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PORP graph of a Dihedral group

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Abstract: Let G be a finite group. The product order relatively prime graph (PORP graph) $\Gamma_{porp}(G)$ is the graph with vertex set $V(\Gamma_{porp}(G)) = G$ and two distinct vertices a and b are adjacent if $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$. In this paper, we investigate the product order relatively prime graph associated with the dihedral group D_n. We study the graph for two important classes of dihedral groups, namely D_{p^α}, where p is a prime and $\alpha \geq 1$, and D_{pq}, where p and q are distinct odd primes. We determine the graph structure and establish several structural properties, including the degree sequence, total number of edges, independence number, clique number and structural decomposition. Furthermore, we investigate graph-theoretic properties such as planarity, bipartiteness, Eulericity, Hamiltonicity, and characterize the conditions under which the graph is complete.

Keywords: product order relatively prime graph; dihedral group; planar graph; bipartite graph

MSC: 05C38, 05C78.

1. Introduction

The interaction between algebra and graph theory has led to the development of several algebraically defined graphs associated with finite groups. In recent years, many researchers have studied graphs constructed from algebraic structures and arithmetic properties of groups. These investigations establish interesting connections between group theoretic concepts and graph theoretic parameters such as connectivity, clique number, chromatic number, spectra, Hamiltonicity and planarity.

Power graphs and their variants form one of the most active research areas in algebraic graph theory. Cameron, Guerra and Jurina studied the power graph of torsion-free groups and investigated several structural properties [1]. Cameron and Jafari discussed the connectivity and independence number of power graphs of groups [2]. Bera *et al.* studied the connectivity of enhanced power graphs of finite groups [3]. Panda *et al.* further investigated enhanced power graphs and obtained several algebraic and graph theoretic results [4]. Ma examined proper connection properties of power graphs of finite groups [5]. Banerjee and Adhikari studied the signless Laplacian spectrum of power graphs of finite cyclic groups [6].

Another important research direction concerns graphs defined using coprimality conditions on element orders. Adhikari and Banerjee introduced the prime coprime graph of a finite group and studied several graph theoretic parameters [7]. Hamm and Way investigated parameters of coprime graphs of groups [8]. Nurhabibah *et al.* studied numerical invariants of coprime graphs associated with generalized quaternion groups [9]. Recently, Romdhini et al. [10] studied the spectral properties of coprime graphs of dihedral groups by determining their adjacency, Laplacian, and signless Laplacian spectra. This work illustrates the importance of spectral techniques in understanding graphs associated with finite groups. Mohamed *et al.* analyzed characteristics of enhanced power coprime graphs of some finite groups [11]. Bello *et al.* introduced commuting order product prime graphs and obtained several structural properties [12]. Ejima, Aremu, Yusuf investigated some algebraic properties and combinatorial structures of the order divisor-power graph $\Gamma_{odp}(G)$ and obtain the conditions under which the order divisor-power graph $\Gamma_{odp}(G)$ can be a star graph and also they exhibit some connection between the order divisor-power graph and the power graph of dihedral groups up to an isomorphism. Furthermore, we prove that the order divisor-power graphs of some classes of dihedral groups are neither bipartite nor tripartite, but it is a complete multipartite graph if the group is a cyclic group [13].

Graphs related to divisibility of element orders have also attracted considerable attention. Hamzeh and Ashrafi introduced the order supergraph of the power graph of a finite group [14]. Zhai and Ma studied perfect codes in proper order divisor graphs of finite groups [15]. The basic concepts and terminologies of graph theory used throughout this paper are taken from Harary [16], while the algebraic preliminaries are referred from Herstein [17]. Furthermore, the recent survey on divisor graphs and divisor function graphs given by Ravi and Desikan [18] provides useful background for the present work. The study of spectral properties and structural invariants of algebraically defined graphs has become an important area of research in modern algebraic graph theory [6,19,20].

Recently, Ponraj and Sutharson introduced the product order divisor graph of a finite group, denoted by $\Gamma_{pod}(G)$, whose vertex set is G and two distinct vertices a and b are adjacent whenever $o(a) \mid o(ab)$ or $o(b) \mid o(ab)$ [21]. This graph provides a natural extension of divisor type graphs associated with finite groups and motivates the study of graphs defined through arithmetic conditions involving products of group elements.

Inspired by these developments, the product order relatively prime (PORP) graph was introduced by the authors in [22], where its definition and fundamental properties were established. However, a systematic investigation of the PORP graph associated with dihedral groups has not been carried out. Since the algebraic structure of the dihedral group D_n depends strongly on the prime factorization of n, it is natural to investigate how this arithmetic structure influences the corresponding PORP graph.

The primary objective of this paper is to study the PORP graph of the dihedral group D_n. We consider two important families, namely D_{p^α}, where p is a prime and $\alpha \geq 1$, and D_{pq}, where p and q are distinct odd primes. These families represent the fundamental prime-power and two-prime cases and provide the first step towards understanding the PORP graph of D_n for arbitrary positive integers n. For these classes of dihedral groups, we determine the graph structure, degree sequence, number of edges, independence number, clique number and several structural properties, including planarity, bipartiteness, Eulericity, and Hamiltonicity. The results obtained in this paper extend the existing theory of PORP graphs and demonstrate how the prime factorization of n influences the structure of $\Gamma_{porp}(D_n)$.

2. Preliminaries

Definition 1. [16] Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two simple graphs with $V_1 \cap V_2 = \emptyset$. The join of Γ_1 and Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$, is the graph obtained from the disjoint union of Γ_1 and Γ_2 by joining every vertex of Γ_1 to every vertex of Γ_2.

Definition 2. [16] Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs. The union of Γ_1 and Γ_2, denoted by $\Gamma_1 \cup \Gamma_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$. That is, $\Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2,\ E_1 \cup E_2)$.

Definition 3. [17] Let G be a group and let $a \in G$. The *order* of the element a, denoted by $o(a)$, is the least positive integer n such that $a^n = e$, where e is the identity element of G.

Definition 4. [17] The dihedral group of order $2n$, denoted by D_n, is defined as $D_n = < r, s : r^n = e, s^2 = e, rs = sr^{-1} >$. The elements of D_n are given by $D_n = \{e, r, r^2, \cdots, r^{n-1}, s, sr, sr^2, \cdots, sr^{n-1}\}$. We denote $a_i = r^i, 0 \leq i \leq n-1$ and $b_j = sr^j, 0 \leq j \leq n-1$.

Definition 5. [17] The greatest common divisor of two integers a and b is denoted by (a, b). Two positive integers a and b are said to be relatively prime (or coprime) if $(a, b) = 1$.

3. Product order relatively prime graph of a group

Definition 6. Let G be a finite group. The product order relatively prime graph (Simply called PORP graph) $\Gamma_{porp}(G)$ is a graph with $V(\Gamma_{porp}(G)) = G$ and two distinct vertices a and b are adjacent in $\Gamma_{porp}(G)$ if either $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$.

Remark 1. The adjacency relation is symmetric. Indeed, $ba = a^{-1}(ab)a$. Hence ab and ba are conjugate elements of G, and therefore $o(ab) = o(ba)$. Thus, $(o(a), o(ab)) = 1$ if and only if $(o(a), o(ba)) = 1$, and similarly for b. Therefore the adjacency relation is symmetric and hence $\Gamma_{porp}(G)$ is a simple undirected graph.

Example 1. $\Gamma_{porp}(D_8)$ is given in Figure 1.

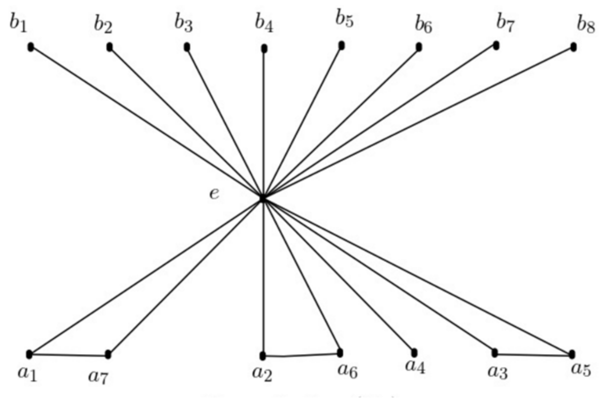

Figure 1. $\Gamma_{porp}(D_8)$

4. Dihedral group of order $2p^\alpha$ (p is prime and $\alpha \geq 1$)

In this section, we consider the dihedral group D_{p^α} which has order $2p^\alpha$ where p is prime and $\alpha \geq 1$.

Theorem 1. *The product order relatively prime graph $\Gamma_{porp}(D_{p^\alpha})$ is complete if and only if $p = 3$ and $\alpha = 1$.*

Proof. Suppose $p^\alpha = 3$. Then $D_3 = \{e, a_1, a_2, b, b_1, b_2\}$. The non-identity rotation elements have order 3 and the reflection elements have order 2. Clearly the identity element is adjacent to every vertex. Further, $aa_2 = e$, so a and a_2 are adjacent. Consider the reflection elements b_i and b_j. Now $(b_i)(b_j) = a_{j-i}$. Therefore $o((b_i)(b_j)) = 3$. Hence every pair of reflection vertices are adjacent. Also, $a_i(b_j) = b_{j-i}$. Clearly $o(a_i(b_j)) = 2$. Hence every rotation vertices are adjacent to every reflection vertices. Therefore every pair of distinct vertices are adjacent, and $\Gamma_{porp}(D_3) \cong K_6$.

Conversely suppose $p^\alpha > 3$. We observe that $o(a_{2^{\alpha-1}}) = 2$, when $p = 2$. Choose two non-identity rotation vertices a_i and a_j such that $i, j \neq 2^{\alpha-1}$, $o(a_i) = p$ and $o(a_j) = p^2$ such vertices are exist, since order of an element of a finite group divides the order of the group. Let $o(a_i a_j) = p^s$ for some s. Since $(o(a_i), o(a_i a_j)) = (p, p^s) > 1$. Therefore a_i and a_j are not adjacent. □

Theorem 2.

$$\Gamma_{porp}(D_{p^\alpha}) \cong \begin{cases} K_1 \vee \left(\frac{p^\alpha-1}{2} K_2 \vee K_{p^\alpha}\right), & p \text{ is odd prime}, \alpha \geq 1, \\ K_1 \vee \left((2^\alpha+1)K_1 \cup (2^{\alpha-1}-1)K_2\right), & p = 2, \alpha \geq 2. \end{cases}$$

Proof. The identity vertex is adjacent to every other vertex, since $(o(e), o(x)) = 1$ for all $x \in V(D_{p^\alpha})$. Consider rotation vertices a_i and a_j. Now $a_i a_j = a_{i+j}$. Since all non-identity rotation orders are powers of p, adjacency occurs only when $a_{i+j} = e$. Hence each non-identity rotation vertex is adjacent exactly to its inverse vertex.

Suppose $p > 2$. Then every non-identity rotation has order p. Consider a rotation element a_i and a reflection element b_j. Now $a_i(b_j) = b_{j-i}$, which has order 2. Hence every non-identity rotation vertex is adjacent to every reflection vertices. Consider two reflection elements (b_i) and (b_j). Now $(b_i)(b_j) = a_{j-i}$, $o((b_i)(b_j)) = p^s$ for some s. Hence all distinct reflection vertices are adjacent. Therefore

$$\Gamma_{porp}(D_{p^\alpha}) \cong K_1 \vee \left(\frac{p^\alpha - 1}{2} K_2 \vee K_{p^\alpha}\right).$$

Now suppose $p = 2$. Every non-identity rotation element has even order. Thus no non-identity rotation vertices are adjacent to a reflection vertices and no two distinct reflection vertices are adjacent. Among rotation vertices, adjacency occurs only between its inverse. Therefore $\Gamma_{porp}(D_{2^\alpha}) \cong K_1 \vee \big((2^\alpha+1)K_1 \cup (2^{\alpha-1}-1)K_2\big)$. $\square$

Theorem 3. *The degree sequence of* $\Gamma_{porp}(D_{p^\alpha})$ *is*

$$\begin{cases} (\underbrace{2p^\alpha-1, 2p^\alpha-1, \cdots, 2p^\alpha-1}_{(p^\alpha+1)times}, \underbrace{p^\alpha+2, p^\alpha+2, \cdots, p^\alpha+2}_{(p^\alpha-1)times}), & p \text{ is odd prime}, \alpha \geq 1, \\ (2^{\alpha+1}-1, \underbrace{2, 2, \cdots, 2}_{(2^\alpha-2)times}, \underbrace{1, 1, \cdots, 1}_{(2^\alpha+1)times}), & p = 2, \alpha \geq 2. \end{cases}$$

Proof. Suppose p is odd prime. The identity vertex is adjacent to every other vertices. Hence $d(e) = 2p^\alpha - 1$. Each non-identity rotation vertices are adjacent to

- the identity vertex,
- its inverse rotation vertices,
- all p^α reflection vertices.

Therefore $d(a_i) = 1 + 1 + p^\alpha = p^\alpha + 2$.
Each reflection vertices are adjacent to

- the identity vertex,
- all other reflection vertices,
- all non-identity rotation vertices.

Thus $d(b_j) = 2p^\alpha - 1$. Hence the degree sequence is

$$(\underbrace{2p^\alpha-1, 2p^\alpha-1, \cdots, 2p^\alpha-1}_{(p^\alpha+1)\text{times}}, \underbrace{p^\alpha+2, p^\alpha+2, \cdots, p^\alpha+2}_{(p^\alpha-1)\text{times}}).$$

Now suppose $p = 2$. The identity vertex is adjacent to all other vertices. Hence $d(e) = 2^{\alpha+1} - 1$. Every non-identity rotation vertices except $a_{2^{\alpha-1}}$ are adjacent to its inverse vertex and the identity vertex. Hence $d(a_i) = 2$, where $i \neq 2^{\alpha-1}$. The element $a_{2^{\alpha-1}}$ is self-inverse, so it is adjacent only to the identity. Thus $d(a_{2^{\alpha-1}}) = 1$. Every reflection vertices are adjacent only to the identity vertex. Hence $d(b_j) = 1$ for all j. Therefore the degree sequence is $(2^{\alpha+1}-1, \underbrace{2, 2, \cdots, 2}_{(2^\alpha-2)\text{times}}, \underbrace{1, 1, \cdots, 1}_{(2^\alpha+1)\text{times}})$. $\square$

Theorem 4. *The size of* $\Gamma_{porp}(D_{p^\alpha})$ *is*

$$|E(\Gamma_{porp}(D_{p^\alpha}))| = \begin{cases} \dfrac{3p^{2\alpha} + 2p^\alpha - 3}{2}, & p \text{ is odd prime}, \\ 5 \cdot 2^{\alpha-1} - 2, & p = 2. \end{cases}$$

Proof. Suppose p is odd prime. The identity vertex contributes $2p^\alpha - 1$ edges. Among the non-identity rotation vertices, each vertex is adjacent only to its inverse. Hence the number of edges among rotations are $\frac{p^\alpha-1}{2}$. The reflection vertices form the complete graph K_p^α. Hence the number of edges among reflection vertices are $\binom{p^\alpha}{2} = \frac{p^\alpha(p^\alpha-1)}{2}$. Each of the $p^\alpha - 1$ non-identity rotation vertices are adjacent to all p^α reflection vertices. Therefore the number of edges between rotation vertices and reflection vertices are $p^\alpha(p^\alpha - 1)$. Thus

$$|E(\Gamma_{porp}(D_{p^\alpha}))| = \frac{3p^{2\alpha} + 2p^\alpha - 3}{2}.$$

Now suppose $p = 2$. The identity vertex is adjacent to all other vertices, contributing $2^{\alpha+1} - 1$ edges. Among the non-identity rotation vertices, adjacency occurs only between inverse pairs. Hence the additional edges are $2^{\alpha-1} - 1$. Therefore $|E(\Gamma_{porp}(D_{2^\alpha}))| = (2^{\alpha+1} - 1) + (2^{\alpha-1} - 1) = 5 \cdot 2^{\alpha-1} - 2$. $\square$

Theorem 5. *1. $\Gamma_{porp}(D_{p^\alpha})$ is planar if and only if $p = 2$.*
2. $\Gamma_{porp}(D_{p^\alpha})$ is not Eulerian.
3. $\Gamma_{porp}(D_{p^\alpha})$ is not bipartite.
4. $\Gamma_{porp}(D_{p^\alpha})$ is Hamiltonian if and only if p is odd prime.

Proof. 1. For $p \geq 3$ and $\alpha \geq 1$, K_5 is a subgraph of $\Gamma_{porp}(D_{p^\alpha})$ and hence $\Gamma_{porp}(D_{p^\alpha})$ is not planar.
For $p = 2$ and $\alpha \geq 2$ $\Gamma_{porp}(D_{2^\alpha}) \cong K_1 \vee \big((2^\alpha+1)K_1 \cup (2^{\alpha-1}-1)K_2\big)$ and hence it is planar.
2. Suppose that p is odd prime and $\alpha \geq 1$. From Theorem 3, the identity vertex has degree $2p^\alpha - 1$, which is odd. Hence the graph contains a vertex of odd degree. Therefore, $\Gamma_{porp}(D_{p^\alpha})$i s not Eulerian. In the case of $p = 2$ and $\alpha \geq 2$, $\Gamma_{porp}(D_{2^\alpha})$ has a pendent vertex and hence it is not Eulerian.
3. Since the graph contains a triangle consisting of the identity vertex and the two endpoints of any K_2 component, the graph contains a cycle of length 3. Hence $\Gamma_{porp}(D_{p^\alpha})$ is not bipartite, where p is prime.
4. For p is odd prime, since $|V| = 2p^\alpha$ and $\delta(\Gamma_{porp}(D_{p^\alpha})) = p^\alpha + 2$, we have $\delta(\Gamma_{porp}(D_{p^\alpha})) = p^\alpha + 2 \geq \frac{|V|}{2} = p^\alpha$. Therefore, by Dirac's theorem, $\Gamma_{porp}(D_{p^\alpha})$ is Hamiltonian. For $p = 2$, $\delta(\Gamma_{porp}(D_{2^\alpha})) = 1$. A Hamiltonian graph has no vertex of degree 1, since every vertex on a Hamiltonian cycle must be incident with two edges of the cycle. Therefore, $\Gamma_{porp}(D_{2^\alpha})$ is not Hamiltonian.
□

Theorem 6. *The independent number of $\Gamma_{porp}(D_{p^\alpha})$,*

$$\alpha(\Gamma_{porp}(D_{p^\alpha})) = \begin{cases} \dfrac{p^\alpha - 1}{2}, & p \text{ is odd prime}, \alpha \geq 1, \\ 3 \cdot 2^{\alpha-1}, & p = 2, \alpha \geq 2, \end{cases}$$

and the clique number of $\Gamma_{porp}(D_{p^\alpha})$,

$$\omega(\Gamma_{porp}(D_{p^\alpha})) = \begin{cases} p^\alpha + 3, & p \text{ is odd prime}, \alpha \geq 1, \\ 3, & p = 2, \alpha \geq 2. \end{cases}$$

Proof. *Case 1:* p is odd prime.

By Theorem 2, $\Gamma_{porp}(D_{p^\alpha}) \cong K_1 \vee \left(\frac{p^\alpha-1}{2}K_2 \vee K_{p^\alpha}\right)$. Since each copy of K_2 contributes at most one vertex to an independent set, an independent set contains at most $\frac{p^\alpha-1}{2}$ rotation vertices. Moreover, every reflection vertex is adjacent to each rotation vertex, and the reflection vertices induce the complete graph K_{p^α}. Hence no reflection vertex can be added to such an independent set. Therefore, $|I| \leq \frac{p^\alpha-1}{2}$. Choosing exactly one vertex from each copy of K_2 gives an independent set of size $\frac{p^\alpha-1}{2}$. Hence,

$$\alpha(\Gamma_{porp}(D_{p^\alpha})) = \frac{p^\alpha - 1}{2}.$$

Since the reflection vertices induce the complete graph K_{p^α}, they are pairwise adjacent. The identity vertex is adjacent to every vertex. Among the rotations, each connected component is a copy of K_2, so a clique can contain the two endpoints of at most one such edge. Hence the largest clique consists of the identity, all p^α reflections, and the two vertices of one inverse pair. Therefore,

$$\omega(\Gamma_{porp}(D_{p^\alpha})) = p^\alpha + 3.$$

Case 2: $p = 2$.

By Theorem 2, $\Gamma_{porp}(D_{2^\alpha}) \cong K_1 \vee \big((2^\alpha+1)K_1 \cup (2^{\alpha-1}-1)K_2\big)$. Since the identity vertex is adjacent to every other vertex, it cannot belong to an independent set. Each isolated vertex may be included in an independent set, while each copy of K_2 contributes at most one vertex. Hence, $|I| \leq (2^\alpha+1) + (2^{\alpha-1}-1) =$

$3 \cdot 2^{\alpha-1}$. Choosing all isolated vertices together with one vertex from each copy of K_2 gives an independent set of the same size. Therefore,

$$\alpha(\Gamma_{porp}(D_{2^\alpha})) = 3 \cdot 2^{\alpha-1}.$$

For the clique number, no two reflection vertices are adjacent. A largest clique consists of the set $S^* = \{e, a_1, a_{n-1}\}$. Therefore $\omega(\Gamma_{porp}(D_{2^\alpha})) = 3$. $\square$

5. Dihedral group of order $2pq$ (p and q are distinct odd primes)

In this section, we consider the dihedral group D_{pq} which has order $2pq$ where p and q are odd primes.

Lemma 1. *Let $x = a_{kq}$, where $1 \leq k < p$. Then $o(x) = p$. Among the vertices of order pq in $V(\Gamma_{porp}(D_{pq}))$, exactly $q-1$ vertices are adjacent to x.*

Proof. Let $y = a_t$, where $(t, pq) = 1$. Then $o(y) = pq$ and $xy = a_{kq+t}$. Since $o(y) = pq$, we have $(o(y), o(xy)) = (pq, o(xy))$. As $o(xy)$ is a divisor of pq, this gcd equals 1 if and only if $o(xy) = 1$. Now, $o(xy) = 1 \iff a_{kq+t} = e \iff kq + t \equiv 0 \pmod{pq}$. Hence $t \equiv -kq \pmod{pq}$. The unique solution in $0 \leq t < pq$ is $t = q(p-k)$, which is divisible by q. Therefore $(t, pq) \neq 1$, so this value does not correspond to a vertex of order pq. Consequently, $(o(y), o(xy)) \neq 1$ for every vertex y of order pq. Thus adjacency is determined only by $(o(x), o(xy)) = 1$. Since $o(x) = p$, $(o(x), o(xy)) = 1 \iff (p, o(xy)) = 1$. Because the order of a rotation divides pq, the only divisor of pq coprime to p is q. Hence $(p, o(xy)) = 1 \iff o(xy) = q$. Now, $o(a_{kq+t}) = \frac{pq}{(kq+t,pq)}$. Therefore, $o(xy) = q \iff (kq + t, pq) = p$. This is equivalent to $p \mid (kq + t)$ and $q \nmid (kq + t)$. Since q is invertible modulo p, $p \mid (kq + t) \iff t \equiv -kq \pmod{p}$, $0 \leq t < pq$. This congruence has exactly q solutions in the interval $0 \leq t < pq$. Exactly one of these solutions is divisible by q, namely $t = q(p-k)$, and hence does not satisfy $(t, pq) = 1$. The remaining $q-1$ solutions are relatively prime to pq, and therefore correspond to vertices of order pq. Hence exactly $q-1$ vertices of order pq are adjacent to x. $\square$

Lemma 2. *Let $x = a_i$ with $o(x) = pq$. Then x is adjacent to exactly one vertex of order p and exactly one vertex of order q in $\Gamma_{porp}(D_{pq})$.*

Proof. Let $y = a_{kq}$, where $1 \leq k < p$. Then $o(y) = p$ and $xy = a_{i+kq}$. Since $o(x) = pq$, we have $(o(x), o(xy)) = (pq, o(xy))$. As $o(xy)$ is a divisor of pq, the equality $(o(x), o(xy)) = 1$ holds if and only if $o(xy) = 1$. Now, $o(xy) = 1 \iff a_{i+kq} = e \iff i + kq \equiv 0 \pmod{pq}$. Since $(i, pq) = 1$, this congruence has no solution. Hence $(o(x), o(xy)) \neq 1$. Therefore, adjacency is determined only by the condition $(o(y), o(xy)) = 1$. Since $o(y) = p$, we have $(o(y), o(xy)) = 1 \iff (p, o(xy)) = 1$. The order of a_{i+kq} is $o(a_{i+kq}) = \frac{pq}{(i+kq,pq)}$. Hence $\gcd(p, o(xy)) = 1 \iff o(xy) = q \iff (i + kq, pq) = p$. This is equivalent to $p \mid (i + kq)$ and $q \nmid (i + kq)$. Since $(i, pq) = 1$, we have $q \nmid i$, and therefore $q \nmid (i + kq)$ automatically. Thus it remains only to solve $i + kq \equiv 0 \pmod{p}$. As q is invertible modulo p, this congruence has a unique solution for k modulo p. Hence exactly one vertex of order p is adjacent to x. Similarly, let $z = a_{lp}$, where $1 \leq l < q$. By the same argument, $(o(z), o(xz)) = 1$ if and only if $q \mid (i + lp)$. Since p is invertible modulo q, the congruence $i + lp \equiv 0 \pmod{q}$ has a unique solution for l modulo q. Therefore exactly one vertex of order q is adjacent to x. $\square$

Lemma 3. *Let $x = a_{iq}$, where $1 \leq i < p$. Then $o(x) = p$. The only vertex of order p adjacent to x is $x^{-1} = a_{(p-i)q}$.*

Proof. Let $x = a_{iq}$, where $1 \leq i < p$, and let $y = a_{kq}$, $1 \leq k < p$. Then $xy = a_{(i+k)q}$. If $k = p - i$, then $xy = a_{pq} = e$. Hence $(o(x), o(xy)) = (p, 1) = 1$, and therefore x is adjacent to $a_{(p-i)q} = x^{-1}$. Now suppose $k \neq p - i$. Then $(i + k)q \not\equiv 0 \pmod{pq}$, so $o(xy) = p$. Consequently, $(o(x), o(xy)) = (p, p) = p \neq 1$, and $(o(y), o(xy)) = (p, p) = p \neq 1$. Hence x and y are not adjacent. Therefore, the only vertex of order p adjacent to x is x^{-1}. $\square$

Theorem 7. *The degree of a vertex x in $V(\Gamma_{porp}(D_{pq}))$ is*

$$d(x) = \begin{cases} 2pq-1, & \text{if } x = e, \\ 2pq-1, & \text{if } x = sr^i,\ 0 \le i < pq, \\ pq+q+1, & \text{if } o(x) = p, \\ pq+p+1, & \text{if } o(x) = q, \\ pq+4, & \text{if } o(x) = pq. \end{cases}$$

Proof. Since $o(e) = 1$, for every non-identity vertex $y \in V(\Gamma_{porp}(D_{pq}))$, $(o(e), o(ey)) = (1, o(y)) = 1$. Hence e is adjacent to every other vertex of $V(\Gamma_{porp}(D_{pq}))$. Therefore $d(e) = 2pq - 1$.

Let $x = b_i$ be a reflection. Then $o(x) = 2$. For another reflection b_j, $(b_i)(b_j) = a_{j-i}$. Since $n = pq$ is odd, every divisor of n is odd. Thus $\big(o(b_i), o((b_i)(b_j))\big) = (2, o(a_{j-i})) = 1$. Hence every reflection is adjacent to every other reflection.

Now let a_t be a rotation. Then $(b_i)a_t = b_{i+t}$, which is again a reflection element and therefore has order 2. Since every rotation element has odd order, $\big(o(a_t), o((b_i)a_t)\big) = (o(a_t), 2) = 1$. Hence every reflection vertices are adjacent to every rotation vertices. Therefore every reflection vertices are adjacent to all other vertices of the graph, and $d(b_j) = 2pq - 1$.

Next let $x = a_q$, so that $o(x) = p$. For every reflection b_j, $a_q(b_j) = b_{j-q}$, which has order 2. Therefore $\big(o(x), o(xb_j)\big) = (p, 2) = 1$. Hence x is adjacent to all n reflection vertices. Also, $(o(e), o(ex)) = (1, p) = 1$, and so x is adjacent to e.

By Lemma 3, the only vertex of order p adjacent to x is x^{-1}. Thus x has exactly one neighbour of order p.

By Lemma 1, among the vertices of order pq, exactly $q - 1$ vertices are adjacent to x.

Further, if y is a vertex of order q, then xy has order pq, and hence $(o(x), o(xy)) = (p, pq) = p \neq 1$, and $(o(y), o(xy)) = (q, pq) = q \neq 1$. Therefore x is not adjacent to any vertex of order q. Consequently, $d(x) = pq + 1 + 1 + (q - 1) = pq + q + 1$. By symmetry, if $o(x) = q$, then $d(x) = pq + p + 1$. Finally, let $x = a_i$, so that $o(x) = pq$. For every reflection b_j, $a_i(b_j) = b_{j-i}$, which has order 2. Hence $(o(x), o(xb_j)) = (pq, 2) = 1$. Therefore x is adjacent to all n reflection vertices. Since $(o(e), o(ex)) = (1, pq) = 1$, the vertex x is adjacent to the identity.

By Lemma 2, x is adjacent to exactly one vertex of order p and exactly one vertex of order q.

Moreover, $x^{-1} = a_{pq-1}$, and $xx^{-1} = e$. Hence $(o(x), o(xx^{-1})) = (pq, 1) = 1$, so x is adjacent to its inverse. Therefore $d(x) = pq + 1 + 1 + 1 + 1 = pq + 4$. Hence the result. □

Theorem 8. *The size of $\Gamma_{porp}(D_{pq})$ is*

$$|E(\Gamma_{porp}(D_{pq}))| = \frac{3p^2q^2 + 6pq - 4p - 4q + 1}{2}.$$

Proof. By the Handshaking Lemma, $2|E(\Gamma_{porp}(D_{pq}))| = \sum_{v \in V(\Gamma_{porp}(D_{pq}))} d(v)$. Using the degree sequence, from Theorem 7, we obtain

$$\begin{aligned} 2|E(\Gamma_{porp}(D_{pq}))| = {} & (2pq-1) + pq(2pq-1) \\ & + (p-1)(pq+q+1) \\ & + (q-1)(pq+p+1) \\ & + (p-1)(q-1)(pq+4). \end{aligned}$$

Substituting $n = pq$ and simplifying yields

$$2|E(\Gamma_{porp}(D_{pq}))| = 3p^2q^2 + 6pq - 4p - 4q + 1.$$

Therefore $|E(\Gamma_{porp}(D_{pq}))| = \frac{3p^2q^2+6pq-4p-4q+1}{2}$. □

Lemma 4. *The subgraph of $\Gamma_{porp}(D_{pq})$ induced by the rotation vertices has clique number* 2*.*

Proof. Let R be the subgraph induced by all rotation vertices of $\Gamma_{porp}(D_{pq})$. We first observe that R contains an edge. Indeed, by Lemma 3, every vertex of order p is adjacent to its inverse. Hence $\omega(R) \geq 2$. It remains to prove that R contains no clique of order 3. Suppose, to the contrary, that $\{u, v, w\}$ is a clique in R. Since every rotation has order p, q, or pq, we consider the following cases.

Case 1. All three vertices have order p.

By Lemma 3, each vertex of order p is adjacent only to its inverse among the vertices of order p. Hence three vertices of order p cannot be pairwise adjacent, a contradiction.

Case 2. All three vertices have order q.

Similarly, each vertex of order q is adjacent only to its inverse. Thus no three vertices of order q are pairwise adjacent.

Case 3. Two vertices have order p and one has order pq.

By Lemma 2, a vertex of order pq is adjacent to exactly one vertex of order p. Hence it cannot be adjacent to two distinct vertices of order p, which is impossible.

Case 4. Two vertices have order q and one has order pq.

Again, by Lemma 2, a vertex of order pq is adjacent to exactly one vertex of order q. Hence this case is impossible.

Case 5. One vertex has order p, one has order q, and one has order pq.

From the proof of Theorem 7, a vertex of order p is not adjacent to any vertex of order q. Hence these three vertices cannot be pairwise adjacent.

Case 6. All three vertices have order pq.

From the proof of Theorem 7, every vertex of order pq is adjacent to exactly one vertex of order pq, namely its inverse. Therefore no vertex of order pq can be adjacent to two distinct vertices of order pq. Hence three vertices of order pq cannot be pairwise adjacent.

Thus every possible case leads to a contradiction. Hence R contains no clique of order 3, and therefore $\omega(R) = 2$. □

Theorem 9. *The clique number of $\Gamma_{porp}(D_{pq})$ is*

$$\omega(\Gamma_{porp}(D_{pq})) = pq + 3.$$

Proof. By Lemma 4, the subgraph induced by the rotation vertices has clique number 2. By Theorem 7, the identity vertex is adjacent to every vertex of $\Gamma_{porp}(D_{pq})$, and all pq reflection vertices induce the complete graph K_{pq}. Hence a largest clique consists of the identity vertex, all pq reflection vertices, and two adjacent rotation vertices. Therefore,

$$\omega(\Gamma_{porp}(D_{pq})) = 1 + pq + 2 = pq + 3.$$

□

Theorem 10. *The graph $\Gamma_{porp}(D_{pq})$ is non-planar.*

Proof. By Theorem 7, the identity vertex is adjacent to every reflection vertex, and all reflection vertices are mutually adjacent. Hence the identity together with the pq reflection vertices induces the complete graph K_{pq+1}. Since $pq \geq 15$, we have $pq + 1 \geq 16$, and therefore K_{pq+1} contains K_5 as a subgraph. Hence $\Gamma_{porp}(D_{pq})$ is non-planar. □

Theorem 11. *The graph $\Gamma_{porp}(D_{pq})$ is not bipartite.*

Proof. By Theorem 7, every pair of reflection vertices is adjacent, and the identity is adjacent to every reflection vertex. Thus the identity together with any two reflection vertices forms a triangle. Since a bipartite graph contains no odd cycle, $\Gamma_{porp}(D_{pq})$ is not bipartite. □

Theorem 12. *The graph $\Gamma_{porp}(D_{pq})$ is not Eulerian.*

Proof. By Theorem 7, the identity vertex has degree $d(e) = 2pq - 1$, which is odd. Since every Eulerian graph has all vertices of even degree, $\Gamma_{porp}(D_{pq})$ is not Eulerian. □

Theorem 13. *The graph $\Gamma_{porp}(D_{pq})$ is Hamiltonian.*

Proof. By Theorem 7, $\delta(\Gamma_{porp}(D_{pq})) = pq + 4$. Also, $|V(\Gamma_{porp}(D_{pq}))| = 2pq$. Since $pq + 4 \geq pq = \frac{|V(\Gamma_{porp}(D_{pq}))|}{2}$, Dirac's theorem implies that $\Gamma_{porp}(D_{pq})$ is Hamiltonian. □

The following Table 1 verifies the theoretical results obtained in this paper for several small dihedral groups. The values agree with the formulas established in the preceding theorems.

Table 1. Verification of the graph invariants of $\Gamma_{porp}(D_n)$ for representative values of n

Group	$\lvert V(\Gamma_{porp}(D_n))\rvert$	Degree Sequence of $\Gamma_{porp}(D_n)$	$\lvert E(\Gamma_{porp}(D_n))\rvert$	$\alpha(\Gamma_{porp}(D_n))$	$\omega(\Gamma_{porp}(D_n))$
D_3	6	$\{5,5,5,5,5,5\}$	15	1	6
D_5	10	$\{7,7,7,7,\underbrace{9,9,\cdots,9}_{6\text{ times}}\}$	41	2	8
D_7	14	$\{\underbrace{13,13,\cdots,13}_{8\text{ times}},\underbrace{9,9,\cdots,9}_{6\text{ times}}\}$	79	3	10
D_8	16	$\{15,\underbrace{2,2,\cdots,2}_{6\text{ times}},\underbrace{1,1,\cdots,1}_{9\text{ times}}\}$	18	10	8
D_{15}	30	$\{\underbrace{29,29,\cdots,29}_{16\text{ times}},21,21,\underbrace{19,19,\cdots,19}_{12\text{ times}}\}$	367	5	18

Remark 2. The data presented in Table 1 agree with the theoretical formulas obtained for the degree sequence, number of edges, independence number, and clique number. These examples provide a verification of the main results for representative prime-power and product-of-primes cases.

6. Discussion

The product order relatively prime (PORP) graph differs significantly from several graph constructions previously associated with finite groups. In the ordinary coprime-order graph, two vertices are adjacent whenever the orders of the corresponding group elements are relatively prime. In contrast, the PORP graph incorporates both the element orders and the group operation by considering the order of the product of two elements. Consequently, the PORP graph reflects not only the distribution of element orders but also the interaction between them under the group operation, making it a finer graph invariant than the ordinary coprime-order graph.

The PORP graph is also closely related to the product order divisor (POD) graph. While the POD graph is defined using divisibility between the orders of elements and the order of their product, the PORP graph replaces the divisibility condition by relative primality. Although both constructions depend on the product of two group elements, they exhibit substantially different graph structures, degree distributions, clique numbers, and connectivity properties.

The results obtained in this paper demonstrate that the PORP graph captures important algebraic information about the dihedral group D_n. In particular, for the families D_{p^α} and D_{pq}, where p and q are distinct odd primes, the graph structure, degree sequence, number of edges, independence number, clique number, and several structural properties are completely determined by the arithmetic structure of n. The comparison between the prime-power case and the product of two distinct primes shows that the PORP graph is sensitive to the prime factorization of n.

An interesting question is whether non-isomorphic finite groups can possess isomorphic PORP graphs. Determining which algebraic properties of a finite group can be recovered uniquely from its PORP graph remains an open problem. Another natural direction is to obtain a unified structural description of $\Gamma_{porp}(D_n)$ for arbitrary n in terms of the prime factorization of n. Such a characterization would considerably extend the present work and provide a deeper understanding of the relationship between the algebraic structure of a finite group and the graph associated with it.

Table 2 compares the product order relatively prime graph with the ordinary coprime-order graph and the product order divisor graph. Unlike the coprime-order graph, which depends only on the orders of group elements, both the POD and PORP graphs incorporate the group operation through the order of the product. The PORP graph further replaces the divisibility condition by relative primality, resulting in different structural properties and graph invariants. This comparison highlights the novelty and significance of the PORP graph.

Table 2. Comparison of graph constructions associated with the dihedral group

Property	**Coprime Graph** Γ_{D_n}	**Product Order Divisor (POD) Graph** $\Gamma_{porp}(D_n)$	**Product Order Relatively Prime (PORP) Graph** $\Gamma_{porp}(D_n)$
Adjacency condition	$\gcd(o(a), o(b)) = 1$	$o(a)\|o(ab)$ or $o(b)\|o(ab)$	$(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$
Uses the group product	No	Yes	Yes
Depends on element orders	Yes	Yes	Yes
Depends on the order of the product	No	Yes	Yes
Graph structure	Complete tripartite or complete bipartite (depending on n)	Depends on the arithmetic structure of n	Depends on the prime factorization of n
Main feature	Depends only on element orders	Uses divisibility between element orders and the order of the product	Uses relative primality between element orders and the order of the product

7. Future work

The present work suggests several directions for future research. One important problem is to determine the structure of the product order relatively prime graph of the dihedral group D_n for arbitrary positive integers n in terms of the prime factorization of n. It is also of interest to investigate whether non-isomorphic finite groups can have isomorphic PORP graphs and to identify the algebraic properties that can be uniquely recovered from these graphs. Furthermore, extending the study of PORP graphs to other families of finite groups, including semidirect products and finite p-groups, remains an interesting direction for future investigation.

8. Conclusions

In this paper we have studied about degree sequence, size, structure of a PORP graph of Dihedral group. Also we have discussed about certain parameters like independent number, clique number. Investigation of topological indices of $\Gamma_{porp}(D_n)$ are open problems for future research work.

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