and $v > 0$ be twice continuously differentiable functions such that $-\Delta v \geq 0$. Then

$$|\Delta u|^p - \Delta\left(\frac{u^p}{v^{p-1}}\right)|\Delta v|^{p-2}\Delta v \geq 0, \tag{5}$$

where Δ is the usual Euclidean Laplacian. He gave applications to p-biharmonic boundary value problems, Sturmian comparison principle and Morse index of solutions. The case $p = 2$ in (5) was obtained by Dwivedi and Tyagi [24] as follows

$$\left(\Delta u - \frac{u}{v}\Delta v\right)^2 - \frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2 = |\Delta u|^2 - \Delta\left(\frac{u^2}{v}\right)\Delta v \geq 0. \tag{6}$$

See [6,8,16,18,20,23,25] for other extensions and generalisations to p-biharmonic operators and applications in different contexts.

In recent years, the study of variable exponent elliptic operators and corresponding problems with non-standard growth condition have gained enormous interest among researchers (see [26–31] for $p(x)$-Laplacian and [32–37] for $p(x)$-biharmonic operator). Models involving $p(x)$-growth condition arise from physical processes such as nonlinear elasticity theory, electrorheological fluids, image processing, etc [38–40]. Several literature as cited above have asserted that $p(x)$-Laplacian and $p(x)$-biharmonic operator are similar in many respects to the classical cases $p(x) = p$-constant, but problems involving these operators do not trivially generalise the p-constant cases. The variable exponent operators are not homogeneous for instance, which thus makes the nonlinearity so much complicated such that many of known approaches to handle the cases p-constant do not hold for $p(x)$-Laplacian and $p(x)$-biharmonic operator.

To this end, there arises a natural question: *Can we extend similar type of Picone identities into non-standard growth problems?* For this course, Abolarinwa and Ali [41], Allegretto [42], Arora, Giacomoni and Warnault [43], Feng and Han [44] and Yoshida [45] have provided answer in the affirmative for the case of problems involving $p(x)$-Laplacian, $\Delta_{p(x)}u := -\nabla\cdot(|\nabla u|^{p(x)-2}\nabla u)$ with $p(x) > 1$. More precisely, Allegretto [42] established variable exponent Picone-type identity for differentiable functions $v > 0$, $0 \leq u \in C_0^\infty(\Omega)$ $\Omega \subset \mathbb{R}^n$ and continuous $p(x) > 1$ as follows:

$$\frac{|\nabla u|^{p(x)}}{p(x)} - \nabla\left[\frac{u^{p(x)}}{p(x)v^{p(x)-1}}\right]|\nabla v|^{p(x)-2}\nabla v \geq 0, \tag{7}$$

on the assumption that $\nabla v \nabla p(x) = 0$. He used the inequality to prove Barta theorem and some other results. Later, Yoshida [45] (see also [46,47]) established similar Picone identities for quasilinear and half-linear elliptic equations involving $p(x)$-Laplacian and pseudo $p(x)$-Laplacian, and consequently developed Sturmian comparison theory for them. Most recently, Feng and Han [44], motivated by Allegretto [42] proved a modified form of (7) and showed that

$$|\nabla u|^{p(x)} - \nabla\left[\frac{u^{p(x)}}{v^{p(x)-1}}\right]|\nabla v|^{p(x)-2}\nabla v \geq 0, \tag{8}$$

if $\nabla v \nabla p(x) = 0$ a.e in Ω, with equality if and only if $\nabla(u/v) = 0$ in $\Omega \subset \mathbb{R}^n$. They proved monotonicity of principal eigenvalue $\lambda_{1,p}$ of $\Delta_{p(x)}$ and a variable exponent Barta inequality for $p(x)$-Laplacian in the form

$$\lambda_{1,p} \geq \inf_{x\in\Omega}\left[\frac{\Delta_{p(x)}v}{v^{p(x)-1}}\right],$$

on the assumption that $\nabla v \nabla p(x) = 0$. In the same spirit, Abolarinwa and Ali [41] extended this result and derived a generalised nonlinear variable exponent Picone-type identities for general vector fields in the sub-elliptic context. In that context the authors [41] considered an n-dimensional smooth manifold $\mathcal{M}$ equipped with a volume form dx and a family of vector fields $\{X_k\}_{k=1}^N$, $n \geq N$. Let $u \geq 0$ and $v > 0$ be differentiable functions almost everywhere in $\Omega \subseteq \mathcal{M}$ with smooth boundary. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1-function satisfying $f(s) > 0$ and $f'(s) \geq (p(x)-1)\left[f(s)^{\frac{p(x)-2}{p(x)-1}}\right]$, then

$$|\nabla_X u|^{p(x)} - \nabla_X\left(\frac{u^{p(x)}}{f(v)}\right)|\nabla_X v|^{p(x)-2}\nabla_X v \geq 0, \tag{9}$$

provided $\nabla_X v \nabla_X p(x) = 0$ with equality if and only if $\nabla_X(u/v) = 0$ almost everywhere in Ω. Here ∇_X is the horizontal gradient for general vector fields. Consequently, the authors [41] gave several applications to qualitative properties of the principal eigenvalue of $p(x)$-sub-Laplacian and derived sub-elliptic variable exponents Caccioppoli estimates in the form

$$\int_\Omega \phi^{p(x)}|\nabla_X v|^{p(x)}dx \leq (p^+)^{p^+}\int_\Omega v^{p(x)}|\nabla_X \phi|^{p(x)}dx,$$

for every nonnegative test function $\phi \in C_0^\infty(\Omega)$, where v is a sub-solution in $\Omega \subset \mathcal{M}$ and $p^+ := ess\sup_\Omega p(x)$.

A generalisation of $p(x)$-Laplacian is the fourth order quasilinear degenerate $p(x)$-biharmonic operator defined by $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$. Up to now and to the best of our knowledge there is no Picone-type identity for $p(x)$-biharmonic operator found in literature. This paper therefore attempts to bridge this gap on one hand, and on the other hand to treat interesting results as discussed above in the setting of quasilinear (sub)-elliptic equations and systems involving $p(x)$-biharmonic operator with the aid of Picone-type identities. All our results are discussed in the context of the general stratified Lie groups (homogeneous Carnot groups). Such groups have been well studied with several classical and modern results. This class of groups includes the Abelian group, Heisenberg group as well as the more general H-type groups as interesting but basic examples (see the books [48–50] and some of the references cited therein). A homogeneous Lie group $\mathbb{G}$ which is nilpotent, connected and simply connected with a stratified Lie algebra g is called a stratified Lie group or homogeneous Carnot group (see discussion in §3).

1.1. Outline of this paper

The rest of this paper is planned as follows: The main results of this work are stated in the next section. §3 is devoted to preliminaries and fixing of notation that appear throughout the paper. Here we give summary description of the stratified Lie group, sub-elliptic operators and their examples. Some notions from the theory of variable Lebesgue and Sobolev spaces are discussed since these spaces naturally form the setting in which all applications will be made. Navier eigenvalue problem is again introduced and characterised in this section. §4 is devoted to the proof of the generalised nonlinear Picone identities of p(x)-sub-biharmonic operator on open bounded domain (Theorem 1) and its applications. The applications discussed here include proofs of variable exponent Hardy-Rellich type inequalities, eigenvalue and domain monotonicity, Barta-type inequality for $p(x)$-biharmonic operator as summarised in Theorem 2. In §5 we establish a generalised Díaz-Saa type inequality for the $p(x)$-biharmonic operator on homogeneous Carnot groups and apply it to establish the uniqueness results for problems involving variable exponent $p(x)$-biharmonic operator with Navier boundary condition.

2. Statement of results

Here we state the generalised nonlinear Picone identities of $p(x)$-sub-biharmonic operator on open bounded domain of $\mathbb{G}$ and later fix the notation in the next section.

Let $\Omega \subseteq \mathbb{G}$ be any bounded domain of a stratified Lie group $\mathbb{G}$ of homogeneous dimension $Q \geq 3$. Denote the horizontal gradient and sub-Laplacian on $\mathbb{G}$ by $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$, respectively (see detail in the next section). The first main result of this paper is the following:

Theorem 1. *Let $u \geq 0$ and $v > 0$ be non-constant differentiable functions a.e. in Ω such that $-\Delta_{\mathbb{G}} v \geq 0$. Suppose $p : \bar{\Omega} \to (0, \infty)$ is a C^2-function for $p(x) > 1$. Define*

$$\mathscr{R}(u,v) = |\Delta_{\mathbb{G}} u|^{p(x)} - \Delta_{\mathbb{G}} \left(\frac{u^{p(x)}}{v^{p(x)-1}} \right) |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v,$$

and

$$\begin{aligned}
\mathscr{L}(u,v) =& |\Delta_{\mathbb{G}} u|^{p(x)} - \frac{u^{p(x)}}{v^{p(x)-1}} \ln \frac{u}{v} \left(\ln \frac{u}{v} |\nabla_{\mathbb{G}} p(x)|^2 + \Delta_{\mathbb{G}} p(x) \right) |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v \\
& - 2 \frac{u^{p(x)-1}}{v^{p(x)-1}} \left(p(x) \ln \frac{u}{v} + 1 \right) \nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p(x) |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v \\
& + 2 \frac{u^{p(x)}}{v^{p(x)}} \left((p(x)-1) \ln \frac{u}{v} + 1 \right) \nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p(x) |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v \\
& + (p(x)-1) \frac{u^{p(x)}}{v^{p(x)}} |\Delta_{\mathbb{G}} v|^{p(x)} - p(x) \frac{u^{p(x)-1}}{v^{p(x)-1}} \Delta_{\mathbb{G}} u |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v \\
& - p(x)(p(x)-1) \frac{u^{p(x)-2}}{v^{p(x)-1}} \left(\nabla_{\mathbb{G}} u - \frac{u}{v} \nabla_{\mathbb{G}} v \right)^2 |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v.
\end{aligned}$$

Then (i) $\mathscr{L}(u,v) = \mathscr{R}(u,v)$.

(ii) $\mathscr{L}(u,v) \geq 0$ *provided* $\nabla_{\mathbb{G}} g \nabla_{\mathbb{G}} p(x) \equiv 0$ *for any differentiable function* g*, and* $\ln \frac{u}{v} |\nabla_{\mathbb{G}} p(x)|^2 + \Delta_{\mathbb{G}} p(x) \equiv 0$ *a.e. in* Ω.

(iii) *Furthermore,* $\mathscr{L}(u,v) = 0$ *a.e in* Ω *if and only if* $u = \alpha v$ *for some constant* $0 < \alpha \in \mathbb{R}$.

Remark 1. The claim that $\mathscr{L}(u,v) \geq 0$ relies on the condition that $\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p(x) = 0$, $\nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p(x) = 0$ and $\ln \frac{u}{v} |\nabla_{\mathbb{G}} p(x)|^2 + \Delta_{\mathbb{G}} p(x) \equiv 0$ a.e. in Ω.

1.
The case which readily comes to mind is the situation where $p(x) = p$ (constant). This case implies that some earlier results for p constant (see [23]) follows directly from the present case of variable $p(x)$ (smooth or at least C^2).

2. The condition $\Delta_{\mathbb{G}} p(x) = 0$ and $\nabla_{\mathbb{G}} p(x) \neq 0$ has been applied successfully by Allegretto [42] (see Lemma 2 and Theorem 2 in [42]). We also remark that from geometric point of view, the condition

$\nabla_{\mathbb{G}} v(x) \nabla_{\mathbb{G}} p(x) = 0$ means that the equation $u(x) = S_1$ and $p(x) = S_2$ represent orthogonal surface on Ω.

3. Examples of test fields $\overrightarrow{f}$ for which $\overrightarrow{f} \cdot \nabla p(x) = 0$ ($\overrightarrow{f}$ is taken as a gradient of scalar field) are given by Mihailescu et al. [51] in the setting of Grushin space.

Let $\Omega \subseteq \mathbb{G}$ be a smooth bounded domain with smooth boundary. For $p \in C^2(\bar{\Omega})$ such that $p(x) > 1$, we consider the Navier eigenvalue problem

$$\begin{cases} \Delta_{\mathbb{G}}(|\Delta_{\mathbb{G}} u|^{p(x)-2} \Delta_{\mathbb{G}} u) = \lambda |u|^{p(x)-2} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta_{\mathbb{G}} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where the pair $(\lambda \in \mathbb{R}, u \in \mathbb{X})$ (with $\mathbb{X} = S^{2,p(x)}(\Omega) \times \overset{\circ}{S}{}^{1,p(x)}(\Omega)$) is called a distributional solution. By the application of Theorem 1 we prove some properties of variable exponent $p(x)$-biharmonic eigenvalues in (10).

Theorem 2. *Let $\lambda_1(\Omega)$ be the first eigenvalue of the $p(x)$-biharmonic operator on $\bar{\Omega}$ in (10).*

(i) *Let there exists $\lambda \in \mathbb{R}$ and a strictly sub-solution $v \in \mathbb{X}$ of (10). Then $\lambda_1(\Omega) > \lambda$.*

(ii) *Let $\lambda_1(\Omega) > 0$ be the principal eigenvalue of (10). Then $\lambda_1(\Omega') \geq \lambda_1(\Omega)$ if and only if $\Omega' \subset \Omega$ and $\Omega' \neq \Omega$.*

(iii) *For any differentiable function $v > 0$ in Ω with $\Delta_{\mathbb{G}}(|\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v) \in C(\bar{\Omega})$ we have*

$$\lambda_1(\Omega) \geq \inf_{0<v\in\mathbb{X}} \inf_{\Omega} \left[\frac{\Delta_{\mathbb{G}}(|\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v)}{v^{p(x)-1}} \right].$$

(iv) *Any eigenfunction v corresponding to an eigenvalue $\lambda \neq \lambda_1(\Omega)$ changes sign.*

(v) *If $v \in \mathbb{X}$ is a positive sup-solution of (10). Then v is a constant multiple of the principal eigenfunction.*

Remark 2. If $\mathbb{G} = \mathbb{H} = (\mathbb{C}^N \times \mathbb{R}, \circ)$ is the Heisenberg group, the homogeneous dimension is $Q = 2N + 2$ and $\Delta_{\mathbb{H}}$ is the Kohn-Laplacian. Also if $\mathbb{G} = \mathbb{E} = (\mathbb{R}^N, +)$ is the Euclidean additive group, then $Q = N$ and $\Delta_{\mathbb{E}}$ is the usual Euclidean Laplacian. For $p(x) = p$(constant) in these settings, Theorem 1 generalises several results in literature (see [18,20–24,52] for instance).

Lastly, we present a generalised Díaz-Saa-type inequality for the $p(x)$-biharmonic operator in this context. We first observe that by reverting to Theorem 1 we can obtain a generalised Picone-type inequality for $p(x) > 1$ as follows

$$\int_{\Omega} \frac{u^{p(x)}}{v^{p(x)-1}} \Delta^2_{\mathbb{G},p(x)} v dx \leq \int_{\Omega} |\Delta_{\mathbb{G}} u|^{p(x)} dx,$$

where $\Omega \subset \mathbb{G}$ is a bounded open domain. By the application of the above inequality we obtain the following generalised Díaz-Saa-type inequality on stratified Lie group.

Proposition 1. *Let $1 < p(x) \leq Q$ and Ω be a bounded open domain in $\mathbb{G}$. Suppose $u_1, u_2 \in \mathbb{X}$ satisfy $u_1, u_2 \geq 0$ a.e. in Ω such that $\nabla_{\mathbb{G}} u_i \nabla_{\mathbb{G}} p = 0$, $\ln u_i |\nabla p|^2 + \Delta_{\mathbb{G}} p \equiv 0$, $l = 1, 2$ and*

$$\Delta^2_{\mathbb{G},p(x)} u_i = g_i(x, u_i), \quad u_i > 0, \quad i = 1, 2 \quad in \ \Omega,$$

where $g_i(x, u_i)$, $i = 1, 2$ is a bounded and measurable function, $g(x, s) \leq A(s^{p(x)-1} + 1)$ for a constant $A > 0$ and all $s \geq 0$ with

$$u_i = \Delta_{\mathbb{G}} u_i = 0, \ i = 1, 2 \quad on \ \partial\Omega.$$

Then

$$0 \leq \int_{\Omega} \left(\frac{\Delta^2_{\mathbb{G},p(x)} u_1}{u_1^{p(x)-1}} - \frac{\Delta^2_{\mathbb{G},p(x)} u_2}{u_2^{p(2)-1}} \right) (u_1^{p(x)} - u_2^{p(x)}) dx. \tag{11}$$

For $p \in C^2(\bar{\Omega})$ such that $p(x) > 1$, we consider the Navier boundary value problem

$$\begin{cases} \Delta^2_{\mathbb{G},p(x)} u = g(x,u), & x \in \Omega \subset \mathbb{G}, \\ u > 0, & x \in \Omega \subset \mathbb{G}, \\ u = \Delta_{\mathbb{G}} u = 0, & x \in \partial\Omega, \end{cases} \tag{12}$$

with the certain hypothesis on the function $g : \bar{\Omega} \times [0,\infty) \to \mathbb{R}^+$. Applying our version of Díaz-Saa-type inequality we prove the next theorem.

Theorem 3. *There exists at most one positive solution u to (12) in $\mathbb{X} \cap L^\infty(\Omega)$ for $1 < p(x) \leq Q$, where $p \in C^2(\bar{\Omega})$, $\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p = 0$ and $\ln u |\nabla p|^2 + \Delta_{\mathbb{G}} p \equiv 0$ a.e. in Ω.*

3. Preliminaries

In this section we recall some preliminaries which will allow us fix necessary notation. First, we give basics of the stratified Lie group (homogeneous Carnot group). Secondly, we discuss some concepts from the theory of variable Lebesgue and Sobolev spaces, and then present Navier eigenvalue problem for $p(x)$-sub-Laplacian on stratified Lie groups.

3.1. Stratified Lie groups

A Lie group $\mathbb{G}$ (on $\mathbb{R}^N$) which is nilpotent, connected and simply connected is called homogeneous if for a dilation δ_γ on $\mathbb{G}$ (see the definition of dilation δ_γ below), the maps $\exp \circ \delta_\gamma \circ \exp^{-1}$ are group of automorphism.

Recall that a stratified Lie algebra g of step r is a Lie algebra with subspaces $V_1, \cdots, V_r$ satisfying

$$g = V_1 \oplus \cdots \oplus V_r, \;\; [V_1, V_j] = V_{j+1}, \;\; j = 1, \cdots, r-1 \text{ and } \; [V_1, V_r] = 0.$$

If a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is nilpotent, connected and simply connected with a stratified Lie algebra g, then $\mathbb{G}$ is called a stratified Lie group (or homogeneous Carnot group) of step r. In fact, if $N_j = \dim V_j$, using the exponential map we can identify $\mathbb{G}$ with $\mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ so that each point $g \in \mathbb{G}$ is identified with a point $x = (x^{(1)}, \cdots, x^{(r)})$ in $\mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ so that $x^j \in \mathbb{R}^{N_j}$.

The stratified Lie groups have a natural family of anisotropic dilations $\delta_\gamma : \mathbb{G} \to \mathbb{G}$ for every $\gamma > 0$ defined by

$$\delta_\gamma(x) \equiv \delta_\gamma(x', x^{(2)}, \cdots, x^{(r)}) := (\gamma x', \gamma^2 x^{(2)}, \cdots, \gamma^r x^{(r)}),$$

which is an automorphism of $\mathbb{G}$. Here $x' = x^{(1)} \in \mathbb{R}^{N_1}$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \cdots, r$. The homogeneous dimension Q of the group $\mathbb{G}$ is given by

$$Q := \sum_{k=1}^{r} k \cdot \dim V_k = \sum_{k=1}^{r} k N_k.$$

The Haar measure on $\mathbb{G}$ is denoted by dx, which can be taken to be the Lebesgue measure on $\mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$. In fact, it can be easily shown that the Lebesgue measure is homogeneous with respect to the dialation δ_γ. Thus given a measurable set $\mathcal{U} \subseteq \mathbb{R}^N$ with its Lebesgue measure denoted by $|\mathcal{U}|$, one then has

$$|\delta_\gamma(\mathcal{U})| = \gamma^Q |\mathcal{U}|.$$

Let $X_1, \cdots, X_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}\Big|_0$ for $k = 1, \cdots, N_1$. Then, for every $x \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$, the Hörmander rank condition

$$\operatorname{rank}(\operatorname{Lie}\{X_1, \cdots, X_{N_1}\}) = N,$$

holds, that is, the iterated commutators of $X_1, \cdots, X_{N_1}$ span the Lie algebra of $\mathbb{G}$. The left invariant vector fields X_k has an explicit form given by (see detail in the book [50]),

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', x^2 \cdots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}},$$

where $a_{k,m}^{(l)}$ is a homogeneous polynomial function of degree $l-1$, r is the step of $\mathbb{G}$, $x = (x', x^{(2)}, \cdots,, x^{(r)})$, and $x^{(l)} = (x_1^{(l)}, \cdots, x_{N_l}^{(l)})$ are the variables in the l^{th}-stratum.

3.2. Horizontal sub-Laplacian

The canonical (horizontal) sub-Laplacian on $\mathbb{G}$ is defined by

$$\mathscr{L}_X := \sum_{k=1}^{N_1} X_k^2,$$

which is a left invariant homogeneous second-order hypoelliptic differential operator. It is elliptic if and only if the step of $\mathbb{G}$ is equal to 1. (The hypoellipticity of $\mathscr{L}_X$ is a special case of Hörmander's sum of square [53]). The horizontal gradient and divergence on $\mathbb{G}$ are respectively denoted by

$$\nabla_{\mathbb{G}} := (X_1, \cdots, X_{N_1}) \quad \text{and} \quad \mathrm{div}_{\mathbb{G}} w := \nabla_{\mathbb{G}} \cdot w,$$

meaning that horizontal sub-Laplacian on $\mathbb{G}$ can be written as

$$\mathscr{L}_X = \Delta_{\mathbb{G}} := \mathrm{div}_{\mathbb{G}} \cdot \nabla_{\mathbb{G}}.$$

Let $p : \bar{\Omega} \to \mathbb{R}$ be a continuous function and $p(x) > 1$ for $x \in \bar{\Omega} \subseteq \mathbb{G}$. We define the $p(x)$-sub-Laplacian (horizontal $p(x)$-Laplacian) on $\mathbb{G}$ by the formula

$$\mathscr{L}_{p(x)} u := \Delta_{\mathbb{G},p(x)} u := \mathrm{div}_{\mathbb{G}}(|\nabla_{\mathbb{G}} u|^{p(x)-2} \nabla_{\mathbb{G}} u),$$

where u is a smooth function. If $p(x) = p$ (p=constant), the operator $\mathscr{L}_p u$ becomes the p-sub-Laplacian, $\Delta_{\mathbb{G},p} u := \mathrm{div}_{\mathbb{G}}(|\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u)$. Here the notation

$$|x'| = (x_1'^2 + \cdots + x_{N_1}'^2)^{\frac{1}{2}},$$

stands for the Euclidean norm in $\mathbb{R}^{N_1}$.

Moreover, the $p(x)$-sub-biLaplacian (or $p(x)$-sub-biharmonic operator) on $\mathbb{G}$ is defined by

$$\Delta^2_{\mathbb{G},p(x)} u := \Delta_{\mathbb{G}}(|\Delta_{\mathbb{G}} u|^{p(x)-2} \Delta_{\mathbb{G}} u),$$

where $p(x)$ is as defined above. By an application of the integration by parts formula it can be seen that for $u, v \in C_0^\infty(\Omega \subseteq \mathbb{G})$ the following formula holds true

$$\int_\Omega \Delta^2_{\mathbb{G},p(x)} u v dx = \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)-2} \Delta_{\mathbb{G}} u \Delta_{\mathbb{G}} v dx \tag{13}$$

and thus

$$\int_\Omega u \Delta^2_{\mathbb{G},p(x)} u dx = \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)} dx,$$

Example 1. Some well-known examples of stratified Lie groups include the Euclidean additive group (Abelian group) $\mathbb{E} := (\mathbb{R}^N, +)$, the Heisenberg group $\mathbb{H} := \mathbb{C}^N \times \mathbb{R}$ as well as more general H-type group $(\mathbb{R}^M \times \mathbb{R}^N)$. See the book [48–50] for further discussions on various examples of stratified Lie groups, particularly, see [54,55] for various results on horizontal Laplacian on Heisenberg groups.

The additive group $\mathbb{E}$ is a homogeneous Carnot group of step 1, whose dilation is given by the usual scalar multiplication, that is, $\delta_\gamma(x) = \gamma x, \gamma > 0$. Its generators are $\partial_{x_1}, \ldots, \partial_{x_N}$, while its canonical sub-Laplacian is the usual Euclidean Laplacian

$$\Delta_{\mathbb{E}} = \sum_{j=1}^{N} \partial_{x_j}^2.$$

The Heisenberg group $\mathbb{H}$ is a homogeneous Carnot group whose underlying manifold is $\mathbb{C}^N \times \mathbb{R} = \mathbb{R}^{2N+1}$ with the non-commutative group law $\circ$ define as follows:

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' - 2(\langle x_j, y_j' \rangle - \langle x_j', y_j \rangle)),$$

where $(x, y, t), (x', y', t') \in \mathbb{H} := \mathbb{R}^{2N+1}$, $N \geq 1$ and $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product in $\mathbb{R}^N$. A basis for the Lie algebra of the left invariant vector fields on $\mathbb{H}$ is given by ($1 \leq j \leq N$)

$$X_j = \partial_{x_j} + 2y_j\partial_t, \;\; Y_j = \partial_{y_j} - 2x_j\partial_t, \quad T = \partial_t.$$

By the Lie bracket definition one can easily check that

$$[X_j, X_k] = [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0, \; j, k = 1, 2 \cdots N,$$

and

$$[X_j, Y_k] = -4T\delta_{jk},$$

which constitute Heisenberg canonical commutation relation of quantum mechanics for position and momentum. These vector fields are known to satisfy the Hörmander rank condition. A family of dilation on $\mathbb{H}$ is defined by $\delta_\gamma(x, y, t) = (\gamma x, \gamma y, \gamma^2 t), \gamma > 0$. Consequently, the horizontal Laplacian or sub-Laplacian on $\mathbb{H}$ (Kohn Laplacian) and p-sub-Laplacian are defined by

$$\Delta_{\mathbb{H}} := \sum_{j=1}^{n} \left(X_j^2 + Y_j^2 \right) = \mathrm{div}_{\mathbb{H}} \nabla_{\mathbb{H}},$$

and

$$\Delta_{\mathbb{H},p} := \mathrm{div}_{\mathbb{H}}(|\nabla_{\mathbb{H}}|^{p-2}\nabla_{\mathbb{H}}), \; p > 1.$$

We further remark that the results of this paper can be extended to other degenerate sub-elliptic operators such as Heisenberg-Greiner operator on $\mathbb{R}^N = \mathbb{R}^{2n+1}$, and Baouendi-Grushin operator on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$. Note that Heisenberg-Greiner vector fields are defined by

$$X_j = \partial_{x_j} + 2\lambda y_j |z|^{2\lambda-2}\partial_t, \;\; Y_j = \partial_{y_j} - 2\lambda x_j |z|^{2\lambda-2}\partial_t, \;\; j = 1, 2, \cdots n,$$

where $\lambda \geq 1$ is a fixed parameter and $z = (x, y) \in \mathbb{R}^{2n}$, with its associated sub-Laplacian given by

$$\Delta_{HG} := \sum_{j=1}^{n} \left(X_j^2 + Y_j^2 \right).$$

While the Baouendi-Grushin vector fields are defined by

$$X_i = \partial_{x_i}, \; i = 1, 2, \cdots, n \;\;\; Y_j = |x|^\alpha \partial_{y_j}, \; j = 1, 2, \cdots m, \;\; \alpha > 0,$$

with its associated gradient operator defined by

$$\nabla_B := (X_1, \cdots, X_n, Y_1, \cdots, Y_m) = (\nabla_x, |x|^{2\alpha}\nabla_y),$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \cdots y_m) \in \mathbb{R}^m$. (See some applications involving variable exponent forms of these operators in [31,51]).

3.3. Variable Lebesgue spaces

In order to discuss generalised solutions, we need some concepts from the theory of variable Lebesgue and Sobolev spaces. Detailed description of these spaces can be found in [56–58].

Let $\Omega \subset \mathbb{G}$ be an open domain and $\mathcal{E}(\Omega)$ denotes the set of all equivalence classes of Haar measurable real-valued functions defined on Ω being equal almost everywhere. For any positive variable exponent $p(x) : \Omega \to [1, \infty)$ such that

$$1 < p^- \leq p(x) \leq p^+ < \infty,$$

where $p^- := ess \inf_{x \in \bar{\Omega}} p(x)$ and $p^+ := ess \sup_{x \in \bar{\Omega}} p(x)$ and any $u \in \mathcal{E}(\Omega)$, the identity

$$|u(x)|^{p(x)} = \exp\left[p(x) \log |u(x)|\right],$$

shows that $|u(x)|^{p(x)}$ is measurable. Since it is also nonnegative, the integral $\int_\Omega |u(x)|^{p(x)} dx$ is well defined. Consider the functional (called the ϱ-modular) $\varrho_{p(\cdot)} : \mathcal{E}(\Omega) \to [0, \infty]$ given by

$$\varrho_{p(\cdot)}(u) := \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} dx + \|u\|_{L^\infty(\Omega_\infty)}.$$

Here we assume that Ω_∞ has zero measure and we stick to

$$\varrho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)} dx.$$

It can be easily seen that the ϱ-modular $\varrho_{p(\cdot)}(u)$ possesses the following useful properties (see [56] for details)

(a) $\varrho_{p(\cdot)}(u) \geq 0$ for all function u with equality if and only if $u = 0$ a.e.
(b) $\varrho_{p(\cdot)}(-u) = \varrho_{p(\cdot)}(u)$ for all u.
(c) $\varrho_{p(\cdot)}(u) \leq \varrho_{p(\cdot)}(v)$ when $|u| \leq |v|$ a.e.
(d) $\varrho_{p(\cdot)}(u)$ is convex.
(e) If $|u| \geq |v|$ a.e in Ω and if $\varrho_{p(\cdot)}(u) < \infty$, then $\varrho_{p(\cdot)}(u) \geq \varrho_{p(\cdot)}(v)$ which is strict if $|u| \neq |v|$.
(f) If $0 < \varrho_{p(\cdot)}(u) < \infty$, then the function $t \mapsto \varrho_{p(\cdot)}(u/t)$ is continuous and decreasing on the interval $[1, \infty)$.
(g) For $\{u_k\} \subset \mathcal{E}(\Omega)$, $\varrho_{p(\cdot)}(\liminf_{k\to\infty} |u_k|) \leq \liminf_{k\to\infty} \varrho_{p(\cdot)}(|u_k|)$ (Fatou's Lemma).

Definition 1. The generalised (variable exponent) Lebesgue space $L^{p(\cdot)}(\Omega)$ is the vector space of measurable functions for which the ϱ-modular is finite, that is

$$L^{p(\cdot)}(\Omega) = \left\{u \in \mathcal{E}(\Omega) : \int_\Omega |u(x)|^{p(\cdot)} dx < \infty\right\}.$$

The space $L^{p(\cdot)}(\Omega)$ is a Banach space when equipped with the (Luxemburg) norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{t > 0 : \varrho_{p(\cdot)}\left(\frac{u}{t}\right) \leq 1\right\}.$$

Obviously, the case where $p(x) = p$ (constant) makes the space $L^{p(\cdot)}(\Omega)$ the classical Lebesgue space $L^p(\Omega)$, and the Luxemburg norm $\|u\|_{p(\cdot)}$ a standard one $\|u\|_{p(\cdot)} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$. Furthermore, ϱ-modular satisfies

$$\varrho_{p(\cdot)}\left(\frac{u}{\|u\|_{p(\cdot)}}\right) \leq 1 \text{ for all } u \text{ with } 0 < \|u\|_{p(\cdot)} < \infty.$$

Indeed, choosing $\gamma_k \searrow \|u\|\|_{p(\cdot)}$, one can apply Fatous lemma to obtain

$$\varrho_{p(\cdot)}\left(\frac{u}{\|u\|_{p(\cdot)}}\right) \leq \liminf_{k\to\infty} \varrho_{p(\cdot)}\left(\frac{u}{\gamma_k}\right) \leq 1.$$

Also if $p^+ < \infty$, then $\varrho_{p(\cdot)}\left(\frac{u}{\|u\|_{p(\cdot)}}\right) = 1$ for all u with $0 < \|u\|_{p(\cdot)} < \infty$. Consequently, if $\|u\|_{p(\cdot)} \leq 1$ then $\varrho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}$.

Another basic result is a generalised Hölder's inequality which can be used to define an equivalent norm. If $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ a.e. on Ω, then for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|dx \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \|u\|_{p(x)}\|v\|_{p'(x)}.$$

3.4. Variable Sobolev spaces

Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$ with $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Setting $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$, where $D_k = X_k$ is the derivative operator in the sense of distribution.

Definition 2. The variable exponent Sobolev space $S^{k,p(x)}(\Omega)$ is the vector space of those functions $u \in L^{p(x)}(\Omega)$ for which $D^\alpha u$ is also $L^{p(x)}(\Omega)$ with $|\alpha| \leq k$, and endowed with norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{k,p(x)}. \tag{14}$$

That is,

$$S^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\right\}.$$

We denote by $\overset{\circ}{S}{}^{k,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $S^{k,p(\cdot)}(\Omega)$ with respect to the norm (14). It can be clearly seen that $S^{k,p(\cdot)}(\Omega)$ and $\overset{\circ}{S}{}^{k,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces if $1 < \inf p(x) < \sup p(x) < \infty$ in Ω.

Precisely, the spaces $S^{1,p(x)}(\Omega)$ and $S^{2,p(x)}(\Omega)$ are respectively endowed with the norms

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla_{\mathbb{G}} u\|_{p(x)},$$

which is equivalent to

$$\|u\|_{1,p(x)} = \inf\left\{t > 0 : \int_\Omega \left(\left|\frac{u(x)}{t}\right|^{p(x)} + \left|\frac{\nabla_{\mathbb{G}} u(x)}{t}\right|^{p(x)}\right) dx \leq 1\right\},$$

and

$$\|u\|_{2,p(x)} = \|u\|_{p(x)} + \|\nabla_{\mathbb{G}} u\|_{p(x)} + \|\Delta_{\mathbb{G}} u\|_{p(x)},$$

which is similarly equivalent to

$$\|u\|_{2,p(x)} = \inf\left\{t > 0 : \int_\Omega \left|\frac{\Delta_{\mathbb{G}} u(x)}{t}\right|^{p(x)} dx \leq 1\right\}.$$

3.5. Navier eigenvalue problem

Let Ω be a smooth bounded domain with smooth boundary in $\mathbb{G}$. For $p \in C^2(\bar{\Omega})$ such that $p(x) > 1$, the Navier eigenvalue problem is the following

$$\begin{cases} \Delta_{\mathbb{G}}(|\Delta_{\mathbb{G}}u|^{p(x)-2}\Delta_{\mathbb{G}}u) = \lambda|u|^{p(x)-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \ \ u = \Delta_{\mathbb{G}}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

The solution of the above is discussed in distributional (weak) sense within the framework of the generalised Sobolev space $\mathbb{X}$ over Ω:

$$\mathbb{X} = \mathbb{X}(\Omega) = S^{2,p(x)}(\Omega) \times \mathring{S}^{1,p(x)}(\Omega).$$

Definition 3. Let $\lambda \in \mathbb{R}$ and $u \in \mathbb{X}$, the pair $(\lambda, u) \in \mathbb{R} \times \mathbb{X}$ is a weak (distributional) solution of (15) provided

$$\int_\Omega |\Delta_{\mathbb{G}}u|^{p(x)-2}\Delta_{\mathbb{G}}u\Delta_{\mathbb{G}}\psi dx = \lambda \int_\Omega |u|^{p(x)-2}u\psi dx, \tag{16}$$

for every $\psi \in C_0^\infty(\Omega)$. In the case u is nontrivial, such pair (λ, u) is called an eigenpair, where λ is an eigenvalue and u is an eigenvalue associated to λ.

Similarly, by the sup-solution of (15) we refer to the pair $(\lambda, u) \in \mathbb{R} \times \mathbb{X}$ satisfying

$$\int_\Omega |\Delta_{\mathbb{G}}u|^{p(x)-2}\Delta_{\mathbb{G}}u\Delta_{\mathbb{G}}\psi dx - \lambda \int_\Omega |u|^{p(x)-2}u\psi dx \geq 0, \tag{17}$$

for every $\psi \in C_0^\infty(\Omega)$. If the inequality in (17) is reversed then the pair (λ, u) is called sub-solution of (15).
Define the spectrum of (15) by the set

$$\mathscr{S}_p := \{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue}\},$$

and by $\lambda_\star$ the infimum of the spectrum $\mathscr{S}_p$, i.e. $\lambda_\star = \inf \mathscr{S}_p$. The principal eigenvalue λ_1 can be characterised by the mini-max procedure resulting into the Rayleigh-type quotient

$$\lambda_1(\Omega) = \inf_{0<u\in\mathbb{X}} \left\{ \frac{\int_\Omega |\Delta_{\mathbb{G}}u|^{p(x)}dx}{\int_\Omega u^{p(x)}dx}, \quad \mathbb{X} = S^{2,p(x)}(\Omega) \times \mathring{S}^{1,p(x)}(\Omega) \right\}.$$

Assume $\lambda_1 > 0$, one can arrange the eigenvalues (the elements of the set $\mathscr{S}_p$) in a nondecreasing unbounded sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \neq \infty \text{ as } n \neq \infty.$$

The above sequence assumes that the first eigenvalue is positive, which is not obvious as seen in the trend of research in this context [29,34,35]. However, sufficient conditions for $\lambda_1 > 0$ have been discussed in [11, 29,33,34,36,37]. Precisely, Ayoujili and Amrouss [34] have proved the existence of infinitely many eigenvalue sequences for (15). They also prove that if there exists an open subset $V \subset \Omega$ and a point $x_0 \in V$ such that $p(x_0) < p(x)$ for all $x \in \partial V$ then $\lambda_\star = \inf \mathscr{S}_p = 0$. Evidently, $\Omega \mapsto \lambda_1(\Omega)$ has the domain monotonicity property for the case $p(x) = p$ (a constant) in the sense that $\Omega' \subset \Omega$ implies $\lambda_1(\Omega') \geq \lambda_1(\Omega)$. Motivated by the above we will apply Picone identity (Theorem 1) to prove Theorem 2.

4. Picone-tye identities and applications

This section contains the proof of Theorem 1 and its applications to the study of nonlinear eigenvalue problems. First, we state Young's inequality in the form that will be applied here and later.

Lemma 1. *Let $p(x) > 1$ be a continuous function on $\bar{\Omega}$ and its Hölder conjugate be $p'(x) = p(x)/(p(x)-1)$. For two real numbers $s \geq 0$ and $t \geq 0$, there holds the inequality*

$$st \leq \frac{s^{p(x)}}{p(x)} + \frac{t^{p'(x)}}{p'(x)}, \tag{18}$$

with equality if and only if $s^{p(x)} = t^{p'(x)}$.

For simplicity sake and without any confusion we drop the subscript $\mathbb{G}$ from sub-gradient and sub-Laplacian operators and we write $\nabla := \nabla_{\mathbb{G}}$, $\Delta := \Delta_{\mathbb{G}}$ in the proof of Theorem 1. Also $p := p(x)$ should be understood as a variable exponent.

Poof of Theorem 1. We start by computing the term

$$\Delta\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right) = \nabla\cdot\nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right).$$

Since $p(x)$ varies, differentiating expressions like $|w|^p(x)$ produces additional terms. This follows from an elementary but a standard rule of differentiation of variable index functions:

$$\nabla|w|^{p(x)} = |w|^{p(x)}\ln|w|\nabla p(x) + p(x)u^{p(x)-2}w\nabla w. \tag{19}$$

Direct computation using the differentiation (19) yields (Note that here ∇ and Δ respectively denote $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$) for $u, v > 0$

$$\begin{aligned}\nabla\left(\frac{u^p}{v^{p-1}}\right) =& \frac{\nabla u^p}{v^{p-1}} - \frac{u^p\nabla v^{p-1}}{v^{2(p-1)}}\\ =& \frac{u^p\ln u\nabla p + pu^{p-1}\nabla u}{v^{p-1}} - \frac{u^p\ln v\nabla p}{v^{p-1}} - (p-1)\frac{u^p\nabla v}{v^p}\\ =& \frac{u^p}{v^{p-1}}\ln\left(\frac{u}{v}\right)\nabla p + \frac{pu^{p-1}\nabla u}{v^{p-1}} - (p-1)\frac{u^p\nabla v}{v^p}.\end{aligned}$$

Thus

$$\Delta\left(\frac{u^p}{v^{p-1}}\right) = \nabla\cdot\left(\frac{u^p}{v^{p-1}}\ln\left(\frac{u}{v}\right)\nabla p\right) + \nabla\cdot\left(\frac{pu^{p-1}\nabla u}{v^{p-1}}\right) - \nabla\cdot\left((p-1)\frac{u^p\nabla v}{v^p}\right). \tag{20}$$

The three terms on the RHS of the last equation are respectively evaluated as follows:

$$\begin{aligned}\nabla\cdot\left(\frac{u^p}{v^{p-1}}\ln\left(\frac{u}{v}\right)\nabla p\right) =& \frac{u^p}{v^{p-1}}\left|\ln\left(\frac{u}{v}\right)\right|^2|\nabla p|^2 + \frac{u^{p-1}}{v^{p-1}}\left(p\ln\left(\frac{u}{v}\right)+1\right)\nabla u\nabla p\\ &- \frac{u^p}{v^p}\left((p-1)\ln\left(\frac{u}{v}\right)+1\right)\nabla v\nabla p + \frac{u^p}{v^{p-1}}\ln\left(\frac{u}{v}\right)\Delta p.\end{aligned} \tag{21}$$

$$\begin{aligned}\nabla\cdot\left(\frac{pu^{p-1}\nabla u}{v^{p-1}}\right) =& \frac{u^{p-1}}{v^{p-1}}\nabla u\nabla p + p\frac{u^{p-1}}{v^{p-1}}\ln\left(\frac{u}{v}\right)\nabla u\nabla p + p(p-1)\frac{u^{p-2}}{v^{p-1}}|\nabla u|^2\\ &+ p\frac{u^{p-1}}{v^{p-1}}\Delta u - p(p-1)\frac{u^{p-1}}{v^p}\nabla u\nabla v,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\nabla\cdot\left((p-1)\frac{u^p\nabla v}{v^p}\right) =& \frac{u^p}{v^p}\nabla v\nabla p + (p-1)\frac{u^p}{v^p}\ln\left(\frac{u}{v}\right)\nabla v\nabla p + (p-1)\frac{u^p}{v^p}\Delta v\\ &+ p(p-1)\frac{u^{p-1}}{v^p}\nabla u\nabla v - p(p-1)\frac{u^p}{v^{p+1}}|\nabla v|^2.\end{aligned} \tag{23}$$

Substituting (21), (22) and (23) into (20) and simplify further we obtain the following expression

$$\begin{aligned}\Delta\left(\frac{u^p}{v^{p-1}}\right) =& \frac{u^p}{v^{p-1}}\left|\ln\left(\frac{u}{v}\right)\right|^2|\nabla p|^2 + 2\frac{u^{p-1}}{v^{p-1}}\left(p\ln\frac{u}{v}+1\right)\nabla u\nabla p\\ &- 2\frac{u^p}{v^p}\left((p-1)\ln\frac{u}{v}+1\right)\nabla v\nabla p + \frac{u^p}{v^{p-1}}\ln\frac{u}{v}\Delta p\end{aligned}$$

$$+ p(p-1)\frac{u^{p-2}}{v^{p-1}}|\nabla u|^2 + p\frac{u^{p-1}}{v^{p-1}}\Delta u - 2p(p-1)\frac{u^{p-1}}{v^p}\nabla u\nabla v + (p-1)\frac{u^p}{v^p}\Delta v + p(p-1)\frac{u^p}{v^{p+1}}|\nabla v|^2. \tag{24}$$

We notice that the 5th, 7th and 9th terms on RHS of (24) can be simplified when combined to get

$$p(p-1)\frac{u^{p-2}}{v^{p-1}}|\nabla u|^2 - 2p(p-1)\frac{u^{p-1}}{v^p}\nabla u\nabla + p(p-1)\frac{u^p}{v^{p+1}}|\nabla v|^2 = p(p-1)\frac{u^{p-2}}{v^{p-1}}\left(\nabla u - \frac{u}{v}\nabla v\right)^2. \tag{25}$$

Combining (24) and (25) and substituting the resulting expression into the expression of $\mathscr{R}(u,v)$ shows that $\mathscr{L}(u,v) = \mathscr{R}(u,v)$.

Next is to prove that $\mathscr{L}(u,v) \geq 0$. In order to show this we write the expression of $\mathscr{L}(u,v)$ as follows

$$\mathscr{L}(u,v) = \mathscr{L}_1(u,v) + \mathscr{L}_2(u,v) + \mathscr{L}_3(u,v) + \mathscr{L}_4(u,v) + \mathscr{L}_5(u,v),$$

where

$$\mathscr{L}_1(u,v) = |\Delta u|^p + (p-1)\frac{u^p}{v^p}|\Delta v|^p - p\frac{u^{p-1}}{v^{p-1}}|\Delta v|^{p-2}|\Delta v||\Delta u|,$$

$$\mathscr{L}_2(u,v) = p\frac{u^{p-1}}{v^{p-1}}|\Delta v|^{p-2}(|\Delta v||\Delta u| - \Delta v\Delta u),$$

$$\mathscr{L}_3(u,v) = -p(p-1)\frac{u^{p-2}}{v^{p-1}}\left(\nabla u - \frac{u}{v}\nabla v\right)^2|\Delta v|^{p-2}\Delta v,$$

$$\mathscr{L}_4(u,v) = 2\frac{u^p}{v^p}\left((p-1)\ln\frac{u}{v} + 1\right)\nabla v\nabla p\,|\Delta v|^{p-2}\Delta v - 2\frac{u^{p-1}}{v^{p-1}}\left(p\ln\frac{u}{v} + 1\right)\nabla u\nabla p\,|\Delta v|^{p-2}\Delta v,$$

$$\mathscr{L}_5(u,v) = -\frac{u^p}{v^{p-1}}\ln\frac{u}{v}\left(\ln\frac{u}{v}|\nabla p|^2 + \Delta p\right)|\Delta v|^{p-2}\Delta v.$$

First, rewriting $\mathscr{L}_1(u,v)$ we obtain

$$\mathscr{L}_1(u,v) = p\left(\frac{|\Delta u|^p}{p} + \frac{p-1}{p}\frac{u^p}{v^p}|\Delta v|^p\right) - p\frac{u^{p-1}}{v^{p-1}}|\Delta v|^{p-1}|\Delta u|.$$

Then applying the Young's inequality in the form (18) by choosing $a = |\Delta u|$ and $b = \left(\frac{u}{v}|\Delta v|\right)^{p-1}$ we can clearly see that $\mathscr{L}_1(u,v) \geq 0$ with equality if and only if $a^{p-1} = b$ which implies

$$\Delta u = \frac{u}{v}\Delta v. \tag{26}$$

Obviously, $\mathscr{L}_2(u,v) \geq 0$ by the virtue of the following inequality $\Delta u\Delta v - |\Delta u||\Delta v| \leq 0$ with equality if and only if

$$\Delta u\Delta v = |\Delta u||\Delta u|. \tag{27}$$

Since $p > 1$, $u \geq 0$ and $v > 0$ and $-\Delta v \geq 0$ a.e. in Ω we have that $\mathscr{L}_3(u,v) \geq 0$ with equality attained if and only if

$$\nabla u - \frac{u}{v}\nabla v = 0. \tag{28}$$

Lastly, considering the assumption that $\nabla g \nabla p \equiv 0$ for any differentiable function g a.e. in Ω we conclude that $\nabla u \nabla p \equiv 0$ and $\nabla v \nabla p \equiv 0$, and then $\mathscr{L}_4(u,v) \equiv 0$. Moreso, the assumption that $\ln \frac{u}{v}|\nabla p|^2 + \Delta p \equiv 0$ leads us to concluding that $\mathscr{L}_5(u,v) \equiv 0$.

Putting all of the above arguments together we therefore conclude that $\mathscr{L}(u,v) \geq 0$ a.e. in Ω with equality if and only if (26), (27) and (28) hold. Now solving (28) we get

$$\nabla\left(\frac{u}{v}\right) = 0. \tag{29}$$

If (29) holds then there exists some non-zero constant $\alpha \in \mathbb{R}$, say $\alpha > 0$, such that

$$u = \alpha v. \tag{30}$$

By (30) Eqs. (26) and (27) hold. We can therefore conclude that $\mathscr{L}(u,v) = 0$ a.e. in Ω if and only if $u = \alpha v$ for some constant $0 < \alpha \in \mathbb{R}$, $\nabla u \nabla p \equiv 0$, $\nabla v \nabla p \equiv 0$ and $\ln \frac{u}{v}|\nabla p|^2 + \Delta p \equiv 0$.

Indeed, if $\mathscr{L}(u,v)(x_0) = 0$ for $x_0 \in \Omega$, then there are two cases to consider. The case $u(x_0) \neq 0$ and the case $u(x_0) = 0$. If $u(x_0) \neq 0$, then $\mathscr{L}(u,v) = 0$ for all $x_0 \in \Omega$, meaning that $\mathscr{L}_k(u,v) = 0$, $k = 1,2,\cdots,5$ as defined above, and we conclude that (26), (27) and (28) hold which eventually implies $u = \alpha v$ a.e. for some constant $\alpha > 0$ for all $x_0 \in \Omega$. On the other hand, if $u(x_0) = 0$, we define a set $\Omega_0 = \{x \in \Omega : u(x) = 0\}$, and suppose $\Omega_0 \neq \Omega$. In this case $u(x_0) = \alpha v(x_0)$ implies $\alpha = 0$ since $u(x_0)$ and $v(x_0) > 0$. The case $u(x) \neq 0$ gives $u(x) = \alpha v(x)$ for all $x \in \Omega \setminus \Omega_0$, then it is impossible to have $\alpha = 0$. This contradiction implies that $\Omega_0 = \Omega$. □

Remark 3. If we allow $p(x) = p$ (a constant) in $\mathscr{R}(u,v)$ and $\mathscr{L}(u,v)$ in Theorem 1, we then recover Dwivedi [23, Lemma 2.2] in the Euclidean setting.

4.1. Some applications of Theorem 1

Here we present series of applications of Picone identity of Theorem 1 to the study of nonlinear eigenvalue problem (10). Let $\Omega \subseteq \mathbb{G}$ be a smooth bounded domain with smooth boundary. For $p \in C^2(\bar{\Omega})$ such that $p(x) > 1$, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the $p(x)$-biharmonic operator on $\bar{\Omega}$ in (10).

First we prove variable exponent Hardy-Rellich type inequality.

Proposition 2. *(Variable exponent Hardy-Rellich type inequality) Let there exists some λ and a strictly positive sup-solution $v \in \mathbb{X}$ of (10) such that $\nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p \equiv 0$. Then for any $0 \leq u \in \mathbb{X}$ satisfying $\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p \equiv 0$ and $\ln(u/v)|\nabla_{\mathbb{G}} p|^2 + \Delta_{\mathbb{G}} p \equiv 0$, it holds that*

$$\int_{\Omega} |\Delta_{\mathbb{G}} u|^{p(x)} dx \geq \lambda \int_{\Omega} |u|^{p(x)} dx,$$

and

$$\lambda_1(\Omega) > \lambda.$$

Proof. Since $0 < v \in \mathbb{X}$ is a sup-solution of (10), reverting to definition of weak sup-solution (17), and setting $\psi = \frac{u^{p(x)}}{v_\epsilon^{p(x)-1}}$, where $v_\epsilon := v + \epsilon$ for some $\epsilon > 0$, (Note that since $\psi \in C_0^\infty(\Omega)$, $\psi = \frac{u^{p(x)}}{v_\epsilon^{p(x)-1}} \in \mathbb{X}$ is compatible as a test function), we have

$$\lambda \int_{\Omega} |v|^{p(x)-2} v \frac{u^{p(x)}}{v_\epsilon^{p(x)-1}} dx \leq \int_{\Omega} |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v \Delta_{\mathbb{G}} \left(\frac{u^{p(x)}}{v_\epsilon^{p(x)-1}}\right) dx$$

$$
\begin{aligned}
&= \int_\Omega \left[|\Delta_{\mathbb{G}} u|^{p(x)} - \mathscr{R}(u, v_\epsilon) \right] dx \\
&= \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)} dx - \int_\Omega \mathscr{L}(u, v_\epsilon) dx.
\end{aligned}
$$

Taking the limit as $\epsilon \to 0$, applying Fatou's lemma and Lebesgue dominated convergence theorem respectively on the left hand side and right hand side of the last inequality yields

$$
\lambda \int_\Omega u^{p(x)} dx \leq \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)} dx - \int_\Omega \mathscr{L}(u, v) dx.
$$

This therefore implies

$$
0 \leq \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)} dx - \lambda \int_\Omega u^{p(x)} dx,
$$

since $\mathscr{L}(u, v) \geq 0$ almost everywhere in Ω.

Next let $u_1 \in \mathbb{X}$ be the eigenfunction corresponding to the principal eigenvalue $\lambda_1(\Omega)$ we have

$$
\int_\Omega |\Delta_{\mathbb{G}} u_1|^{p(x)-2} \Delta_{\mathbb{G}} u_1 \Delta_{\mathbb{G}} \psi dx = \lambda_1(\Omega) \int_\Omega |u_1|^{p(x)-2} u_1 \psi dx, \tag{31}
$$

for any $\psi \in C_0^\infty(\Omega)$. For $\epsilon > 0$ and Picone identity with u_1 and v where $v \in \mathbb{X}$ is a sup-solution, using $\psi = \frac{u_1^p(x)}{v_\epsilon^{p(x)-1}} \in \mathbb{X}$ in (17) we have

$$
\begin{aligned}
0 &\leq \int_\Omega \mathscr{L}(u_1, v_\epsilon) dx = \int_\Omega \mathscr{R}(u_1, v_\epsilon) dx \\
&= \int_\Omega |\Delta_{\mathbb{G}} u_1|^{p(x)} dx - \int_\Omega \Delta_{\mathbb{G}} \left(\frac{u_1^p(x)}{v_\epsilon^{p(x)-1}} \right) |\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v dx \\
&= \int_\Omega |\Delta_{\mathbb{G}} u_1|^{p(x)} dx - \int_\Omega \frac{u_1^p(x)}{v_\epsilon^{p(x)-1}} \Delta_{\mathbb{G}} \left(|\Delta_{\mathbb{G}} v|^{p(x)-2} \Delta_{\mathbb{G}} v \right) dx \\
&\leq \int_\Omega |\Delta_{\mathbb{G}} u_1|^{p(x)} dx - \lambda \int_\Omega \frac{u_1^p(x)}{v_\epsilon^{p(x)-1}} |v|^{p(x)-2} v dx.
\end{aligned}
$$

Setting $\psi = u_1 \in \mathbb{X}$ in (31), since $u_1 \in \mathbb{X}$ is the associated eigenfunction to $\lambda_1(\Omega)$ we arrive at

$$
0 \leq \lambda_1(\Omega) \int_\Omega |u_1|^{p(x)} dx - \lambda \int_\Omega \frac{u_1^p(x)}{v_\epsilon^{p(x)-1}} |v|^{p(x)-2} v dx.
$$

Applying Fatou's lemma and Lebesgue dominated convergence theorem after sending $\epsilon \to 0$ we obtain

$$
0 \leq [\lambda_1(\Omega) - \lambda] \int_\Omega u_1^{p(x)} dx,
$$

which implies $\lambda_1(\Omega) > \lambda$. $\square$

Secondly, we prove domain (eigenvalue) monotonicity.

Proposition 3. *(Domain monotonicity) Suppose $\Omega_1 \subset \Omega_2 \subseteq \Omega$ and $\Omega_1 \neq \Omega_2$. If both $\lambda_1(\Omega_1)$ and $\lambda_1(\Omega_2)$ exist, then*

$$
\lambda_1(\Omega_1) > \lambda_1(\Omega_2),
$$

provided $\nabla_{\mathbb{G}} u_1 \nabla_{\mathbb{G}} p(x) \equiv 0$, $\nabla_{\mathbb{G}} u_2 \nabla_{\mathbb{G}} p(x) \equiv 0$ and $\ln(u_1/u_2)|\nabla_{\mathbb{G}} p|^2 + \Delta_{\mathbb{G}} p \equiv 0$, where u_1 and u_2 are the eigenfunctions associated with $\lambda_1(\Omega_1)$ and $\lambda_1(\Omega_2)$, respectively.

Proof. Let u_1 and u_2 be positive eigenfunction associated with $\lambda_1(\Omega_1)$ and $\lambda_1(\Omega_2)$, respectively. Clearly, we have by Picone identity and integration by parts formula (13) that

$$\begin{aligned}
0 &\leq \int_{\Omega_1} \mathscr{L}(u_1,u_2)dx = \int_{\Omega_1} \mathscr{R}(u_1,u_2)dx \\
&= \int_{\Omega_1} |\Delta_{\mathbb{G}} u_1|^{p(x)}dx - \int_{\Omega_1} \Delta_{\mathbb{G}} \left(\frac{u_1^p(x)}{u_2^{p(x)-1}} \right) |\Delta_{\mathbb{G}} u_2|^{p(x)-2}\Delta_{\mathbb{G}} u_2 dx \\
&= \int_{\Omega_1} |\Delta_{\mathbb{G}} u_1|^{p(x)}dx - \int_{\Omega_1} \frac{u_1^p(x)}{u_2^{p(x)-1}} \Delta_{\mathbb{G}} \left(|\Delta_{\mathbb{G}} u_2|^{p(x)-2}\Delta_{\mathbb{G}} u_2 \right) dx \\
&= \int_{\Omega_1} |\Delta_{\mathbb{G}} u_1|^{p(x)}dx - \lambda_1(\Omega_2) \int_{\Omega_1} u_1^{p(x)} dx \\
&= [\lambda_1(\Omega_1) - \lambda_1(\Omega_2)] \int_{\Omega_1} u_1^{p(x)} dx.
\end{aligned}$$

This implies

$$\lambda_1(\Omega_1) - \lambda_1(\Omega_2) \geq 0.$$

Noting that if $\lambda_1(\Omega_1) = \lambda_1(\Omega_2)$, then $\mathscr{L}(u_1,u_2) = 0$ almost everywhere in Ω_1 and thus $u_1 = \alpha u_2$ for some constant $\alpha > 0$. This is impossible since $\Omega_1 \neq \Omega_2$. □

Next we prove a variable exponent Barta type inequality. For various Barta type inequalities see [3] for p-Laplacian (p constant) and [44] for $p(x)$-Laplacian. This type of inequality was first derived by Barta in [59] and generalised by Protter and Weinbeger [60]. This result is new for $p(x)$-biharmonic operator even for p constant.

Proposition 4. *(Barta type inequality) Suppose $u \in \mathbb{X}$ be a positive solution to (10). Then for any differentiable function $v > 0$ in $\bar{\Omega}$ with $-\Delta_{\mathbb{G}} v \geq 0$, $\Delta_{\mathbb{G}} \left(|\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v \right) \in C(\bar{\Omega})$, $\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p(x) \equiv 0$, $\nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p(x) \equiv 0$ and $\ln(u/v)|\nabla_{\mathbb{G}} p|^2 + \Delta_{\mathbb{G}} p \equiv 0$ for $p(x) > 1$, we have*

$$\lambda_1 \geq \inf_{x\in\Omega} \frac{\Delta_{\mathbb{G}} \left(|\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v \right)}{v^{p(x)-1}}.$$

Equality holds if and only if v is a constant multiple of eigenfunction corresponding to λ_1.

Proof. Note that the assumption on u and v make them admissible for Picone identity. Thus by Picone identity and integration by parts formula (13)

$$\begin{aligned}
0 &\leq \int_{\Omega} \mathscr{L}(u,v)dx = \int_{\Omega} \mathscr{R}(u,v)dx \\
&= \int_{\Omega} |\Delta_{\mathbb{G}} u|^{p(x)}dx - \int_{\Omega} \Delta_{\mathbb{G}} \left(\frac{u^{p(x)}}{v^{p(x)-1}} \right) |\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v dx \\
&= \int_{\Omega} |\Delta_{\mathbb{G}} u|^{p(x)}dx - \int_{\Omega} \frac{u^{p(x)}}{v^{p(x)-1}} \Delta_{\mathbb{G}} \left(|\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v \right) dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\Omega} |\Delta_{\mathbb{G}} u|^{p(x)}dx &\geq \int_{\Omega} u^{p(x)} \frac{\Delta_{\mathbb{G}} \left(|\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v \right)}{v^{p(x)-1}} dx \\
&\geq \inf_{x\in\Omega} \left[\frac{\Delta_{\mathbb{G}} \left(|\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v \right)}{v^{p(x)-1}} \right] \int_{\Omega} u^{p(x)} dx.
\end{aligned}$$

We therefore conclude that

$$\frac{\int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)}dx}{\int_\Omega u^{p(x)}dx} \geq \inf_{x\in\Omega}\left[\frac{\Delta_{\mathbb{G}}\left(|\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v\right)}{v^{p(x)-1}}\right],$$

which proves the desired result. □

Now, we discuss the sign changing nature of eigenfunctions.

Proposition 5. *Any eigenfunction v corresponding to an eigenvalue $\lambda \neq \lambda_1$ changes sign provided $\nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p(x) = 0$, $\nabla_{\mathbb{G}} u_1 \nabla_{\mathbb{G}} p(x) = 0$ and $\ln(u_1/v)|\nabla_{\mathbb{G}} p|^2 + \Delta_{\mathbb{G}} p \equiv 0$ for $p(x) > 1$, where u_1 is the eigenfunction associated with $\lambda_1(\Omega)$.*

Proof. Suppose that $v > 0$ does not change sign (the case $v \leq 0$ can be proved similarly) with $\nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p(x) \equiv 0$. Let $u_1 > 0$ be the eigenfunction associated with $\lambda_1(\Omega)$ satisfying $\nabla_{\mathbb{G}} u_1 \nabla_{\mathbb{G}} p(x) \equiv 0$. For any $\epsilon > 0$ we have

$$\begin{aligned}0 \leq \int_\Omega \mathscr{L}(u_1, v_\epsilon)dx &= \int_\Omega \mathscr{R}(u_1, v_\epsilon)dx\\ &= \int_\Omega |\Delta_{\mathbb{G}} u_1|^{p(x)}dx - \int_\Omega \Delta_{\mathbb{G}}\left(\frac{u_1^{p(x)}}{v_\epsilon^{p(x)-1}}\right)|\Delta_{\mathbb{G}} v|^{p(x)-2}\Delta_{\mathbb{G}} v dx\\ &=\lambda_1(\Omega)\int_\Omega u_1^{p(x)}dx - \lambda\int_\Omega \frac{u_1^{p(x)}}{v_\epsilon^{p(x)-1}}|v|^{p(x)-2}v dx.\end{aligned}$$

Sending $\epsilon \to 0$, applying Fatou's Lemma as usual we have

$$0 \leq [\lambda_1(\Omega) - \lambda]\int_\Omega u_1^{p(x)}dx,$$

which is a contradiction since $\int_\Omega u_1^{p(x)}dx > 0$. Thus v must change sign. □

5. Díaz-Saa-type inequality and applications

In this section we present a generalised Díaz-Saa type inequality for the $p(x)$-biharmonic operator on homogeneous Carnot groups. As an application, the uniqueness of solution of the positive solution to Navier boundary value problem is proved.

Lemma 2. *Let Ω be a bounded domain in $\mathbb{G}$. Let $0 \leq u \in \mathbb{X}$ and $v \in \mathbb{X}$ be such that $v \geq \delta > 0$, $-\Delta_{\mathbb{G}} v \geq 0$, $\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p = 0$, $\nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p = 0$ and $\ln(u/v)|\nabla_{\mathbb{G}} p|^2 + \Delta_{\mathbb{G}} p \equiv 0$ a.e. in Ω.Then for all $p > 1$ we have*

$$\int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}}\Delta^2_{\mathbb{G},p} v dx \leq \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)}dx. \tag{32}$$

Proof. Be density argument we can choose $v_k \in \mathbb{X}$, $k = 1, 2, \cdots$, such that $v_k > \delta/2$ in Ω, $v_k \to v$ in $\mathbb{X}$ and $\Delta_{\mathbb{G}} v_k \to \Delta_{\mathbb{G}} v$ a.e. and $\nabla_{\mathbb{G}} v_k \nabla_{\mathbb{G}} p = 0$. Considering also $0 \leq u \in \mathbb{X}$ and choosing u_n such that $u_n \geq 0$ for each n, and $u_n \to u$ and $\nabla_{\mathbb{G}} u_n \nabla_{\mathbb{G}} p = 0$ in $\mathbb{X}$. By the application of Picone identity (Theorem 1) we have

$$\int_\Omega L(u_n, v_k)dx \geq 0,$$

for each n and each k. This thus implies

$$\int_\Omega \frac{u_n^{p(x)}}{v_k^{p(x)-1}}\Delta^2_{\mathbb{G},p} v_k dx \leq \int_\Omega |\Delta_{\mathbb{G}} u_n|^{p(x)}dx.$$

Therefore, by Fatou's Lemma and Lebesgue dominated convergence theorem, we have

$$\int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}} \Delta^2_{\mathbb{G},p} v dx \leq \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)} dx.$$

This completes the proof. □

By applying the above Lemma we establish the following generalised Picone-type inequality.

Theorem 4. *Let $\Omega \subset \mathbb{G}$ be a bounded open domain and let $g(x,s) \leq A(s^{p(x)-1}+1)$ for constant $A > 0$ and all $s \geq 0$. If the functions $u, v \in \mathbb{X}$ with $v > 0$ ($\not\equiv 0$) a.e. in Ω satisfying the assumption of Lemma 2 are such that*

$$\Delta^2_{\mathbb{G},p(x)} v = g(x,v).$$

Then

$$\int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}} \Delta^2_{\mathbb{G},p} v dx \leq \int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)} dx,$$

for all $p(x) > 1$.

Proof. The proof follows easily from Lemma 2. □

As a consequence of the above generalised Picone inequality we obtain the generalised Díaz-Saa-type inequality on stratified Lie groups (Proposition 1).

Proof of Proposition 1. Let $u_i \geq 0$ ($u_i \not\equiv 0$) for $i = 1, 2$ satisfy the hypothesis of the theorem. Then by Picone-type identitty

$$\int_\Omega L(u_1, u_2) dx \geq 0,$$

and we have

$$\int_\Omega \frac{u_1^{p(x)}}{u_2^{p(x)-1}} \Delta^2_{\mathbb{G},p(x)} u_2 dx \leq \int_\Omega |\Delta_{\mathbb{G}} u_1|^{p(x)} dx.$$

Applying integration by parts formula (see (13)) on the right hand side of the last inequality we have

$$0 \leq \int_\Omega \left(\frac{\Delta^2_{\mathbb{G},p(x)} u_1}{u_1^{p(x)-1}} - \frac{\Delta^2_{\mathbb{G},p(x)} u_2}{u_2^{p(x)-1}} \right) u_1^{p(x)} dx. \tag{33}$$

Applying the Picone-type identity again by interchanging the position of u_1 and u_2 we obtain

$$\int_\Omega L(u_2, u_1) dx \geq 0,$$

implying

$$\int_\Omega \frac{u_2^{p(x)}}{u_1^{p(x)-1}} \Delta^2_{\mathbb{G},p(x)} u_1 dx \leq \int_\Omega |\Delta_{\mathbb{G}} u_2|^{p(x)} dx,$$

and then

$$0 \leq \int_\Omega \left(\frac{\Delta^2_{\mathbb{G},p(x)} u_2}{u_2^{p(x)-1}} - \frac{\Delta^2_{\mathbb{G},p(x)} u_1}{u_1^{p(x)-1}} \right) u_2^{p(x)} dx. \tag{34}$$

Adding (33) and (34) we have

$$0 \leq \int_\Omega \left(\frac{\Delta^2_{\mathbb{G},p(x)} u_1}{u_1^{p(x)-1}} - \frac{\Delta^2_{\mathbb{G},p(x)} u_2}{u_2^{p(x)-1}} \right) (u_1^p - u_2^p) dx,$$

which is the desired inequality. □

5.1. Applications of Díaz-Saa type inequality

The application of Díaz-Saa-type inequality that will be presented in the next is on the uniqueness of solution to the following Navier boundary value problem on a stratified Lie group.

Let $\Omega \subseteq \mathbb{G}$ be a smooth bounded domain with smooth boundary. For $p \in C^2(\bar{\Omega})$ such that $p(x) > 1$, we consider

$$\begin{aligned} &\Delta^2_{\mathbb{G},p(x)} u = g(x,u), \quad x \in \Omega \subset \mathbb{G}, \\ &u > 0, \qquad\qquad\quad x \in \Omega \subset \mathbb{G}, \\ &u = \Delta_{\mathbb{G}} u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{35}$$

with the following hypothesis on the positive function $g : \bar{\Omega} \times [0,\infty) \to \mathbb{R}^+$:

H1: The function $s \mapsto g(x,s)$ is continuous on $[0,\infty)$ for a.e. $x \in \Omega$ and for all $s \geq 0$. The function $x \mapsto g(x,s)$ is bounded and measurable.

H2: The function $s \mapsto \dfrac{g(x,s)}{s^{p(x)-1}}$, $p(x) \in (1,\infty)$ is strictly decreasing in $(0,\infty)$.

H3: There exists $A > 0$ such that $g(x,s) \leq A(s^{p(x)-1} + 1)$ for a.e. $x \in \Omega$ and for all $s \geq 0$.

Definition 4. A nonnegative function $u \in \mathbb{X} \cap L^\infty(\Omega)$ is said to be a nonnegative weak solution of (35) if

$$\int_\Omega |\Delta_{\mathbb{G}} u|^{p(x)-2} \Delta_{\mathbb{G}} u \Delta_{\mathbb{G}} \phi = \int_\Omega g(x,u) \phi dx, \tag{36}$$

holds for every $\phi \in \mathbb{X}$. If $u > 0$ throughout Ω we say u is a positive weak solution.

Theorem 5 (Uniqueness of a solution)**.** *There exists at most one positive solution u to (35) in $\mathbb{X} \cap L^\infty(\Omega)$ for $1 < p(x) \leq Q$, where $p \in C^2(\bar{\Omega})$, $\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p = 0$ and $\ln u |\nabla_{\mathbb{G}} p|^2 + \Delta_{\mathbb{G}} p \equiv 0$ a.e. in Ω.*

Proof. Suppose u_1 and u_2 are two different positive solution of (35), $\nabla_{\mathbb{G}} u_l \nabla_{\mathbb{G}} p = 0$, and $\ln u_l |\nabla p|^2 + \Delta_{\mathbb{G}} p \equiv 0$, $l = 1,2$, a.e. in Ω. First, we observe that Díaz-Saa-type inequality (11) (Proposition 1) implies

$$0 \leq \int_\Omega \left(\frac{\Delta^2_{\mathbb{G},p(x)} u_1}{u_1^{p(x)-1}} - \frac{\Delta^2_{\mathbb{G},p(x)} u_2}{u_2^{p(x)-1}} \right) (u_1^p - u_2^p) dx.$$

On the other hand, with the assumption that $g(x,s) > 0$ and that $s \mapsto \dfrac{g(x,s)}{s^{p(x)-1}}$ is strictly decreasing on $(0,\infty)$ (since $g(x,s)$ satisfies Assumptions H1 - H3), it follows that

$$\int_\Omega \left(\frac{g(x,u_1)}{u_1^{p(x)-1}} - \frac{g(x,u_2)}{u_2^{p(x)-1}} \right) (u_1^{p(x)} - u_2^{p(x)}) dx < 0.$$

Clearly we see that

$$\begin{aligned} 0 &\leq \int_\Omega \left(\frac{\Delta^2_{\mathbb{G},p(x)} u_1}{u_1^{p(x)-1}} - \frac{\Delta^2_{\mathbb{G},p(x)} u_2}{u_2^{p(x)-1}} \right) (u_1^{p(x)} - u_2^{p(x)}) dx \\ &= \int_\Omega \left(\frac{g(x,u_1)}{u_1^{p(x)-1}} - \frac{g(x,u_2)}{u_2^{p(x)-1}} \right) (u_1^{p(x)} - u_2^{p(x)}) dx < 0, \end{aligned}$$

which is a contradiction to the assumption that $u_1 \neq u_2$. $\square$

The following weak comparison principle for sub-sup solutions is a direct consequence of the above proof (we leave detail for the readers).

Proposition 6. *Suppose $u, v \in \mathbb{X} \cap L^\infty(\Omega)$ are positive function satisfying $\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} p = 0$, $\nabla_{\mathbb{G}} v \nabla_{\mathbb{G}} p = 0$ and $\ln(u/v)|\nabla p|^2 + \Delta_{\mathbb{G}} p \equiv 0$ a.e. in Ω*

$$\begin{cases} \Delta^2_{\mathbb{G},p(x)} u \geq g(x,u), & x \in \Omega \\ \Delta^2_{\mathbb{G},p(x)} v \leq g(x,v), & x \in \Omega \\ u > 0, v > 0, & x \in \Omega \\ u = v = \Delta_{\mathbb{G}} u = \Delta_{\mathbb{G}} v = 0, & x \in \partial\Omega, \end{cases}$$

where $g(x,s) > 0$ is a bounded and measurable function such that $\dfrac{g(x,s)}{s^{p(x)-1}}$, $1 < p(x) \leq Q$, is strictly increasing in $(0,\infty)$ for a.e. $x \in \Omega$. Then $v \leq u$ a.e. in Ω.

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