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Rarely Convergent Nets in Topological Spaces

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Abstract: A subset of a topological space is called rare if it has empty interior. Rare convergence of sequences, introduced in earlier work, allows a sequence to leave a neighbourhood of its limit, provided that the values outside it lie in a rare set, and leads to the rarely sequential topology. Sequences, however, do not describe the topology of general spaces, and in the real line every sequence rarely converges to every point. This paper develops rare convergence for nets. We show that a net rarely converges to a point if and only if, for every neighbourhood of the point, some tail of the net lies in that neighbourhood up to a set with empty interior. Consequently, the set of rare limits of a net is closed and, in a Hausdorff space, is empty, a singleton, or the whole space. Unlike for sequences, all three cases occur for nets in the real line, for instance for corrupted continuous paths. In general, rare convergence is independent of statistical convergence and of ideal convergence for proper ideals containing an infinite set, is preserved by continuous maps sending rare sets to rare sets, and passes from coordinates to products, but not conversely. We then construct the rarely net-open topology. Since the union of all rare sets is the set of non-isolated points, this topology and the rarely sequential topology can be described explicitly. This characterisation determines when the rarely net-open topology is non-trivial and which of the separation axioms T_0, T_1 and T_2 it satisfies; it is contained in the rarely sequential topology, with equality exactly when every sequentially closed set of isolated points is closed. Thus nets give rare convergence a non-trivial meaning even in metric spaces such as the real line, whereas the associated topologies are coarse invariants determined by the isolated points.

Keywords: rare sets, rare continuity, rarely convergent nets, rarely net-open topology, rare limit set, sequential spaces

MSC: 54A20, 54C08, 54D10, 54D55, 54B10, 40A35

1. Introduction and Preliminaries

The notion of a rare set — a subset R of a topological space (X, τ) satisfying $\operatorname{int}(R) = \emptyset$ — was introduced by Popa [1] as a tool for relaxing the classical notion of continuity. Unlike nowhere dense sets, which require $\operatorname{int}(\operatorname{cl}(R)) = \emptyset$, rare sets impose only the weaker condition that R contains no non-empty open subset; in particular, dense sets with empty interior, such as $\mathbb{Q} \subset \mathbb{R}$, are rare but not nowhere dense.

Popa's rare continuity has since generated a substantial literature. Functions and multifunctions satisfying weaker or related forms of rare continuity have been studied in connection with g-continuity [2,3], β-continuity [4], b-continuity [5], α-continuity [6], s-precontinuity [7], and further variants [8–13]. The convergence-theoretic counterpart of this programme was initiated in [14], where Öner introduced rarely convergent sequences and constructed from them the rarely sequential topology τ_{rs}. That work established that rare sequential convergence is strictly weaker than ordinary convergence, and characterised τ_{rs} as the collection of sequentially open sets that additionally absorb all rare sets.

The sequential framework of [14], however, is inherently limited to sequential phenomena. In a topological space that is not sequential (a situation that necessarily entails failure of first countability), sequences are unable to detect the full topological structure: there exist sets that are not closed yet contain the sequential limits of all their sequences, and continuity cannot be characterised by sequential preservation alone. The standard remedy is to pass from sequences to nets (Moore–Smith sequences) [15]. A net indexed by an arbitrary directed set encodes the same closure and convergence information as the topology itself, without any countability hypothesis. Uncountable directed sets also arise naturally as index sets in some

applied settings, for instance in continuous-time processes where the temporal domain is $(0,\infty)$ rather than $\mathbb{N}$, even when the state space itself is a first-countable metric space and sequential methods would suffice for topological convergence (cf. Proposition 5). For rare convergence the passage to nets matters even in $\mathbb{R}$: by [14, Example 6] every sequence in $\mathbb{R}$ rarely converges to every point, whereas a net in $\mathbb{R}$ may have no rare limit, exactly one, or all points of $\mathbb{R}$ as rare limits.

The present paper develops the theory of rarely convergent nets in arbitrary topological spaces. Several of its results are net versions of results of [14]: they are obtained by replacing sequences by nets and the sequential topology τ_s by $\tau_n = \tau$, and we include them, with proofs, for completeness. In the following summary we indicate which results are of this kind and which are new.

a) *Rarely convergent nets (§2).* Definition 3, Remarks 4 and 5, Propositions 2 and 3 and the subnet theorem (Theorem 4) are net versions of [14, Definition 4, Example 6, Remark 5, Proposition 10 and the discussion following it, Example 12, Theorem 8]. The new results start from a tail characterisation (Lemma 1), which formalises the argument of [14, Example 6]: x_λ rarely converges to y if and only if, for every open $G \ni y$, some tail of the net lies in G up to a set with empty interior. Consequently, the set $\mathcal{L}_r(x_\lambda)$ of rare limits of a net is closed (Proposition 4), and in a Hausdorff space it is empty, a singleton or the whole space (Theorem 3). This distinction identifies a central difference between nets and sequences in the present theory. Every sequence in $\mathbb{R}$ rarely converges to every point [14, Example 6], so for sequences in $\mathbb{R}$ the notion carries no information; for nets in $\mathbb{R}$ all three cases occur (Remark 6, Examples 3 and 5). Proposition 5 exhibits all three for continuous real paths whose values are corrupted at the times of a sequence $t_k \to \infty$. We also show that rare convergence is independent of statistical convergence and of $\mathcal{I}$-convergence for every proper ideal $\mathcal{I}$ containing an infinite set (Proposition 1), and that in the Arens–Fort space a rare limit of a net need not be a rare limit of any sequence with values in the same set (Example 1).

b) *Continuity (§3).* Examples 7 and 8 and Theorem 6 are net versions of [14, Examples 13, 15 and Theorem 16]. Theorem 5 extends [14, Theorem 14] from injective maps to continuous maps sending rare sets to rare sets. The new results include Theorem 7, which gives sufficient conditions under which rarely continuous maps preserve rare convergence, together with Example 9 and the fact that neither of its two additional hypotheses can be dropped (Remark 9), and the behaviour under products: coordinatewise rare convergence implies rare convergence in arbitrary products, but not conversely (Theorem 8, Example 10).

c) *The rarely net-open topology (§4).* The characterisation of τ_{rn} (Theorem 9), the fact that rare convergence in τ implies convergence in τ_{rn} (Theorem 11), and Example 11, Corollary 3 and Propositions 7 and 8 are net versions of results of [14, Theorems 21 and 30, Examples 18, 25 and 26, Remarks 27 and 28, Proposition 29]; since $\tau_n = \tau$, the proofs are simpler than in the sequential case. Theorem 12, which has no counterpart in [14], follows directly from Theorems 6 and 11. A simple observation, not made explicit in [14], is that the union of all rare sets is exactly the set of non-isolated points (Remark 1). This makes τ_{rn} explicit: it consists of $\emptyset$ and the complements of clopen sets of isolated points (Proposition 6). Using this description, we determine when τ_{rn} is non-trivial (Theorem 10) and which of the separation axioms T_0, T_1 and T_2 it satisfies (Proposition 9), and we show that $(\tau_{rn})_{rn}$ is indiscrete for every non-discrete space (Theorem 13).

d) *Comparison with [14] (§5).* The same observation describes τ_{rs} explicitly. We prove that $\tau_{rn} \subseteq \tau_{rs}$ always, with equality if and only if every sequentially closed set of isolated points is closed (Theorem 15). In particular, equality holds in all sequential spaces, while the Arens–Fort space gives a strict inclusion. These results also improve the sequential theory of [14]: the hypothesis of [14, Corollary 35] that X has at least two rare elements can be weakened to the existence of one rare element, that is, to non-discreteness, which is also necessary for spaces with at least two points (Corollary 5), and the example of [14, Remark 32], a rare element that becomes isolated in τ_{rs}, is explained by Theorem 15 (Example 16(b)).

Several of the new results, namely Lemma 1, Proposition 4, Theorem 3, Corollary 2 and Theorems 7 and 8, hold verbatim for sequences. What is specific to nets is that the singleton and empty cases of Theorem 3(ii) can occur in spaces in which every countable set has empty interior, such as $\mathbb{R}$, although in such spaces every sequence rarely converges to every point (Corollary 1); see Remark 6, Examples 5 and 6 and Proposition 5. Also specific to nets are Example 1 and the comparison of τ_{rn} with τ_{rs} in §5.

A key observation, standard in net theory [15], is that in any topological space the net topology τ_n, consisting of the sets A such that every net converging to a point of A is eventually in A, coincides with the given topology ($\tau_n = \tau$), whereas in general only $\tau \subseteq \tau_s$ holds, with equality exactly for sequential spaces. We exploit this fact throughout to work directly with τ without tracking a separate convergence topology.

We conclude this section by recalling the definitions and results from the literature that will be used throughout the paper. Throughout, rare means having empty interior in the specified ambient space; it does not mean nowhere dense. This distinction is essential because rare sets need not remain rare under finite unions, as the later examples demonstrate. Likewise, ordinary convergence and rare convergence are always understood relative to the explicitly stated topology, rather than to an implicit refinement or coarsening of it.

A directed set $(D, \leq)$ is a non-empty set D equipped with a preorder $\leq$ such that for every $\alpha, \beta \in D$ there exists $\gamma \in D$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. A net in a topological space (X, τ) is a function $s : D \to X$ for some directed set D, written $(x_\lambda)_{\lambda \in D}$. The net (x_λ) converges to $x \in X$, written $x_\lambda \to x$, if for every open set $G \ni x$ there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in G$. A net $(y_\mu)_{\mu \in E}$ is a subnet of $(x_\lambda)_{\lambda \in D}$ in the sense of Kelley [15] if there is a function $h : E \to D$ such that $y_\mu = x_{h(\mu)}$ for all $\mu \in E$, and for every $\lambda_0 \in D$ there exists $\mu_0 \in E$ such that $\mu \geq \mu_0$ implies $h(\mu) \geq \lambda_0$. For $\lambda_0 \in D$ we write $T_{\lambda_0} = \{x_\lambda : \lambda \geq \lambda_0\}$ for the corresponding tail of the net; note that $T_{\lambda_1} \subseteq T_{\lambda_0}$ whenever $\lambda_1 \geq \lambda_0$. For $x \in X$ we denote by $\mathcal{N}(x)$ the set of open neighbourhoods of x; when it is used as a directed set, written $(\mathcal{N}(x), \supseteq)$, it is directed by reverse inclusion ($U \geq V$ if and only if $U \subseteq V$).

A subset A of X is sequentially open if every sequence converging to a point of A is eventually in A, and sequentially closed if its complement is sequentially open, equivalently, if A contains the limits of all convergent sequences in A. The space X is sequential if every sequentially open set is open; every first-countable space is sequential [16]. The sequentially open sets form a topology $\tau_s \supseteq \tau$.

Definition 1 (Popa [1])**.** A function $f : (X, \tau_1) \to (Y, \tau_2)$ between topological spaces is called rarely continuous at $x \in X$ if for every open set G containing $f(x)$ there exist a rare set $R_G \subset Y$ with $G \cap \mathrm{cl}(R_G) = \emptyset$ and an open set $U \ni x$ such that $f(U) \subseteq G \cup R_G$. The function f is rarely continuous if it is rarely continuous at every point of X.

Theorem 1 (Long and Herrington [17])**.** *Let $f : X \to Y$ be a continuous and injective function between topological spaces. If $R \subset X$ is rare, then $f(R)$ is rare in Y.*

Definition 2 (Öner [14, Definitions 4 and 17])**.** Let (X, τ) be a topological space.

(i) A sequence (x_n) in X rarely converges to $x \in X$ if for every open set G containing x there exist a rare set R_G with $G \cap \mathrm{cl}(R_G) = \emptyset$ and $n_0 \in \mathbb{N}$ such that $x_n \in G \cup R_G$ for all $n \geq n_0$.

(ii) A set $A \subseteq X$ is rarely sequentially open if for every sequence (x_n) rarely converging to a point of A there is n_0 with $x_n \in A$ for all $n \geq n_0$. The rarely sequentially open sets form a topology τ_{rs}, the rarely sequential topology [14].

Definition 3 below reduces to (i) when the directed set is $D = \mathbb{N}$.

Theorem 2 (Öner [14, Theorem 21])**.** *Let (X, τ) be a topological space. A non-empty subset $A \subseteq X$ is rarely sequentially open if and only if it is sequentially open and contains every rare set in (X, τ).*

The empty set is (vacuously) rarely sequentially open, so Theorem 2 describes τ_{rs} completely; the restriction to non-empty sets, which is implicit in [14], is needed because $\emptyset$ contains no non-empty rare set.

Remark 1. A point $x \in X$ is called a rare element of (X, τ) if the singleton $\{x\}$ is rare, equivalently, if x is not an isolated point of (X, τ). In the discrete topology no rare elements exist; at the other extreme, in any space with no isolated points every point is a rare element. This notion will play a central role in Propositions 2 and 3, where it guarantees that the constant net at such a point rarely converges to every point. Recall that a point $x \in X$ is an isolated point of X if it is not an accumulation point of X, that is, some open neighbourhood of x contains no other point of X; equivalently, $\{x\}$ is open in X. We write $\mathrm{Iso}(X)$ for the set of isolated points of X. A set containing an isolated point x has non-empty interior (it contains the open set $\{x\}$), so every rare

set is contained in $X \setminus \mathrm{Iso}(X)$; conversely every point of $X \setminus \mathrm{Iso}(X)$ is a rare element. Hence the union of all rare sets of X, which we call the rare locus, is

$$U_{\mathrm{rare}} := \bigcup \{R \subseteq X : R \text{ rare}\} = X \setminus \mathrm{Iso}(X),$$

the set of rare elements of X.

2. Rarely Convergent Nets

Definition 3. Let (X, τ) be a topological space and $(x_\lambda)_{\lambda \in D}$ be a net in X. Then (x_λ) is said to be rarely convergent to $x \in X$, written $x_\lambda \xrightarrow{r} x$, if for every open set G containing x there exists a rare set R_G in X such that:

(i) $G \cap \mathrm{cl}(R_G) = \emptyset$, and
(ii) there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in G \cup R_G$.

The following reformulation will be used throughout the paper.

Lemma 1 (Tail Characterisation). *Let $(x_\lambda)_{\lambda \in D}$ be a net in a topological space (X, τ) and let $y \in X$.*

(i) *$x_\lambda \xrightarrow{r} y$ if and only if for every open set $G \ni y$ there exists $\lambda_0 \in D$ such that $\mathrm{int}(T_{\lambda_0} \setminus G) = \emptyset$.*
(ii) *If $\mathcal{B}$ is a base of open neighbourhoods of y, then it suffices to verify the condition in (i), or in Definition 3, for $G \in \mathcal{B}$.*

Proof. (i) Suppose $x_\lambda \xrightarrow{r} y$ and let $G \ni y$ be open, with R_G and λ_0 as in Definition 3. Then $T_{\lambda_0} \subseteq G \cup R_G$, so $T_{\lambda_0} \setminus G \subseteq R_G$ and $\mathrm{int}(T_{\lambda_0} \setminus G) \subseteq \mathrm{int}(R_G) = \emptyset$. Conversely, given an open $G \ni y$, choose λ_0 with $\mathrm{int}(T_{\lambda_0} \setminus G) = \emptyset$ and put $R_G := T_{\lambda_0} \setminus G$. Then R_G is rare by the choice of λ_0. Since $R_G \subseteq X \setminus G$ and $X \setminus G$ is closed, we have $\mathrm{cl}(R_G) \subseteq X \setminus G$, that is, $G \cap \mathrm{cl}(R_G) = \emptyset$, which is condition (i) of Definition 3. Finally, $x_\lambda \in T_{\lambda_0} \subseteq G \cup R_G$ for all $\lambda \geq \lambda_0$, which is condition (ii).

(ii) Let $G \ni y$ be open and choose $B \in \mathcal{B}$ with $B \subseteq G$. If $\mathrm{int}(T_{\lambda_0} \setminus B) = \emptyset$, then $T_{\lambda_0} \setminus G \subseteq T_{\lambda_0} \setminus B$ gives $\mathrm{int}(T_{\lambda_0} \setminus G) = \emptyset$. Now apply (i). □

Remark 2. Lemma 1 shows that condition (i) of Definition 3 can always be achieved by replacing R_G with $R_G \setminus G$: the essential requirement is that the values of a tail lying outside G form a set with empty interior. In particular, the closure condition does not force the exceptional values to stay away from the target; they may accumulate at it, or even be dense (see the comment after Proposition 5). The same trimming argument shows that a function f is rarely continuous at x if and only if for every open $G \ni f(x)$ there is an open $U \ni x$ with $\mathrm{int}(f(U) \setminus G) = \emptyset$.

Corollary 1. *If some tail T_{λ_0} of a net (x_λ) in (X, τ) is rare, then $x_\lambda \xrightarrow{r} y$ for every $y \in X$. In particular, in a space without non-empty countable open sets (for example $\mathbb{R}^n$), every net with countable range, and hence every sequence, rarely converges to every point.*

Proof. For every open G we have $\mathrm{int}(T_{\lambda_0} \setminus G) \subseteq \mathrm{int}(T_{\lambda_0}) = \emptyset$; apply Lemma 1. A countable set in such a space has empty interior. □

Remark 3. Rare net convergence is conceptually distinct from the principal existing generalisations of convergence. In ideal convergence, introduced for sequences in [18] and extended to nets in [19], a net (x_λ) converges to x with respect to an ideal $\mathcal{I}$ of subsets of D if, for every open $G \ni x$, the set of indices λ with $x_\lambda \notin G$ belongs to $\mathcal{I}$; filter convergence [20], [21, Chapter I, §7] is the dual formulation, in which the set of indices λ with $x_\lambda \in G$ belongs to a filter $\mathcal{F}$ on D. In both, the deviation is controlled by restricting the index set. In rare net convergence, by contrast, no restriction is placed on the index set: the net may lie outside G cofinally, provided that the values outside G are confined to a fixed rare set R_G in the target space. The deviation is thus controlled topologically in X, not combinatorially in D.

Similarly, statistical convergence [22], which is defined for sequences, requires the set of exceptional indices to have natural density zero, again a condition on the index set; for nets, a notion of density requires additional structure on D [23], and ideal convergence [19] provides the general framework. Rare convergence

imposes no density condition whatsoever on the index set; a rarely convergent sequence may fail to lie in G along a set of indices of natural density 1, as long as those exceptional values belong to R_G (Proposition 1(a)(i)).

These comparisons distinguish two ways of controlling exceptional terms. Filter, ideal, and statistical convergence impose conditions on the indices at which a net or sequence lies outside a neighbourhood. Rare convergence instead allows exceptional values whose set has empty interior in the target space, replacing the neighbourhood G by $G \cup R_G$ for the relevant tail. Proposition 1 below shows that neither notion implies the other in general. In particular spaces they may be comparable (for instance, in $\mathbb{R}$ every sequence rarely converges to every point), and for the corrupted paths of Proposition 5(b) rare convergence coincides with $\mathcal{I}_E$-convergence for a suitable ideal $\mathcal{I}_E$ on the index set (Remark 8(iii)).

Proposition 1. *Recall that a sequence (x_n) converges statistically to x if, for every open $G \ni x$, the set $\{n : x_n \notin G\}$ has natural density 0, where a set $A \subseteq \mathbb{N}$ has natural density $d(A) = \lim_{n\to\infty} |A \cap \{1, \dots, n\}|/n$ when this limit exists.*

(a) *In general, rare convergence and statistical convergence are independent. In $X = \{0\} \cup [1, 2]$ with the subspace topology of $\mathbb{R}$:*

(i) *the constant sequence $x_n = \frac{3}{2}$ rarely converges to 0 but does not converge statistically to 0;*
(ii) *the sequence $x_n = 0$ if n is a perfect square and $x_n = 1$ otherwise converges statistically to 1 but does not rarely converge to 1.*

(b) *More generally, let $\mathcal{I}$ be a proper ideal of subsets of $\mathbb{N}$ that contains an infinite set A, and recall that (x_n) is $\mathcal{I}$-convergent to x [18] if $\{n : x_n \notin G\} \in \mathcal{I}$ for every open $G \ni x$. Then rare convergence and $\mathcal{I}$-convergence are independent. In the space X of (a):*

(i) *the constant sequence $x_n = \frac{3}{2}$ rarely converges to 0 but is not $\mathcal{I}$-convergent to 0;*
(ii) *the sequence $x_n = 0$ if $n \in A$ and $x_n = 1$ otherwise is $\mathcal{I}$-convergent to 1 but does not rarely converge to 1.*

Proof. (a) In X the point 0 is isolated, while every point of $[1, 2]$ is a rare element; thus $\mathrm{Iso}(X) = \{0\}$. (i) The singleton $\{\frac{3}{2}\}$ is rare, so the sequence rarely converges to every point by Corollary 1. For the open neighbourhood $G = \{0\}$ of 0 we have $\{n : x_n \notin G\} = \mathbb{N}$, so the sequence does not converge statistically to 0. (ii) For every open $G \ni 1$ the set $\{n : x_n \notin G\}$ is contained in the set of perfect squares, which has density 0; hence it has density 0 as well, and the sequence converges statistically to 1. On the other hand, for the open neighbourhood $G = [1, \frac{3}{2})$ of 1 in X and every n_0, the tail $\{x_n : n \geq n_0\}$ equals $\{0, 1\}$, so $T_{n_0} \setminus G = \{0\}$ is open and non-empty. By Lemma 1, the sequence does not rarely converge to 1.

(b) (i) Rare convergence to 0 was shown in (a)(i). For the open neighbourhood $G = \{0\}$ of 0 we have $\{n : x_n \notin G\} = \mathbb{N}$, which does not belong to $\mathcal{I}$ because $\mathcal{I}$ is proper. Hence the sequence is not $\mathcal{I}$-convergent to 0. (ii) For every open $G \ni 1$ we have $x_n = 1 \in G$ whenever $n \notin A$, so $\{n : x_n \notin G\} \subseteq A$; since $A \in \mathcal{I}$ and subsets of members of $\mathcal{I}$ belong to $\mathcal{I}$, the sequence is $\mathcal{I}$-convergent to 1. On the other hand, A is infinite, so for every n_0 there is $n \geq n_0$ with $n \in A$, and the tail T_{n_0} contains the value 0. As in (a)(ii), for the open neighbourhood $G = [1, \frac{3}{2})$ of 1 the set $T_{n_0} \setminus G = \{0\}$ is open and non-empty, so by Lemma 1 the sequence does not rarely converge to 1. □

For the ideal of sets of natural density zero, $\mathcal{I}$-convergence is statistical convergence, so (a) is a special case of (b). Some hypothesis on $\mathcal{I}$ is needed: for the ideal of finite sets, $\mathcal{I}$-convergence is ordinary convergence [18], which implies rare convergence (Remark 5).

Remark 4. Rare convergence does not imply that the limit point is a cluster point of the net. Recall that x is a cluster point of (x_λ) if for every open $G \ni x$ and every $\lambda_0 \in D$ there exists $\lambda \geq \lambda_0$ with $x_\lambda \in G$.

To see explicitly that rare convergence does not imply this cluster-point property, consider $X = \mathbb{R}$ with the standard topology and the sequence $(x_n) = (1, 2, 3, \dots)$. We claim $(x_n) \xrightarrow{r} 0$. Let G be any open neighbourhood of 0. Set $R_G = \mathbb{N} \setminus G = \{n \in \mathbb{N}^+ : n \notin G\}$. Since $R_G \subseteq \mathbb{N}$ and $\mathbb{N}$ has empty interior in $\mathbb{R}$, R_G is rare. Since $R_G \subseteq G^c$ and G^c is closed, $\mathrm{cl}(R_G) \subseteq G^c$, so $G \cap \mathrm{cl}(R_G) = \emptyset$. For every $n \geq 1$, either $x_n = n \in G$ or $x_n = n \in \mathbb{N} \setminus G = R_G$, so $x_n \in G \cup R_G$ for all n. Hence $(x_n) \xrightarrow{r} 0$.

However, 0 is not a cluster point of (x_n): for $G = (-\frac{1}{2}, \frac{1}{2})$ and any threshold n_0, every $n \geq n_0$ satisfies $x_n = n \geq n_0 > \frac{1}{2}$, so $x_n \notin G$.

Thus a rare limit need not be a cluster point. This can only happen in the universal regime: by Theorem 3(iii), if a rare limit of a net is not a cluster point, then some tail of the net is rare, so the net rarely converges to every point, as the present sequence does.

Remark 5. Ordinary convergence of nets implies rare convergence. Indeed, if $x_\lambda \to x$ in the ordinary sense, then for any open $G \ni x$ we may take $R_G = \emptyset$, which is a rare set, and the conditions are trivially satisfied. The converse is not true in general, as shown in Remark 4.

Proposition 2. *Let R be a rare set in (X, τ) and $r \in R$. Then:*

(i) *The constant net $x_\lambda = r$ for all λ rarely converges to every $x \in X$.*
(ii) *For any $x \in X$, any net that is eventually in $\{x, r\}$ rarely converges to x.*

Proof. (i) Let G be any open set containing x. If $r \in G$ then the condition holds with $R_G = \emptyset$. If $r \notin G$ then choose $R_G = \{r\} \subseteq R$, which is rare since $R_G \subseteq R$ and $\mathrm{int}(R) = \emptyset$. Since $R_G \subseteq G^c$ we have $G \cap \mathrm{cl}(R_G) = \emptyset$. For all λ we have $x_\lambda = r \in G \cup R_G$, so $(x_\lambda) \xrightarrow{r} x$.

(ii) If (x_λ) is eventually in $\{x, r\}$, then there exists λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in \{x, r\}$. For any open $G \ni x$, if $r \in G$ we take $R_G = \emptyset$; otherwise $R_G = \{r\}$ as above. In either case, $x_\lambda \in G \cup R_G$ for all $\lambda \geq \lambda_0$. □

Proposition 3. *Let (X, τ) be a topological space.*

(i) *If X has at least one rare element, then some net (for instance the constant net at a rare element) rarely converges to every point of X; in particular, rare limits are not unique.*
(ii) *If X has no rare elements (i.e., (X, τ) is discrete), then every rarely convergent net has a unique limit.*

Proof. (i) If $r \in X$ is a rare element, the constant net $x_\lambda = r$ rarely converges to every $x \in X$ by Proposition 2(i). Since $\{r\}$ is not open, $X \neq \{r\}$; thus X has at least two points, and uniqueness of limits fails.

(ii) In the discrete topology, every subset is open, meaning there are no non-empty rare sets. Therefore, any rare set R_G must be empty ($\emptyset$), and Definition 3 therefore reduces to ordinary net convergence. Since a space equipped with the discrete topology is Hausdorff (T_2), ordinary net convergence has unique limits. □

Example 1. (A rare limit of a net in $X \setminus \{(0,0)\}$ that is not a rare limit of any sequence in $X \setminus \{(0,0)\}$.) Let $X = (\mathbb{N}^+ \times \mathbb{N}^+) \cup \{(0,0)\}$ be the Arens–Fort space [24, Counterexample 26]: every point (m, n) with $m, n \geq 1$ is isolated, and a set $U \ni (0,0)$ is open if and only if, for all but finitely many $m \in \mathbb{N}^+$, the set $C_m \setminus U$ is finite, where $C_m = \{(m, n) : n \in \mathbb{N}^+\}$ is the m-th column. The space is Hausdorff and is not first countable at $(0,0)$.

Rare sets. Every open set containing $(0,0)$ is infinite, so $(0,0)$ is not isolated. By Remark 1 the rare locus is $X \setminus \mathrm{Iso}(X) = \{(0,0)\}$, so the rare sets are exactly $\emptyset$ and $\{(0,0)\}$.

Rare convergence to $(0,0)$ is ordinary convergence. Let G be an open neighbourhood of $(0,0)$ and T a tail of a net. The set $T \setminus G$ consists of isolated points, so $\mathrm{int}(T \setminus G) = \emptyset$ if and only if $T \setminus G = \emptyset$. By Lemma 1, a net rarely converges to $(0,0)$ if and only if it converges to $(0,0)$.

A net with rare limit set $\{(0,0)\}$. Let D be the set of open neighbourhoods of $(0,0)$, directed by reverse inclusion, and choose $x_U \in U \setminus \{(0,0)\}$ for each $U \in D$. Then $x_U \to (0,0)$, hence $x_U \xrightarrow{r} (0,0)$. If y is an isolated point, then $X \setminus \{y\} \in D$, and for every $U_0 \in D$ the tail T_{U_0} contains the isolated point $x_{U_0 \setminus \{y\}} \neq y$; thus $T_{U_0} \setminus \{y\}$ has non-empty interior, and by Lemma 1 (with $G = \{y\}$) the net does not rarely converge to y. Hence $(0,0)$ is the only rare limit of (x_U).

No sequence in $X \setminus \{(0,0)\}$ rarely converges to $(0,0)$. It is classical that no sequence in $X \setminus \{(0,0)\}$ converges to $(0,0)$ [24, Counterexample 26]; for completeness we recall the argument. Let (s_k) be a sequence in $X \setminus \{(0,0)\}$. Let S be $\{y\}$ if some value y occurs infinitely often. Otherwise, if some column C_m contains infinitely many terms of the sequence, fix one such column and let S be the set of terms lying in it. Otherwise let S be the range of the sequence. In each case $X \setminus S$ is an open neighbourhood of $(0,0)$ (in the second case $S \subseteq C_m$ for a single m; in the last case $S \cap C_m$ is finite for every m), and infinitely many terms lie in S. So (s_k) does not converge to $(0,0)$, and therefore, since rare convergence to $(0,0)$ is ordinary convergence, it does not rarely converge to $(0,0)$.

Thus $(0,0)$ is the unique rare limit of a net with values in $X \setminus \{(0,0)\}$, but no sequence in $X \setminus \{(0,0)\}$ has $(0,0)$ as a rare limit. This is the rare-convergence counterpart of the classical fact that sequences do not suffice in non-sequential spaces.

Remark 6. It was shown in [14] that in $\mathbb{R}$ with the standard topology, every sequence rarely converges to every point. This holds because the range of any sequence is at most countable, which implies that every subset of its range has empty interior (i.e., it is a rare set) in $\mathbb{R}$.

For nets, however, this property fails in general. Consider the directed set $D = (\mathbb{R}, \leq)$ and the net $x_\lambda = \lambda$. We claim this net does not rarely converge to any $x \in \mathbb{R}$. Suppose for contradiction that $x_\lambda \xrightarrow{r} x$. Let $G = (x-1, x+1)$. Then there must exist a rare set R_G with $G \cap \mathrm{cl}(R_G) = \emptyset$ and $\lambda_0 \in \mathbb{R}$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in G \cup R_G$. Since $x_\lambda = \lambda$ and G is bounded, we have $x_\lambda \notin G$ for all $\lambda \geq \lambda_1$ where $\lambda_1 = \max(\lambda_0, x+1)$. Hence $[\lambda_1, \infty) \subseteq R_G$, so $\mathrm{int}(R_G) \supseteq (\lambda_1, \infty) \neq \emptyset$, contradicting R_G being rare. Therefore $x_\lambda \xrightarrow{r} x$ fails for every $x \in \mathbb{R}$.

This shows that the universal rare convergence of sequences in $\mathbb{R}$ does not extend to arbitrary nets. By Corollary 1 and Theorem 3, a net in $\mathbb{R}$ rarely converges to every point if and only if some tail has empty interior; the tails of $x_\lambda = \lambda$ are rays, which have non-empty interior. In the sequential setting of [14] every tail is countable, so every sequence in $\mathbb{R}$ has all of $\mathbb{R}$ as its set of rare limits, and the limit concept carries no information there. For nets in $\mathbb{R}$ the set of rare limits can be empty (the present net), a single point (Examples 5 and 6) or all of $\mathbb{R}$ (Example 3), and by Theorem 3 there are no other possibilities. This is one reason for passing from sequences to nets even in metric spaces such as $\mathbb{R}$.

Example 2. (An oscillating net in the Sierpiński space.) Let $X = \{0,1\}$ with the Sierpiński topology $\tau = \{\emptyset, \{1\}, X\}$. The only non-empty rare set is $\{0\}$ since $\mathrm{int}(\{0\}) = \emptyset$.

Let $D = (\mathbb{R}, \leq)$ and define

$$x_\lambda = \begin{cases} 0 & \lambda \in \mathbb{R} \setminus \mathbb{Q}, \\ 1 & \lambda \in \mathbb{Q}. \end{cases}$$

Ordinary Convergence to 1 Fails. The only non-trivial open set is $\{1\}$. Since $\mathbb{R} \setminus \mathbb{Q}$ is unbounded above, for any threshold λ_0 there exists $\lambda \geq \lambda_0$ with $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, giving $x_\lambda = 0 \notin \{1\}$. Hence (x_λ) is not eventually in $\{1\}$ and does not converge to 1.

Rare convergence holds. The only proper open neighbourhood of 1 is $G = \{1\}$. For the other open neighbourhood, $G = X$, the convergence condition is immediate with $R_G = \emptyset$. For $G = \{1\}$, set $R_G = \{0\}$. Then $\mathrm{int}(R_G) = \emptyset$, and since $\{1\}$ is open, $\{0\}$ is closed in (X, τ), so $\mathrm{cl}(R_G) = \{0\}$ and $G \cap \mathrm{cl}(R_G) = \{1\} \cap \{0\} = \emptyset$. For any $\lambda \in D$, $x_\lambda \in \{0,1\} = G \cup R_G$, so the condition holds with no threshold required. Hence $(x_\lambda) \xrightarrow{r} 1$. Since the only open neighbourhood of 0 is X, the net also converges ordinarily, and therefore rarely, to 0, so $\mathcal{L}_r(x_\lambda) = X$, although no tail of the net is rare. Thus the last assertion of Theorem 3(ii) fails without the Hausdorff assumption. The uncountable index set plays no role here: the sequence $0, 1, 0, 1, \ldots$ behaves in the same way (cf. [14, Example 9(iv)]).

Example 3. (A free ultrafilter net.) Let $X = \mathbb{R}$ with the standard topology. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, directed by reverse inclusion ($A \geq B \iff A \subseteq B$). Define the net $(x_A)_{A \in \mathcal{U}}$ by choosing $x_A \in A$ for each $A \in \mathcal{U}$. Since every $A \in \mathcal{U}$ is a subset of $\mathbb{N}$, the net takes values in $\mathbb{N} \subset \mathbb{R}$.

Ordinary convergence fails. We show that the net (x_A) does not converge in the ordinary sense to any $y \in \mathbb{R}$.

Case 1: $y \notin \mathbb{N}$. Then $d := d(y, \mathbb{N}) > 0$, so the open neighbourhood $G = (y - d/2, y + d/2)$ satisfies $G \cap \mathbb{N} = \emptyset$. Since $x_A \in \mathbb{N}$ for every A, we have $x_A \notin G$ for all $A \in \mathcal{U}$, so the net is never eventually in the open set G and cannot converge to y.

Case 2: $y = n \in \mathbb{N}$. Suppose $(x_A) \to n$ in the ordinary sense. The set $G = (n - \frac{1}{2}, n + \frac{1}{2})$ is an open neighbourhood of n, so there exists $A_0 \in \mathcal{U}$ such that $x_A \in G$ for all $A \subseteq A_0$. Since $x_A \in \mathbb{N}$ and $G \cap \mathbb{N} = \{n\}$, this forces $x_A = n$ for all $A \subseteq A_0$.

However, since $\mathcal{U}$ is a free ultrafilter, $\{n\}$ is finite, and hence $\mathbb{N} \setminus \{n\}$ is cofinite, so $\mathbb{N} \setminus \{n\} \in \mathcal{U}$. The set $A_1 = A_0 \cap (\mathbb{N} \setminus \{n\}) \in \mathcal{U}$ satisfies $A_1 \subseteq A_0$ (i.e., $A_1 \geq A_0$), yet $x_{A_1} \in A_1 \subseteq \mathbb{N} \setminus \{n\}$, giving $x_{A_1} \neq n$. This contradicts $x_A = n$ for all $A \subseteq A_0$. Combining the two cases, (x_A) does not converge ordinarily to any $y \in \mathbb{R}$.

Rare convergence holds. The range of the net is contained in $\mathbb{N}$, which has empty interior in $\mathbb{R}$. By Corollary 1, the net rarely converges to every real number, although it converges ordinarily to none.

Example 4. (A geometric net: jumping to a boundary in $\mathbb{R}^2$.) Consider $X = \mathbb{R}^2$ with the standard topology and target point $x^* = (0,0)$. Define the directed set

$$D = \mathbb{R}^2 \setminus \{(0,0)\}, \quad u \succeq v \iff \|u\| \leq \|v\|.$$

This is a valid directed set: for any $u, v \in D$, choosing any $w \in D$ with $\|w\| \leq \min(\|u\|, \|v\|)$ (e.g., $w = \frac{\min(\|u\|,\|v\|)}{2} \cdot \frac{u}{\|u\|}$) gives $w \succeq u$ and $w \succeq v$.

Let $S^1 = \{u \in \mathbb{R}^2 : \|u\| = 1\}$ be the unit circle.

Define the net $(p_u)_{u \in D}$ by

$$p_u = \begin{cases} u, & \text{if } \|u\| \in \mathbb{Q}, \\ \dfrac{u}{\|u\|}, & \text{if } \|u\| \notin \mathbb{Q}. \end{cases}$$

When $\|u\|$ is irrational, p_u is the radial projection of u onto S^1.

Ordinary convergence fails. For any $u_0 \in D$, pick any irrational r with $0 < r \leq \|u_0\|$ and set $u = (r, 0)$. Then $u \succeq u_0$ but $p_u = u/\|u\| \in S^1$, so $\|p_u\| = 1$ and the net is not eventually in $B(0, \frac{1}{2})$; hence it does not converge to $(0,0)$.

Rare convergence holds at every point. Although the net was constructed with $(0,0)$ in mind, $(0,0)$ is not a distinguished rare limit. Every value p_u has rational norm (when $\|u\|$ is irrational, $\|p_u\| = 1$), so the range is contained in $A = \{v \in \mathbb{R}^2 : \|v\| \in \mathbb{Q}_{>0}\}$, a countable union of circles. The set A is dense in $\mathbb{R}^2$ but has empty interior, since every open ball contains points of irrational norm. By Corollary 1, (p_u) rarely converges to every point of $\mathbb{R}^2$, in particular to $(0,0)$. The example shows that density of the values near a target does not prevent universal rare convergence; in a Hausdorff space with at least two points, such as $\mathbb{R}^2$, what matters is whether some tail is rare (Theorem 3(ii)).

Example 5. (A thick net with a unique rare limit.) Let $X = \mathbb{R}$ with the standard topology. Consider the directed set

$$D = \{(U, x) : U \in \tau_{\text{std}}, 5 \in U, x \in U\}, \quad (U_1, x_1) \geq (U_2, x_2) \iff U_1 \subseteq U_2,$$

and define the net $(y_\lambda)_{\lambda \in D}$ by $y_{(U,x)} = x$.

Ordinary convergence to 5. For any open $V \ni 5$, take $\lambda_0 = (V, 5)$. For all $(U, x) \geq \lambda_0$, $U \subseteq V$ and $x \in U \subseteq V$, so $(y_\lambda) \to 5$, and consequently $(y_\lambda) \xrightarrow{r} 5$ by Remark 5.

Rare convergence fails at every $y \neq 5$. Let $y \neq 5$. Since $\mathbb{R}$ is Hausdorff, there exist disjoint open sets $G \ni y$ and $V \ni 5$. We show $(y_\lambda) \not\xrightarrow{r} y$.

Suppose for contradiction there exist a rare set R with $G \cap \mathrm{cl}(R) = \emptyset$ and $\lambda_0 = (U_0, x_0)$ such that $y_{(U,x)} \in G \cup R$ for all $(U, x) \geq \lambda_0$. For every $x \in U_0$, $(U_0, x) \geq \lambda_0$, so $U_0 \subseteq G \cup R$. Since $5 \in U_0$ and $5 \notin G$, we have $5 \in R$. Let $W = V \cap U_0$, which is open and contains 5. Since $W \subseteq U_0 \subseteq G \cup R$ and $W \cap G \subseteq V \cap G = \emptyset$, we get $W \subseteq R$. Hence $\mathrm{int}(R) \supseteq W \neq \emptyset$, contradicting the assumption that R is rare.

Therefore $(y_\lambda) \xrightarrow{r} y$ if and only if $y = 5$: the net rarely converges to exactly one point, showing that the set of rare limits can be a proper non-empty subset of X. The argument above is an instance of the thickness criterion of Corollary 2 below: for $\lambda_0 = (U_0, x_0)$ the tail T_{λ_0} equals U_0, and the non-empty open set $W = V \cap U_0$ used above lies in this tail and outside G. The nets of Examples 3 and 4, by contrast, have tails with empty interior and therefore rarely converge to every point (Corollary 1).

Example 6. (A selective rarely convergent net.) Let $X = \mathbb{R}$ with the standard topology. Define the directed set

$$D = \{(U, x) : U \in \mathcal{N}(0), x \in U \cup \{10\}\}, \quad (U_1, x_1) \geq (U_2, x_2) \iff U_1 \subseteq U_2.$$

This is a valid directed set: for any (U_1, x_1) and (U_2, x_2), the pair $(U_1 \cap U_2, 0)$ is an upper bound. Define the net $(y_\lambda)_{\lambda \in D}$ by $y_{(U,x)} = x$.

Rare convergence to 0 holds. Let G be any open neighbourhood of 0. Set $R_G = \{10\} \setminus G$. Since $\{10\}$ is a singleton, R_G is either $\{10\}$ or $\emptyset$, so $\mathrm{int}(R_G) = \emptyset$ and $G \cap \mathrm{cl}(R_G) = \emptyset$ always holds. Take $\lambda_0 = (G, 0)$. For

any $(U, x) \geq \lambda_0$, we have $U \subseteq G$. By definition of D, $x \in U \cup \{10\}$. Thus, $x \in G \cup \{10\}$. Since $G \cup \{10\} = G \cup (\{10\} \setminus G) = G \cup R_G$, we have $y_\lambda \in G \cup R_G$. Hence $(y_\lambda) \xrightarrow{r} 0$.

Ordinary convergence to 0 **fails.** Let $G = (-1, 1)$. For any $\lambda_0 = (U_0, x_0) \in D$, the index $\lambda_1 = (U_0, 10)$ belongs to D, satisfies $\lambda_1 \geq \lambda_0$, and $y_{\lambda_1} = 10 \notin G$. Hence the net is not eventually in G, and it does not converge to 0.

Rare convergence to any $y \neq 0$ **fails.** Let $y \neq 0$. Choose disjoint open sets $V \ni y$ and $W \ni 0$. Suppose $(y_\lambda) \xrightarrow{r} y$. Then there exist a rare set R^* with $V \cap \mathrm{cl}(R^*) = \emptyset$ and a threshold $\lambda_0 = (U_0, x_0)$ such that $y_\lambda \in V \cup R^*$ for all $\lambda \geq \lambda_0$.

Let $U_1 = U_0 \cap W$. For every $x \in U_1$, the index (U_1, x) is in D and satisfies $(U_1, x) \geq \lambda_0$. Thus $y_{(U_1,x)} = x \in V \cup R^*$. This implies $U_1 \subseteq V \cup R^*$. Since $U_1 \subseteq W$ and $W \cap V = \emptyset$, we have $U_1 \cap V = \emptyset$. Therefore, $U_1 \subseteq R^*$. However, U_1 is a non-empty open set, giving $\mathrm{int}(R^*) \supseteq U_1 \neq \emptyset$, contradicting the rareness of R^*.

Hence $(y_\lambda) \xrightarrow{r} y$ if and only if $y = 0$: the net has the unique rare limit 0, although, since it takes the value 10 cofinally, it does not converge to 0 in the ordinary sense (nor, by Remark 5, to any other point). As in Example 5, the argument above is an instance of Corollary 2: for $\lambda_0 = (U_0, x_0)$ the tail T_{λ_0} equals $U_0 \cup \{10\}$, and the non-empty open set $U_1 = U_0 \cap W$ used above lies in this tail and outside V. So every tail contains non-empty open sets arbitrarily close to 0, whereas the anomalous value 10 forms a rare set.

2.1. The Set of Rare Limits

Definition 4. For a net (x_λ) in (X, τ), the rare limit set of (x_λ) is

$$\mathcal{L}_r(x_\lambda) := \{y \in X : x_\lambda \xrightarrow{r} y\}.$$

Proposition 4. *For every net (x_λ) in (X, τ), the set $\mathcal{L}_r(x_\lambda)$ is closed in X.*

Proof. Let $y \notin \mathcal{L}_r(x_\lambda)$. By Lemma 1 there is an open set $G \ni y$ such that $\mathrm{int}(T_{\lambda_0} \setminus G) \neq \emptyset$ for every $\lambda_0 \in D$. The same G is an open neighbourhood of every $z \in G$, so by Lemma 1 no $z \in G$ is a rare limit of the net. Hence $G \subseteq X \setminus \mathcal{L}_r(x_\lambda)$, and $X \setminus \mathcal{L}_r(x_\lambda)$ is open. □

Theorem 3. *Let (x_λ) be a net in a topological space (X, τ).*

(i) *If $\mathcal{L}_r(x_\lambda)$ contains two points that have disjoint open neighbourhoods, then some tail of the net is rare, and consequently $\mathcal{L}_r(x_\lambda) = X$.*
(ii) *If X is Hausdorff and has at least two points, then $\mathcal{L}_r(x_\lambda)$ is either empty, a singleton, or X; moreover, $\mathcal{L}_r(x_\lambda) = X$ if and only if some tail of the net is rare.*
(iii) *If no tail of (x_λ) is rare, in particular if $\mathcal{L}_r(x_\lambda) \neq X$, then every rare limit of (x_λ) is a cluster point of (x_λ).*

Proof. (i) Let $y_1, y_2 \in \mathcal{L}_r(x_\lambda)$ have disjoint open neighbourhoods G_1, G_2. By Lemma 1 there are $\lambda_1, \lambda_2 \in D$ with $\mathrm{int}(T_{\lambda_i} \setminus G_i) = \emptyset$; choose $\lambda_0 \geq \lambda_1, \lambda_2$. Since tails decrease, $\mathrm{int}(T_{\lambda_0} \setminus G_i) = \emptyset$ for $i = 1, 2$. Suppose $W := \mathrm{int}(T_{\lambda_0}) \neq \emptyset$. If $W \cap G_1 = \emptyset$, then $W \subseteq T_{\lambda_0} \setminus G_1$, which is impossible. Hence $W \cap G_1$ is a non-empty open set, and since $G_1 \cap G_2 = \emptyset$ we get $W \cap G_1 \subseteq T_{\lambda_0} \setminus G_2$, again impossible. Therefore T_{λ_0} is rare, and $\mathcal{L}_r(x_\lambda) = X$ by Corollary 1.

(ii) If $\mathcal{L}_r(x_\lambda)$ has at least two points, (i) gives $\mathcal{L}_r(x_\lambda) = X$. If some tail is rare, then $\mathcal{L}_r(x_\lambda) = X$ by Corollary 1. Conversely, if $\mathcal{L}_r(x_\lambda) = X$, pick two distinct points of X; they have disjoint open neighbourhoods, and (i) yields a rare tail.

(iii) Suppose that no tail of (x_λ) is rare, and suppose, for a contradiction, that some $y \in \mathcal{L}_r(x_\lambda)$ is not a cluster point. Then there are an open $G \ni y$ and $\lambda_0 \in D$ with $x_\lambda \notin G$ for all $\lambda \geq \lambda_0$. By Lemma 1 there is λ_1 with $\mathrm{int}(T_{\lambda_1} \setminus G) = \emptyset$. For $\lambda_2 \geq \lambda_0, \lambda_1$ we get $T_{\lambda_2} \subseteq T_{\lambda_1} \setminus G$, so the tail T_{λ_2} is rare, which contradicts the hypothesis. Finally, if $\mathcal{L}_r(x_\lambda) \neq X$, then no tail is rare by Corollary 1. □

Corollary 2 (Thickness Criterion). *Let (X, τ) be Hausdorff, $x^* \in X$, and let (x_λ) be a net such that $\mathrm{int}(T_{\lambda_0} \cap W) \neq \emptyset$ for every open $W \ni x^*$ and every $\lambda_0 \in D$. Then $\mathcal{L}_r(x_\lambda) \subseteq \{x^*\}$; if moreover $x_\lambda \xrightarrow{r} x^*$, then $\mathcal{L}_r(x_\lambda) = \{x^*\}$.*

Proof. Let $y \neq x^*$ and choose disjoint open sets $V \ni y$ and $W \ni x^*$. Then $T_{\lambda_0} \cap W \subseteq T_{\lambda_0} \setminus V$, so $\mathrm{int}(T_{\lambda_0} \setminus V) \neq \emptyset$ for every λ_0, and $y \notin \mathcal{L}_r(x_\lambda)$ by Lemma 1. □

Remark 7. By Proposition 2(i), as soon as X has a rare element, some nets (for instance, the constant net at a rare element) rarely converge to every point. Thus rare limits are not unique in general, and the rare limit is best regarded as the set-valued object $\mathcal{L}_r(x_\lambda)$. By Proposition 4 and Theorem 3, this set is always closed, and in a Hausdorff space with at least two points there are only three regimes, all of which occur already in $\mathbb{R}$ (Remark 6, Examples 3, 5 and 6):

(a) *Universal* ($\mathcal{L}_r = X$), which happens exactly when some tail is rare: Examples 3 and 4 (tails contained in $\mathbb{N}$, respectively in a dense countable union of circles), every sequence in $\mathbb{R}$, and every net that is eventually constant at a rare element.
(b) *Selective* ($\mathcal{L}_r$ a singleton): Examples 1, 5 and 6. In the last two the tails contain non-empty open sets near the target (Corollary 2); in the first, rare convergence to $(0,0)$ reduces to ordinary convergence.
(c) *Empty* ($\mathcal{L}_r = \emptyset$): the net $x_\lambda = \lambda$ of Remark 6.

Proposition 5 below exhibits all three regimes for a family of nets in $\mathbb{R}$ indexed by a half-line. In a Hausdorff space with at least two points, universality is decided by whether some tail is rare (Theorem 3(ii)), not by whether the values are dense near a target; without the Hausdorff assumption, universality need not imply that some tail is rare (Example 2). By Proposition 3, every rarely convergent net in X has a unique rare limit exactly when X is discrete; nevertheless, as (b) shows, a single net in a non-discrete space can have a unique rare limit.

The Hausdorff hypothesis in Theorem 3(ii) cannot be dropped. In $X = \{x, y, z\}$ with $\tau = \{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}$ (see [14, Example 9(i)]), the only rare set other than $\emptyset$ is $\{z\}$, and the constant net at x rarely converges exactly to x and z; its rare limit set $\{x, z\}$ is closed but is neither a singleton nor X.

Theorem 4 (Subnet Theorem)**.** *Let (X, τ) be a topological space. If $(x_\lambda)_{\lambda \in D}$ rarely converges to x, then every subnet of (x_λ) rarely converges to x.*

Proof. Let $(y_\mu)_{\mu \in E}$ be a subnet of (x_λ) via $h : E \to D$. Let G be any open set containing x. Since $x_\lambda \xrightarrow{r} x$, there exist a rare set R_G with $G \cap \mathrm{cl}(R_G) = \emptyset$ and $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in G \cup R_G$. By the subnet condition, there exists $\mu_0 \in E$ such that $\mu \geq \mu_0$ implies $h(\mu) \geq \lambda_0$. Hence for $\mu \geq \mu_0$ we have $y_\mu = x_{h(\mu)} \in G \cup R_G$. Therefore $(y_\mu) \xrightarrow{r} x$. □

Consequently, $\mathcal{L}_r(x_\lambda) \subseteq \mathcal{L}_r(y_\mu)$ for every subnet (y_μ) of (x_λ).

2.2. Corrupted Continuous Paths in $\mathbb{R}$

The following proposition concerns nets in $\mathbb{R}$ indexed by a half-line: a continuous path whose values are replaced by arbitrary values at the times of a strictly increasing sequence $t_k \to \infty$. All three cases of Theorem 3(ii) occur for such nets, whereas every sequence in $\mathbb{R}$ rarely converges to every point (Corollary 1).

Proposition 5. *Let $D = [a, \infty)$ or $D = (a, \infty)$ with $a \in \mathbb{R}$, directed by $\leq$, and let $\psi : D \to \mathbb{R}$ be continuous. Let $(t_k)_{k \geq 1}$ be a strictly increasing sequence in D with $t_k \to \infty$, and let $r_k \in \mathbb{R}$ be arbitrary. Define the net $\varphi : D \to \mathbb{R}$ by $\varphi(t) = \psi(t)$ for $t \notin \{t_k : k \geq 1\}$ and $\varphi(t_k) = r_k$.*

(a) *If $\psi(t) \to y$, then $\varphi \xrightarrow{r} y$.*
(b) *If ψ is not eventually constant, then for every $y \in \mathbb{R}$, $\varphi \xrightarrow{r} y$ if and only if $\psi(t) \to y$. In particular, if $\psi(t) \to x^*$ then $\mathcal{L}_r(\varphi) = \{x^*\}$, and if ψ does not converge then $\mathcal{L}_r(\varphi) = \emptyset$.*
(c) *If ψ is eventually constant, then $\mathcal{L}_r(\varphi) = \mathbb{R}$.*
(d) *If $\psi(t) \to x^*$, then $\varphi(t) \to x^*$ in the ordinary sense if and only if $r_k \to x^*$.*

Proof. We write $T_s = \{\varphi(t) : t \geq s\}$ for the tails of φ.

(a) Let $G \ni y$ be open and choose s_0 with $\psi(t) \in G$ for all $t \geq s_0$. Then $T_{s_0} \setminus G \subseteq \{r_k : k \geq 1\}$, a countable set, which has empty interior in $\mathbb{R}$; now Lemma 1 applies.

(b) By (a) it remains to show that $\varphi \xrightarrow{r} y$ implies $\psi(t) \to y$. For each k let $J_k = \psi([t_k, t_{k+1}])$, a compact interval. If J_k were degenerate for all $k \geq K$, then, since consecutive intervals $[t_k, t_{k+1}]$ share endpoints and $t_k \to \infty$, ψ would be constant on $[t_K, \infty)$. Hence infinitely many J_k are non-degenerate.

Let $\varepsilon > 0$ and $G = (y-\varepsilon, y+\varepsilon)$. By Lemma 1 there is s_0 with $\operatorname{int}(T_{s_0} \setminus G) = \emptyset$. Let $t_k \geq s_0$. If J_k is non-degenerate, then $\psi((t_k, t_{k+1}))$ is an interval with closure J_k, and it is contained in T_{s_0} because no t_m lies in (t_k, t_{k+1}). Its part outside G is contained in $T_{s_0} \setminus G$ and therefore has empty interior. This forces $\psi((t_k, t_{k+1}))$ to lie in $[y-\varepsilon, y+\varepsilon]$: if it contained a point $p > y+\varepsilon$, then, being a non-degenerate interval, it would meet $(y+\varepsilon, \infty)$ in a non-degenerate interval contained in its part outside G, and similarly if it contained a point $p < y-\varepsilon$. Hence J_k, its closure, is also contained in $[y-\varepsilon, y+\varepsilon]$. If J_k is degenerate, let $m > k$ be the least index with J_m non-degenerate; then $J_k = J_{k+1} = \cdots = J_{m-1} = \{\psi(t_m)\}$ and $\psi(t_m) \in J_m \subseteq [y-\varepsilon, y+\varepsilon]$. Thus $|\psi(t) - y| \leq \varepsilon$ for all $t \geq t_{k_0}$, where k_0 is the least index with $t_{k_0} \geq s_0$. As ε was arbitrary, $\psi(t) \to y$. The remaining assertions follow because limits in $\mathbb{R}$ are unique.

(c) If $\psi(t) = c$ for all $t \geq s_0$, then $T_{s_0} \subseteq \{c\} \cup \{r_k : k \geq 1\}$, a countable and hence rare set. By Corollary 1, $\mathcal{L}_r(\varphi) = \mathbb{R}$. Together with (b), this shows that $\mathcal{L}_r(\varphi) = \mathbb{R}$ if and only if ψ is eventually constant.

(d) If $r_k \to x^*$ and $G \ni x^*$ is open, choose s_0 with $\psi(t) \in G$ for $t \geq s_0$ and K with $r_k \in G$ for $k \geq K$; then $\varphi(t) \in G$ for $t \geq \max(s_0, t_K)$. Conversely, if $\varphi(t) \to x^*$, then $r_k = \varphi(t_k) \to x^*$ because $t_k \to \infty$. □

No condition on the anomalous values r_k is needed: they form a countable set, which is automatically rare in $\mathbb{R}$. In particular they may accumulate at the limit of ψ or be dense in $\mathbb{R}$. They only affect part (d), that is, whether the anomalies destroy ordinary convergence. Of the assumptions on the corruption, parts (a) and (c) use only that the anomalous values form a countable set. The proof of (b) uses in addition that $t_k \to \infty$, so that ψ is observed without corruption on each interval (t_k, t_{k+1}).

Remark 8. (i) *Observed scalar flows.* Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz, let x^* be a globally asymptotically stable equilibrium of $\dot{x} = g(x)$, and let ψ be the solution with $\psi(0) = x_0$, which is defined on $[0, \infty)$ and satisfies $\psi(t) \to x^*$; for instance $\psi(t) = x^* + (x_0 - x^*)e^{-t}$ when $g(x) = -(x - x^*)$. Suppose that ψ is recorded continuously, but that at the times t_k the recorded value is replaced by a spurious value r_k, where (t_k) and (r_k) are as in Proposition 5 with $D = [0, \infty)$, so that the recorded signal is the net φ of Proposition 5. If ψ were constant on some interval, its value there would be an equilibrium and, by uniqueness of solutions, ψ would be constant for all time; since $\psi(t) \to x^*$, this forces $x_0 = x^*$. Hence, by Proposition 5(b) and (c), $\mathcal{L}_r(\varphi) = \{x^*\}$ if $x_0 \neq x^*$ and $\mathcal{L}_r(\varphi) = \mathbb{R}$ if $x_0 = x^*$, for every choice of the values r_k; and $\varphi(t) \to x^*$ if and only if $r_k \to x^*$ (Proposition 5(d)).

(ii) *Comparison with the ω-limit set.* The ω-limit set $\omega(\varphi) = \bigcap_s \operatorname{cl}(\varphi([s, \infty)))$ of the recorded signal in (i) is its set of cluster points; it consists of x^* together with the cluster points of (r_k). It collects limits of subnets, whereas $\mathcal{L}_r(\varphi)$ collects rare limits of the whole net. By Theorem 3(iii), $\mathcal{L}_r(\varphi) \subseteq \omega(\varphi)$ unless $\mathcal{L}_r(\varphi) = \mathbb{R}$, and neither inclusion holds for all choices of x_0 and (r_k): if $r_k = r \neq x^*$ for all k, then $\omega(\varphi) = \{x^*, r\}$, while $\mathcal{L}_r(\varphi) = \{x^*\}$ for $x_0 \neq x^*$ and $\mathcal{L}_r(\varphi) = \mathbb{R}$ for $x_0 = x^*$.

(iii) *Relation to ideal convergence.* Let $E = \{t_k : k \geq 1\}$ and $\mathcal{I}_E = \{A \subseteq D : A \setminus E \text{ is bounded}\}$, a proper ideal of subsets of D because $D \setminus E$ is unbounded. A net $(x_t)_{t \in D}$ is $\mathcal{I}_E$-convergent to y if and only if it converges to y along $D \setminus E$. For every net $(x_t)_{t \in D}$ in $\mathbb{R}$, $\mathcal{I}_E$-convergence to y implies rare convergence to y: for an open $G \ni y$ and large s, the values of the tail T_s outside G are taken at times in the countable set E, so $T_s \setminus G$ is countable and Lemma 1 applies. For the net φ, since ψ is continuous and $D \setminus E$ is dense in D, φ converges to y along $D \setminus E$ if and only if $\psi(t) \to y$. Hence, under the hypotheses of Proposition 5(b), φ rarely converges to y if and only if it is $\mathcal{I}_E$-convergent to y. In case (c), rare convergence does not imply $\mathcal{I}_E$-convergence: if $\psi(t) = c$ for large t, then φ rarely converges to every y but is $\mathcal{I}_E$-convergent only to c. A conceptual difference is that $\mathcal{I}_E$ depends on E, whereas Definition 3 does not refer to E. What the proof of Proposition 5(b) uses is that each arc $\psi((t_k, t_{k+1}))$ is observed without corruption and that infinitely many of these arcs are non-degenerate intervals; here this follows from $t_k \to \infty$ together with the assumption that ψ is not eventually constant. By contrast, for sequences in general spaces rare convergence and $\mathcal{I}$-convergence are independent (Proposition 1(b)).

(iv) *State resets.* Proposition 5 concerns corrupted observations, not impulsive dynamics. Suppose instead that, in (i) with $g(x) = -(x - x^*)$, the state itself is reset to r_k at each time t_k and then follows the flow, so that $x(t) = x^* + (x_0 - x^*)e^{-t}$ for $0 \leq t < t_1$ and $x(t) = x^* + (r_k - x^*)e^{-(t-t_k)}$ for $t \in [t_k, t_{k+1})$. Then, in this reset model, $x(t)$ rarely converges to x^* if and only if $r_k \to x^*$, that is, if and only if $x(t)$ converges to x^* in the ordinary sense. Indeed, if $r_k \to x^*$, then $|x(t) - x^*| \leq |r_k - x^*|$ on $[t_k, t_{k+1})$, and $x(t) \to x^*$. Otherwise there are $d > 0$ and infinitely many k with $|r_k - x^*| \geq d$. Let $G = (x^* - d/2, x^* + d/2)$ and

$\delta_k = \min(\ln 2,\, t_{k+1} - t_k)$. For each such k and $t \in [t_k, t_k + \delta_k)$ we have $|x(t) - x^*| > d/2$, and x maps this interval onto a non-degenerate interval. So every tail minus G has non-empty interior, and by Lemma 1 $x(t)$ does not rarely converge to x^*.

(v) *Higher dimensions.* Let $\psi : [0,\infty) \to \mathbb{R}^n$, $n \geq 2$, be locally Lipschitz (for example a solution, defined on $[0,\infty)$, of $\dot{x} = f(x)$ with f continuous; such a solution is C^1), and define φ as in Proposition 5 with arbitrary values $r_k \in \mathbb{R}^n$. Each set $\psi([m, m+1])$ has finite one-dimensional Hausdorff measure and hence Lebesgue measure zero, so $\psi([0,\infty)) \cup \{r_k : k \geq 1\}$ is a Lebesgue null set and has empty interior. Hence every tail of φ is rare and $\mathcal{L}_r(\varphi) = \mathbb{R}^n$ by Corollary 1, whether or not ψ converges. Thus for locally Lipschitz ψ, in particular for such solutions, only the universal case occurs when $n \geq 2$; for such ψ a rare limit set $\{x^*\}$ as in (i) can occur only for $n = 1$. For continuous paths that are not locally Lipschitz, a singleton rare limit set can occur in $\mathbb{R}^2$. For $j \geq 1$ let $Q_j \subseteq B(x^*, 1/j)$ be closed squares with non-empty interior, and let $\psi : [2,\infty) \to \mathbb{R}^2$ be a continuous path that maps $[2j, 2j+1]$ onto Q_j (a Peano curve) and runs along the line segment from $\psi(2j+1) \in Q_j$ to $\psi(2j+2) \in Q_{j+1}$ on $[2j+1, 2j+2]$; then $\psi(t) \to x^*$. Choose the exceptional times outside $\bigcup_j [2j, 2j+1]$, for instance $t_k = 2k + \frac{3}{2}$, and define φ as in Proposition 5 with $D = [2,\infty)$ and arbitrary values $r_k \in \mathbb{R}^2$. Every open $W \ni x^*$ contains Q_j for all large j, so $\operatorname{int}(T_s \cap W) \neq \emptyset$ for every tail T_s of φ. Moreover $\varphi \xrightarrow{r} x^*$, as in Proposition 5(a), since countable sets are rare in $\mathbb{R}^2$. By Corollary 2, $\mathcal{L}_r(\varphi) = \{x^*\}$.

3. Rare Net Convergence and Continuity

We now examine how rare net convergence interacts with continuous and rarely continuous functions. The situation is analogous to the sequential case in [14], but now holds for arbitrary nets, removing the index countability constraints of the sequential case.

Example 7. (Continuous functions need not preserve rare net convergence.) Let $X = (0,1) \cup (2,3)$ with the subspace topology inherited from $\mathbb{R}$ equipped with its standard topology, and $Y = \{0,1\}$ with the discrete topology. Define $f : X \to Y$ by $f(x) = 0$ for $x \in (0,1)$ and $f(x) = 1$ for $x \in (2,3)$. Since $f^{-1}(\{0\}) = (0,1)$ and $f^{-1}(\{1\}) = (2,3)$ are both open in X, the function f is continuous.

Let $x^* = \frac{5}{2} \in (2,3)$ and let $D = (\mathcal{N}(x^*), \supseteq)$ be the set of open neighbourhoods of x^* directed by reverse inclusion. Define the net $(x_U)_{U \in D}$ by $x_U = \frac{1}{2}$ for all $U \in D$.

Ordinary convergence fails in X. For the open neighbourhood $(2,3)$ of x^* we have $x_U = \frac{1}{2} \notin (2,3)$ for every $U \in D$, so the net is not eventually in this neighbourhood of x^*.

Rare convergence in X. Every singleton of X has empty interior, so $\frac{1}{2}$ is a rare element of X and the net is constant at it. By Proposition 2(i), (x_U) rarely converges to every point of X; in particular $(x_U) \xrightarrow{r} x^*$.

Rare convergence of the image fails in Y. We have $f(x_U) = 0$ for all $U \in D$. Consider the open neighbourhood $G_Y = \{1\}$ of $f(x^*) = 1$ in Y. Since Y is discrete, the only rare set is $\emptyset$, so any R_{G_Y} must be $\emptyset$. But $f(x_U) = 0 \notin G_Y \cup \emptyset = \{1\}$, so $(f(x_U))$ does not rarely converge to $f(x^*) = 1$.

Hence f is continuous but does not preserve rare net convergence. Here f maps the rare set $\{\frac{1}{2}\}$ onto the open set $\{0\}$ of Y. Theorem 5 below shows that a continuous map which sends rare sets to rare sets does preserve rare convergence. That condition is sufficient but not necessary: a constant map $\mathbb{R} \to \{0,1\}$ (discrete) sends the rare singletons of $\mathbb{R}$ onto an open singleton, yet every image net is constant and therefore convergent.

Theorem 5. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a continuous function that maps rare sets of X to rare sets of Y; by Theorem 1, this holds in particular if f is one-to-one. If $(x_\lambda) \xrightarrow{r} x$ in X, then $(f(x_\lambda)) \xrightarrow{r} f(x)$ in Y.*

Proof. Let G be an open set containing $f(x)$ in Y. Then $f^{-1}(G)$ is an open neighbourhood of x, so by Lemma 1 there is λ_0 such that $T_{\lambda_0} \setminus f^{-1}(G)$ is rare in X. The tail of the image net from λ_0 on is $f(T_{\lambda_0})$, and

$$f(T_{\lambda_0}) \setminus G = f\big(T_{\lambda_0} \setminus f^{-1}(G)\big),$$

which is rare in Y by hypothesis. By Lemma 1, $(f(x_\lambda)) \xrightarrow{r} f(x)$. □

For one-to-one maps, Theorem 5 is the net version of [14, Theorem 14]; the proof above shows that injectivity is used only to guarantee that rare sets are mapped to rare sets.

Example 8. (Rarely continuous functions need not preserve ordinary net convergence.) Let $X = Y = \mathbb{R}$ with the standard topology. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

This function is rarely continuous [17].

Let $x^* \in \mathbb{Q}^c$ and let $D = (\mathcal{N}(x^*), \supseteq)$ be the set of open neighbourhoods of x^* directed by reverse inclusion. For each $U \in D$, choose $x_U \in U \cap \mathbb{Q}$ (possible since $\mathbb{Q}$ is dense). Then $(x_U) \to x^*$ in $\mathbb{R}$, since for any open $G \ni x^*$ the net is eventually in G. However, $f(x_U) = 0$ for all $U \in D$, so $(f(x_U))$ does not converge to $f(x^*) = 1$.

Nevertheless, $(f(x_U)) \xrightarrow{r} f(x^*)$, as guaranteed by Theorem 6 below. We stress that Theorem 6 concerns a specific pairing: it takes an *ordinarily* convergent net in X and produces a *rarely* convergent net in Y. It does *not* assert that rarely continuous functions preserve rare net convergence in the same sense (input rare, output rare). The latter fails in general, as Example 9 below demonstrates. The three preservation questions — (i) ordinary $\to$ ordinary, (ii) ordinary $\to$ rare, (iii) rare $\to$ rare — are logically distinct and are addressed separately in this section.

Theorem 6. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a rarely continuous function. If $(x_\lambda)_{\lambda \in D}$ converges to x in X, then $(f(x_\lambda))$ rarely converges to $f(x)$ in Y.*

Proof. Let G be an open set containing $f(x)$ in Y. Since f is rarely continuous, there exist a rare set R_G in Y with $G \cap \mathrm{cl}(R_G) = \emptyset$ and an open set U containing x in X such that $f(U) \subseteq G \cup R_G$. Since $(x_\lambda) \to x$ and U is a neighbourhood of x, there exists λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$. Hence $f(x_\lambda) \in f(U) \subseteq G \cup R_G$ for all $\lambda \geq \lambda_0$. Therefore $(f(x_\lambda)) \xrightarrow{r} f(x)$. □

Example 9. (A rarely continuous function failing to preserve rare net convergence.) To demonstrate that a rarely continuous function need not preserve rare net convergence, we construct a counterexample within the standard Euclidean space $\mathbb{R}$. (Example 7 already gives a continuous, hence rarely continuous, map with this property; the present example shows the same phenomenon for a continuous self-map of $\mathbb{R}$.)

Let $X = \mathbb{R}$ and $Y = \mathbb{R}$, both equipped with the standard topology. Let $C \subset [0, 1]$ be the standard Cantor ternary set. Since C is closed and nowhere dense, $\mathrm{int}(C) = \emptyset$, making C a rare set in X. Let $f : \mathbb{R} \to \mathbb{R}$ be the Cantor–Lebesgue function (the Devil's staircase) [25], the non-decreasing continuous function satisfying $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$. This function is uniformly continuous on $\mathbb{R}$ and maps the Cantor set C surjectively onto $[0, 1]$: $f(C) = [0, 1]$ [25].

f is rarely continuous everywhere. Since the Cantor–Lebesgue function is uniformly continuous on $\mathbb{R}$ in the standard ordinary sense, it trivially satisfies the definition of rare continuity at every point (by taking the anomaly set $R_G = \emptyset$).

Rare net convergence holds in the domain. Let $a = 2$. Clearly, $f(a) = 1$. Consider the directed set $D = \{(U, x) : U \in \mathcal{N}(a), x \in U \cup C\}$ ordered by $(U_1, x_1) \geq (U_2, x_2) \iff U_1 \subseteq U_2$. Define the net $(x_\lambda)_{\lambda \in D}$ by $x_{(U,x)} = x$.

Let G be any open neighbourhood of a. We define the anomaly set as $R_G = C \setminus G$. Because C is a closed nowhere dense set and G is open, R_G is closed. Thus, $\mathrm{cl}(R_G) = R_G = C \setminus G$, which guarantees $G \cap \mathrm{cl}(R_G) = \emptyset$. Furthermore, since $R_G \subseteq C$, it inherits the nowhere dense property, meaning $\mathrm{int}(R_G) = \emptyset$. Hence, R_G is a valid rare set for G.

Let $\lambda_0 = (G, a)$. For any $\lambda = (U, x) \geq \lambda_0$, we have $U \subseteq G$. By the definition of the directed set, $x \in U \cup C$. Since $U \subseteq G$, we obtain $x \in G \cup C$. Notice that $G \cup C = G \cup (C \setminus G) = G \cup R_G$. Therefore, $x_\lambda \in G \cup R_G$ for all $\lambda \geq \lambda_0$. This proves $(x_\lambda) \xrightarrow{r} a$.

Rare convergence fails in the codomain. We examine whether the image net $(f(x_\lambda))$ rarely converges to $f(a) = 1$. Let $V = (0.8, 1.2)$ be an open neighbourhood of 1. Suppose $(f(x_\lambda)) \xrightarrow{r} 1$. Then there exist a rare set $R^* \subset \mathbb{R}$ and an index $\lambda_0 = (U_0, x_0)$ such that for all $(U, x) \geq \lambda_0$, we have $f(x) \in V \cup R^*$.

For any $c \in C$, the index (U_0, c) satisfies $(U_0, c) \geq (U_0, x_0)$ since $U_0 \subseteq U_0$. Thus, $f(c) \in V \cup R^*$ for all $c \in C$. This implies that $f(C) \subseteq V \cup R^*$.

A fundamental property of the Cantor–Lebesgue function is that it maps the nowhere dense Cantor set C onto the entire non-degenerate closed interval $[0,1]$ [25]. Thus, $f(C) = [0,1]$.

Substituting this equality into the preceding inclusion, we obtain $[0,1] \subseteq V \cup R^*$. Since $V = (0.8, 1.2)$, the closed interval $[0, 0.8]$ is disjoint from V. Therefore, the entire interval $[0, 0.8]$ must be absorbed by R^*, yielding $[0, 0.8] \subseteq R^*$. However, $[0, 0.8]$ contains a non-empty open interval (e.g., $(0, 0.8)$), meaning $\text{int}(R^*) \supseteq (0, 0.8) \neq \emptyset$. This contradicts the assumption that R^* is a rare set.

Hence, $(f(x_\lambda)) \overset{r}{\not\to} f(a)$. In this example the failure comes from two features together: f maps the rare set C onto $[0,1]$, which has non-empty interior, and every tail of the net contains all of C. We do not claim that mapping a rare set onto a set with non-empty interior always destroys rare convergence; the constant map of Example 7 shows that it does not. Theorem 7 below gives sufficient conditions for preservation, and Remark 9 shows that neither of them can be dropped.

Theorem 7. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a rarely continuous function that maps rare sets to rare sets. Suppose further that (Y, τ_2) is a space in which the union of any two rare sets is rare. Then f preserves rare net convergence: if $(x_\lambda) \overset{r}{\to} x$ in X, then $(f(x_\lambda)) \overset{r}{\to} f(x)$ in Y.*

Proof. Let G be an open neighbourhood of $f(x)$ in Y. Since f is rarely continuous, there exist a rare set $R_G \subset Y$ with $G \cap \text{cl}(R_G) = \emptyset$ and an open neighbourhood U of x in X such that $f(U) \subseteq G \cup R_G$.

Since $(x_\lambda) \overset{r}{\to} x$ and U is an open neighbourhood of x, there exist a rare set $R_U \subset X$ with $U \cap \text{cl}(R_U) = \emptyset$ and λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U \cup R_U$.

For all $\lambda \geq \lambda_0$, either $x_\lambda \in U$, in which case $f(x_\lambda) \in f(U) \subseteq G \cup R_G$, or $x_\lambda \in R_U \setminus U$, in which case $f(x_\lambda) \in f(R_U)$. In both cases,

$$f(x_\lambda) \in (G \cup R_G) \cup f(R_U).$$

By hypothesis, $f(R_U)$ is rare in Y. Since (Y, τ_2) admits rare-union closure, $R^* = R_G \cup f(R_U)$ is rare. Set $R_{\text{fin}} = R^* \setminus G$. Then R_{fin} is rare, $R_{\text{fin}} \subseteq G^c$ gives $G \cap \text{cl}(R_{\text{fin}}) = \emptyset$, and $G \cup R^* = G \cup R_{\text{fin}}$. Therefore $f(x_\lambda) \in G \cup R_{\text{fin}}$ for all $\lambda \geq \lambda_0$, proving $(f(x_\lambda)) \overset{r}{\to} f(x)$. □

Remark 9. Neither of the two additional hypotheses of Theorem 7, that f maps rare sets to rare sets and that the union of two rare sets of Y is rare, can be omitted.

(a) Preservation of rare sets. In Example 7 the codomain Y is discrete, so its only rare set is $\emptyset$ and the union condition holds trivially; f is continuous, hence rarely continuous. However, f does not map rare sets to rare sets: it maps the rare set $\{\frac{1}{2}\}$ of X onto the open set $\{0\}$ of Y. As shown there, f does not preserve rare convergence.

(b) Rare-union closure. Let $X = \{1,2,3\}$ with $\tau_X = \{\emptyset, \{3\}, \{1,3\}, \{2,3\}, X\}$, let $Y = \{a,b,c,d\}$ with $\tau_Y = \{\emptyset, \{d\}, \{a,c\}, \{b,d\}, \{a,c,d\}, Y\}$, and let $f : X \to Y$ be given by $f(1) = a$, $f(2) = b$, $f(3) = c$.

Every non-empty open set of X contains 3, so a subset of X is rare if and only if it does not contain 3. Thus the rare sets of X are

$$\emptyset,\ \{1\},\ \{2\},\ \{1,2\}.$$

The minimal non-empty open sets of Y are $\{d\}$ and $\{a,c\}$, so a subset of Y is rare if and only if it contains neither d nor both a and c. Thus the rare sets of Y are

$$\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{b,c\}.$$

Hence f maps the rare sets $\emptyset, \{1\}, \{2\}, \{1,2\}$ of X onto the rare sets $\emptyset, \{a\}, \{b\}, \{a,b\}$ of Y, so f maps rare sets to rare sets. However, Y does not satisfy the rare-union condition: the rare sets $\{a\}$ and $\{c\}$ of Y have the open union $\{a,c\}$.

The map f is rarely continuous. It is continuous at 1 and at 3: every open set of Y containing $f(1) = a$ or $f(3) = c$ contains $\{a,c\}$, and the open sets $U = \{1,3\}$ and $U = \{3\}$ of X satisfy $f(U) \subseteq \{a,c\}$, so $R_G = \emptyset$ works. At 2, the only open set of Y containing $f(2) = b$ other than Y is $G = \{b,d\}$. Every open set of X containing 2 also contains 3, so we take the open set $U = \{2,3\}$ of X and $R_G = \{c\}$; this set is rare in Y, $\text{cl}_Y(\{c\}) = \{a,c\}$ is disjoint from G, and $f(U) = \{b,c\} \subseteq G \cup R_G$.

The sequence $1,3,1,3,\dots$ rarely converges to 2 in X: for the open set $G=\{2,3\}$ of X, every tail satisfies $T\setminus G=\{1\}$, which is rare in X, and for $G=X$ there is nothing to prove (Lemma 1). Its image $a,c,a,c,\dots$ does not rarely converge to $f(2)=b$ in Y: for the open set $G=\{b,d\}$ of Y, every tail satisfies $T\setminus G=\{a,c\}$, which is open in Y.

The rare-union closure condition is restrictive. It fails already in $\mathbb{R}$: $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are rare, but their union is $\mathbb{R}$; in particular, Theorem 7 does not apply to maps into $\mathbb{R}$. It also fails in the cofinite topology on $\mathbb{N}$, where the even and the odd natural numbers are rare but their union $\mathbb{N}$ is open. It holds in every space in which the rare locus $X\setminus \mathrm{Iso}(X)$ has empty interior (then every rare set is contained in this rare set, and so is every union of rare sets); examples are the Sierpiński space, whose only non-empty rare set is $\{0\}$, and the finite space $(\{0,1,2\},\tau)$ with $\tau=\{\emptyset,\{0\},\{0,1\},\{0,1,2\}\}$, whose rare sets are exactly the subsets of $\{1,2\}$.

3.1. Products

Theorem 8. *Let $\{(X_\alpha,\tau_\alpha):\alpha\in I\}$ be a family of topological spaces, let $X=\prod_{\alpha\in I}X_\alpha$ carry the product topology, and let D be a directed set. For each α let $(x_\lambda^\alpha)_{\lambda\in D}$ be a net in X_α with $x_\lambda^\alpha \xrightarrow{r} x^\alpha$. Then the net $x_\lambda=(x_\lambda^\alpha)_{\alpha\in I}$ rarely converges to $x=(x^\alpha)_{\alpha\in I}$ in X.*

Proof. By Lemma 1(ii) it suffices to consider basic open neighbourhoods $B=\prod_\alpha G_\alpha$ of x, where G_α is open in X_α and $G_\alpha=X_\alpha$ for α outside a finite set $F\subseteq I$. Write T_μ^α for the tails of the α-th coordinate net. For $\alpha\in F$ choose λ_α with $\mathrm{int}(T_{\lambda_\alpha}^\alpha\setminus G_\alpha)=\emptyset$, and choose $\lambda_0\geq\lambda_\alpha$ for all $\alpha\in F$. Put $A_\alpha=T_{\lambda_0}^\alpha\setminus G_\alpha$; then $\mathrm{int}(A_\alpha)=\emptyset$. If $\lambda\geq\lambda_0$ and $x_\lambda\notin B$, then $x_\lambda^\alpha\notin G_\alpha$ for some $\alpha\in F$, so $x_\lambda\in\pi_\alpha^{-1}(A_\alpha)$. Hence

$$T_{\lambda_0}\setminus B\subseteq\bigcup_{\alpha\in F}\pi_\alpha^{-1}(A_\alpha).$$

This finite union has empty interior. Indeed, a non-empty open subset of X contains a non-empty basic open set $W=\prod_\alpha W_\alpha$. For $\alpha\in F$ the non-empty open set W_α is not contained in A_α, so we may pick $q_\alpha\in W_\alpha\setminus A_\alpha$; for $\alpha\notin F$ pick any $q_\alpha\in W_\alpha$. Then $q=(q_\alpha)\in W$ lies outside every $\pi_\alpha^{-1}(A_\alpha)$, $\alpha\in F$. Therefore $\mathrm{int}(T_{\lambda_0}\setminus B)=\emptyset$, and Lemma 1 completes the proof. □

Example 10. (The converse of Theorem 8 fails.) Let $D=(\mathbb{R},\leq)$ and consider the net $x_\lambda=(\lambda,0)$ in $\mathbb{R}^2$. Its tails lie in the x-axis $\mathbb{R}\times\{0\}$, which has empty interior in $\mathbb{R}^2$, so by Corollary 1 the net rarely converges to every point of $\mathbb{R}^2$. Its first coordinate $x_\lambda=\lambda$, however, does not rarely converge to any point of $\mathbb{R}$ (Remark 6). In particular, the projection $\pi_1:\mathbb{R}^2\to\mathbb{R}$, although continuous, open and surjective, does not preserve rare convergence; this is consistent with Theorem 5, since π_1 maps the rare set $\mathbb{R}\times\{0\}$ onto $\mathbb{R}$.

4. Rarely Net-Open Sets and the Rarely Net-Open Topology

A central question is how rare net convergence relates to ordinary net convergence in a suitable topology on X. To answer this, we introduce the notion of a *rarely net-open* set (Definition 5) and, via Theorem 9, show that the collection τ_{rn} of such sets forms a topology on X characterised by a simple condition: a non-empty set belongs to τ_{rn} if and only if it is open and contains every rare set. The relationship between rare convergence in (X,τ) and ordinary convergence in (X,τ_{rn}) is then examined in the rest of the section. Theorem 11 below establishes *one* direction of the desired equivalence: rare convergence in (X,τ) implies ordinary convergence in (X,τ_{rn}). The converse direction, which would be required to say that τ_{rn} is exactly the topology induced by rare convergence, does *not* hold in general. In fact, as we shall see in Example 11 below, on $\mathbb{R}$ with the usual topology the rarely net-open topology τ_{rn} is the indiscrete topology $\{\emptyset,\mathbb{R}\}$, in which every net converges (ordinarily) to every point; yet by Remark 6 the net $x_\lambda=\lambda$ indexed by $(\mathbb{R},\leq)$ does not rarely converge to any $x\in\mathbb{R}$ in $(\mathbb{R},\tau)$. Hence rare convergence in (X,τ) implies, but is in general not equivalent to, ordinary convergence in (X,τ_{rn}), and τ_{rn} is best regarded as a topology *induced by* rare convergence in the one-sided sense above, not as a full topologisation of the convergence structure.

By Theorem 9, $\tau_{rn}\subseteq\tau$ (Remark 10), with equality if and only if τ is discrete or indiscrete (Propositions 7 and 8).

Definition 5. A subset A of a topological space (X, τ) is called rarely net-open if for every net (x_λ) in X that rarely converges to a point $a \in A$, there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in A$.

We now characterise rarely net-open sets analogously to Theorem 2.

Theorem 9. *Let (X, τ) be a topological space. A subset $A \subseteq X$ is rarely net-open if and only if $A = \emptyset$ or ($A \in \tau$ and A contains all rare sets in (X, τ)).*

Proof. We use the fact that in any topological space, net-open sets coincide with open sets: $\tau_n = \tau$ [15]. The empty set $\emptyset$ is vacuously rarely net-open, satisfying the theorem. Hence, assume $A \neq \emptyset$.

($\Rightarrow$) Suppose $A \neq \emptyset$ is rarely net-open.

A is open. Let $a \in A$ and let $(x_\lambda) \to a$ ordinarily in (X, τ). Since ordinary convergence trivially implies rare convergence, by the definition of a rarely net-open set, there exists λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in A$. Since every net converging to a is eventually in A, and since $\tau_n = \tau$, we conclude $A \in \tau$.

A contains all rare sets. Let R be any rare set, and let $a \in A$ (which exists since $A \neq \emptyset$). Take any $r \in R$. Since R is a rare set, no element of R can be an isolated point in X (otherwise $\{r\}$ would be an open set contained in R, contradicting $\text{int}(R) = \emptyset$). Hence, each $r \in R$ is a rare element.

Therefore, Proposition 2(ii) applies: the alternating net $r, a, r, a, \ldots$ rarely converges to a. Because A is rarely net-open and $a \in A$, this net must eventually remain within A. However, the net frequently visits r, which forces $r \in A$. Since $r \in R$ was arbitrary, we conclude $R \subseteq A$.

($\Leftarrow$) Suppose $A \neq \emptyset$, $A \in \tau$, and A contains all rare sets. Let $a \in A$ and suppose $(x_\lambda) \xrightarrow{r} a$. Since A is τ-open and $a \in A$, we may apply Definition 3 with the open neighbourhood $G = A$. There exist a rare set R_A with $A \cap \text{cl}(R_A) = \emptyset$ and an index λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in A \cup R_A$.

Since A contains all rare sets, we know $R_A \subseteq A$. However, we also have $R_A \subseteq \text{cl}(R_A)$. Thus, $R_A \subseteq A \cap \text{cl}(R_A) = \emptyset$, which in turn forces $R_A = \emptyset$. Therefore, the condition $x_\lambda \in A \cup R_A$ reduces exactly to $x_\lambda \in A$ for all $\lambda \geq \lambda_0$. Hence, A is rarely net-open. □

Remark 10. Theorem 9 implies that τ_{rn} is a topology on X. Let $U = \bigcup\{R : R \text{ is rare in } X\}$ be the union of all rare sets in X (by Remark 1, $U = U_{\text{rare}} = X \setminus \text{Iso}(X)$). We verify the three axioms.

(i) $\emptyset \in \tau_{rn}$ by Theorem 9, and $X \in \tau_{rn}$ since $X \in \tau_n$ and $U \subseteq X$.

(ii) *Arbitrary unions.* Let $\{A_i\}_{i \in I} \subseteq \tau_{rn}$. Empty members do not change the union, and the union of an empty family is $\emptyset \in \tau_{rn}$, so we may assume that every A_i is non-empty. Then each A_i is net-open and satisfies $U \subseteq A_i$. Since τ_n is a topology, $\bigcup_i A_i \in \tau_n$. Since $U \subseteq A_i \subseteq \bigcup_i A_i$ for each i, we have $U \subseteq \bigcup_i A_i$. Hence $\bigcup_i A_i \in \tau_{rn}$.

(iii) *Finite intersections.* Let $A_1, A_2 \in \tau_{rn}$. If one of them is empty, so is $A_1 \cap A_2$. Otherwise $A_1 \cap A_2 \in \tau_n$ (since τ_n is a topology) and $U \subseteq A_1$, $U \subseteq A_2$ implies $U \subseteq A_1 \cap A_2$. Hence $A_1 \cap A_2 \in \tau_{rn}$.

Moreover, by Theorem 9, $\tau_{rn} \subseteq \tau$ always.

Definition 6. Let (X, τ) be a topological space. The topology τ_{rn} of all rarely net-open sets is called the rarely net-open topology, and (X, τ_{rn}) is called the rarely net-open space.

Proposition 6. *Let (X, τ) be a topological space. Then*

$$\tau_{rn} = \{\emptyset\} \cup \{X \setminus F : F \subseteq \text{Iso}(X) \text{ and } F \text{ is closed in } (X, \tau)\}.$$

Every subset of $\text{Iso}(X)$ *is open, so the sets F occurring here are exactly the clopen subsets of X consisting of isolated points, and every member of τ_{rn} is clopen in (X, τ).*

Proof. By Theorem 9 and Remark 1, a non-empty set A belongs to τ_{rn} if and only if $A \in \tau$ and $A \supseteq X \setminus \text{Iso}(X)$, that is, if and only if $F := X \setminus A$ is closed and $F \subseteq \text{Iso}(X)$. A subset of $\text{Iso}(X)$ is a union of open singletons, hence open; so such an F is clopen, and so is $A = X \setminus F$. □

Example 11. Consider $(\mathbb{R}, \tau_u)$ with the standard topology.

Rare sets. Since $\text{int}(\{x\}) = \emptyset$ for every $x \in \mathbb{R}$, every singleton is rare. Hence the union of all rare sets is $U = \mathbb{R}$.

The rarely net-open topology. Since $\tau_n = \tau_u$, by Theorem 9 any non-empty $A \in \tau_{rn}$ must satisfy $\mathbb{R} \subseteq A$, hence $A = \mathbb{R}$. Therefore

$$\tau_{rn} = \{\emptyset, \mathbb{R}\},$$

the indiscrete topology, even though τ_u is not indiscrete.

Example 12. Let X be a set with at least two elements, $p \in X$ a fixed point, and $\tau_p = \{\emptyset\} \cup \{A \subseteq X : p \in A\}$ the particular point topology on X.

Rare sets. A set $R \subseteq X$ is rare in (X, τ_p) if and only if $p \notin R$: indeed, $\text{int}(R) \neq \emptyset$ iff R contains a non-empty open set iff $p \in R$. Hence the union of all rare sets is $U = X \setminus \{p\}$.

The rarely net-open topology. Since $\tau_n = \tau_p$, by Theorem 9 a non-empty $A \in \tau_{rn}$ must satisfy $A \in \tau_p$ (so $p \in A$) and $X \setminus \{p\} \subseteq A$. The only set satisfying both conditions is $A = X$. Therefore

$$\tau_{rn} = \{\emptyset, X\},$$

the indiscrete topology. Here p is the only isolated point, but $\{p\}$ is not closed (its closure is X), so the only closed set of isolated points is $\emptyset$; this is in accordance with Proposition 6 and Theorem 10 below.

Example 13. Let $X = \{1, 2, 3, 4\}$ with topology $\tau = \{\emptyset, \{1\}, \{2, 3, 4\}, X\}$.

Rare sets. A set R is rare iff $\text{int}(R) = \emptyset$, i.e., R contains neither $\{1\}$ nor $\{2, 3, 4\}$ as a subset. Hence R is rare iff $1 \notin R$ and $\{2, 3, 4\} \not\subseteq R$. The rare sets are:

$$\mathcal{R} = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},$$

with union $U = \{2, 3, 4\}$.

The rarely net-open topology. By Theorem 9, a non-empty set A belongs to τ_{rn} if and only if $A \in \tau$ and $\{2, 3, 4\} \subseteq A$. Checking each element of τ: $\{1\} \not\supseteq \{2, 3, 4\}$, so $\{1\} \notin \tau_{rn}$; both $\{2, 3, 4\}$ and X contain U, so both belong to τ_{rn}. Therefore

$$\tau_{rn} = \{\emptyset, \{2, 3, 4\}, X\},$$

which lies strictly between the indiscrete and τ: $\{\emptyset, X\} \subsetneq \tau_{rn} \subsetneq \tau$. This shows that τ_{rn} need not be indiscrete and filters out open sets that do not absorb the rare locus.

Example 14. Let $X = \{1, 2, 3, 4, 5\}$ with topology

$$\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, X\}.$$

Rare sets. The minimal non-empty open sets are $\{1\}$, $\{2\}$, and $\{3, 4, 5\}$. A set R is rare iff it contains none of these, i.e., $1 \notin R$, $2 \notin R$, and $\{3, 4, 5\} \not\subseteq R$. Hence the rare sets are all subsets of $\{3, 4, 5\}$ except $\{3, 4, 5\}$ itself:

$$\mathcal{R} = \{\emptyset, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\},$$

with union $U = \{3, 4, 5\}$.

The rarely net-open topology. By Theorem 9, a non-empty set A belongs to τ_{rn} if and only if $A \in \tau$ and $\{3, 4, 5\} \subseteq A$. The elements of τ containing $\{3, 4, 5\}$ are $\{3, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$, and X. Therefore

$$\tau_{rn} = \{\emptyset, \{3, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, X\},$$

which satisfies $\{\emptyset, X\} \subsetneq \tau_{rn} \subsetneq \tau$, so τ_{rn} is neither indiscrete nor equal to τ. Both this example and Example 13 agree with Proposition 6: there $\text{Iso}(X) = \{1\}$ and $\{1\}$ is clopen; here $\text{Iso}(X) = \{1, 2\}$ and each of its subsets is clopen.

Example 15. (Spaces with a clopen singleton.) (a) Let $K = \{0\} \cup \{1/n : n \geq 1\}$ with the subspace topology of $\mathbb{R}$. Then $\mathrm{Iso}(K) = \{1/n : n \geq 1\}$, and a subset of $\mathrm{Iso}(K)$ is closed in K if and only if it is finite (an infinite subset accumulates at 0). By Proposition 6,

$$\tau_{rn} = \{\emptyset\} \cup \{K \setminus F : F \text{ a finite subset of } \{1/n : n \geq 1\}\}.$$

This coincides with the rarely sequential topology computed in [14, Example 20], as it must by Corollary 4, since K is metrizable.

(b) Let $X = [0,1] \cup \{2\}$ with the subspace topology of $\mathbb{R}$. Then $\mathrm{Iso}(X) = \{2\}$, which is clopen, and $\tau_{rn} = \{\emptyset, [0,1], X\}$.

Theorem 10. *Let (X,τ) be a non-discrete topological space. Then $\tau_{rn} \neq \{\emptyset, X\}$ if and only if (X,τ) has at least one clopen singleton, that is, an isolated point y of (X,τ) such that $\{y\}$ is closed in (X,τ). In particular, if (X,τ) is T_1, then $\tau_{rn} \neq \{\emptyset, X\}$ if and only if X has at least one isolated point.*

Proof. By Proposition 6, $\tau_{rn} \neq \{\emptyset, X\}$ if and only if there is a closed set $F \subseteq \mathrm{Iso}(X)$ with $F \neq \emptyset$ and $F \neq X$; since X is not discrete, $\mathrm{Iso}(X) \neq X$, so the condition $F \neq X$ is automatic. If such an F exists, pick $y \in F$; then $\mathrm{cl}(\{y\}) \subseteq F \subseteq \mathrm{Iso}(X)$. For $w \in \mathrm{cl}(\{y\})$ the set $\{w\}$ is an open neighbourhood of w, so it meets $\{y\}$ and $w = y$; hence $\{y\}$ is closed. Conversely, if $\{y\}$ is a clopen singleton, take $F = \{y\}$. In a T_1 space every singleton is closed. □

Corollary 3. *Let (X,τ) be a topological space with no isolated points. Then $\tau_{rn} = \{\emptyset, X\}$.*

Proof. Here $\mathrm{Iso}(X) = \emptyset$, so the only closed subset of $\mathrm{Iso}(X)$ is $\emptyset$, and Proposition 6 gives $\tau_{rn} = \{\emptyset, X\}$. □

Proposition 7. *Let (X,τ) be a topological space.*

(i) If τ is the discrete topology, then $\tau_{rn} = \tau$.
(ii) If τ is the indiscrete topology, then $\tau_{rn} = \tau$.

Proof. (i) In the discrete topology every singleton is open, so there are no non-empty rare sets and $U = \emptyset$. By Theorem 9, $A \in \tau_{rn}$ iff $A \in \tau_n$ and $\emptyset \subseteq A$, so $\tau_{rn} = \tau_n = \tau$.

(ii) If X has at most one point, then τ is also discrete and (i) applies. Otherwise, in the indiscrete topology $\mathrm{int}(R) = \emptyset$ for every $R \subsetneq X$, so $U = X$. By Theorem 9, any non-empty $A \in \tau_{rn}$ must satisfy $X \subseteq A$, hence $A = X$ and $\tau_{rn} = \{\emptyset, X\} = \tau$. □

Proposition 8. *Let (X,τ) be a topological space. If τ is different from both the discrete and the indiscrete topology, then $\tau \neq \tau_{rn}$.*

Proof. Assume for contradiction that $\tau_{rn} = \tau$.

Since τ is not discrete, there exists a point $x \in X$ such that $\{x\} \notin \tau$, i.e., $\mathrm{int}(\{x\}) = \emptyset$, so $\{x\}$ is rare.

Since τ is not indiscrete, there exists a proper non-empty open set $V \in \tau \setminus \{\emptyset, X\}$. Since $\tau_{rn} = \tau$, every non-empty open set must contain all rare sets by Theorem 9. In particular $V \in \tau_{rn}$, so $\{x\} \subseteq V$, i.e., $x \in V$.

Now consider any $y \in V^c$ (non-empty since V is proper). If $\{y\}$ were rare, then $V \in \tau_{rn}$ would require $y \in V$ by Theorem 9, contradicting $y \in V^c$. Hence $\{y\}$ is not rare, so $\mathrm{int}(\{y\}) \neq \emptyset$, which gives $\{y\} \in \tau = \tau_{rn}$.

Since $\{y\} \in \tau_{rn}$, Theorem 9 requires $\{y\}$ to contain every rare set in (X,τ). In particular, since $\{x\}$ is rare, we need $\{x\} \subseteq \{y\}$, which forces $x = y$. But $x \in V$ and $y \in V^c$, a contradiction. □

Remark 11. By Remark 10 and Proposition 8, $\tau_{rn} \subsetneq \tau$ whenever τ is neither discrete nor indiscrete, so information is lost in passing to τ_{rn}. By Corollary 3, τ_{rn} is even indiscrete in every space without isolated points, for instance in every metric space without isolated points. Theorem 10 identifies exactly where τ_{rn} retains some information: a non-discrete space has a non-trivial rarely net-open topology if and only if it has a clopen singleton. This is the case in Examples 13, 14 and 15, and in every T_1 space having both isolated and non-isolated points. It is not the case in any space without isolated points. Nor is it the case in a space X with at least two points that has a point g lying in every non-empty open set, such as the point 1 of the Sierpiński

space or the point p of the particular point topology of Example 12: no singleton other than $\{g\}$ is open, and $\{g\}$ is not closed because its closure is X. Finally, it is not the case if X has at least two points and some point has X as its only open neighbourhood, such as the point 0 of the Sierpiński space: such a point is a rare element, so every non-empty member of τ_{rn} is an open set containing it and therefore equals X. Even when τ_{rn} is non-trivial, it only records which clopen sets of isolated points exist (Proposition 6), and for non-discrete τ the space (X, τ_{rn}) is hyperconnected (Proposition 9(iii)). The topology τ_{rn} should therefore be regarded as a coarse invariant, and the results of this section should be read with this in mind.

Proposition 9. *Let (X, τ) be a topological space with at least two points.*

(i) *(X, τ_{rn}) is T_1 if and only if it is T_2, if and only if τ is discrete.*
(ii) *(X, τ_{rn}) is T_0 if and only if either τ is discrete, or $X \setminus \mathrm{Iso}(X) = \{u\}$ for some $u \in X$ and $\{y\}$ is closed in (X, τ) for every $y \neq u$ (equivalently, $u \notin \mathrm{cl}_\tau(\{y\})$ for every $y \neq u$).*
(iii) *If τ is not discrete, then any two non-empty members of τ_{rn} intersect; thus (X, τ_{rn}) is hyperconnected and, in particular, connected.*

Proof. If τ is discrete, then $\tau_{rn} = \tau$ by Proposition 7(i), and all three separation axioms hold. Suppose τ is not discrete and let $u \in X \setminus \mathrm{Iso}(X)$. By Theorem 9 and Remark 1, every non-empty member of τ_{rn} contains $X \setminus \mathrm{Iso}(X)$, and in particular u. This proves (iii). It also shows that for $y \neq u$ no member of τ_{rn} contains y but not u, so (X, τ_{rn}) is not T_1, hence not T_2; this proves (i).

(ii) Suppose τ is not discrete, so that $X \setminus \mathrm{Iso}(X) \neq \emptyset$. If $X \setminus \mathrm{Iso}(X) = \{u\}$, then for an isolated point y the conditions $u \notin \mathrm{cl}(\{y\})$, $\mathrm{cl}(\{y\}) \subseteq \mathrm{Iso}(X)$ and "$\{y\}$ is closed" are equivalent, as in the proof of Theorem 10.

($\Rightarrow$) Suppose (X, τ_{rn}) is T_0. Two points of $X \setminus \mathrm{Iso}(X)$ would belong to exactly the same non-empty members of τ_{rn}, so $X \setminus \mathrm{Iso}(X) = \{u\}$ for some u. Let $y \neq u$. Since every non-empty member of τ_{rn} contains u, a member distinguishing u and y contains u and not y; by Proposition 6 it has the form $X \setminus F$ with $y \in F \subseteq \mathrm{Iso}(X)$ and F closed. Then $\mathrm{cl}(\{y\}) \subseteq F \subseteq \mathrm{Iso}(X)$, so $\{y\}$ is closed.

($\Leftarrow$) Suppose $X \setminus \mathrm{Iso}(X) = \{u\}$ and $\{y\}$ is closed for every $y \neq u$. For $y \neq u$, the set $F = \{y\}$ is closed and consists of an isolated point, so $X \setminus \{y\} \in \tau_{rn}$ by Proposition 6; this set contains every point other than y. Given two distinct points of X, at least one of them, say y, differs from u, and $X \setminus \{y\}$ contains the other point but not y. Hence (X, τ_{rn}) is T_0. □

For instance, (K, τ_{rn}) in Example 15(a) is T_0 but not T_1, whereas the rarely net-open topologies of Examples 13 and 14 are not T_0. By (iii), for non-discrete τ the space (X, τ_{rn}) is connected; hence (X, τ) and (X, τ_{rn}) have the same number of connected components if and only if τ is discrete or (X, τ) is connected.

Theorem 11. *Let (X, τ) be a topological space. If (x_λ) rarely converges to x in (X, τ), then (x_λ) converges to x in (X, τ_{rn}).*

Proof. Let G be a τ_{rn}-open set containing x. By Theorem 9, G is net-open in (X, τ) and contains every rare set in (X, τ).

Since $(x_\lambda) \xrightarrow{r} x$ in (X, τ), for the open set $G \ni x$ there exist a rare set R_G in (X, τ) with $G \cap \mathrm{cl}(R_G) = \emptyset$ and λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in G \cup R_G$. Since R_G is rare and G contains every rare set, $R_G \subseteq G$. Therefore $G \cup R_G = G$, and $x_\lambda \in G$ for all $\lambda \geq \lambda_0$. Hence $(x_\lambda) \to x$ in (X, τ_{rn}). □

Remark 12. Theorem 11 is a one-sided statement: rare convergence in (X, τ) implies ordinary convergence in (X, τ_{rn}). The converse implication — that ordinary convergence in (X, τ_{rn}) implies rare convergence in (X, τ) — fails in general. A concrete counterexample was already discussed in the introduction to this section: for $X = \mathbb{R}$ with the standard topology, $\tau_{rn} = \{\emptyset, \mathbb{R}\}$ is indiscrete, so every net in $\mathbb{R}$ converges to every point in (X, τ_{rn}); but by Remark 6 the net $x_\lambda = \lambda$ indexed by $(\mathbb{R}, \leq)$ does not rarely converge to any point in (X, τ). Hence, in this example, some nets that converge in (X, τ_{rn}) do not rarely converge in (X, τ). In fact, rare convergence in $\mathbb{R}$ with the standard topology τ is not the convergence of any topology σ on $\mathbb{R}$; that is, there is no topology σ on $\mathbb{R}$ such that, for every net and every point y, the net rarely converges to y in $(\mathbb{R}, \tau)$ if and only if it σ-converges to y. Suppose there were. Every point $r \in \mathbb{R}$ is a rare element of $(\mathbb{R}, \tau)$, so the constant net at r rarely converges to every y (Proposition 2(i)) and hence, by assumption, σ-converges to every y. Thus

every σ-neighbourhood of every y contains every r, and σ is indiscrete. But then the net $x_\lambda = \lambda$ σ-converges to every point, so by assumption it rarely converges in $(\mathbb{R}, \tau)$ to every point, contradicting Remark 6.

Theorem 12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function between topological spaces. If f is rarely continuous, then $f : (X, \tau) \to (Y, \sigma_{rn})$ is continuous.*

Proof. Suppose $f : (X, \tau) \to (Y, \sigma)$ is rarely continuous. To show $f : (X, \tau) \to (Y, \sigma_{rn})$ is continuous, it is sufficient to show that for every net (x_λ) converging to x in (X, τ), the net $(f(x_\lambda))$ converges to $f(x)$ in (Y, σ_{rn}).

Let $(x_\lambda) \to x$ in (X, τ). Since f is rarely continuous, by Theorem 6, the image net $(f(x_\lambda))$ rarely converges to $f(x)$ in (Y, σ). Now, by Theorem 11, since $(f(x_\lambda)) \xrightarrow{r} f(x)$ in (Y, σ), it necessarily follows that the net $(f(x_\lambda))$ converges to $f(x)$ in (Y, σ_{rn}) in the ordinary sense. Thus, $f : (X, \tau) \to (Y, \sigma_{rn})$ is continuous. □

Remark 13. Theorem 12 says that every rarely continuous $f : (X, \tau) \to (Y, \sigma)$ is automatically continuous into the coarser codomain topology σ_{rn}. In view of Remark 11, the significance of this statement depends heavily on how coarse σ_{rn} is: whenever σ_{rn} is indiscrete (which happens, for example, when (Y, σ) has no isolated points), every function into (Y, σ_{rn}) is continuous, so the conclusion of Theorem 12 holds trivially. If σ is discrete, then $\sigma_{rn} = \sigma$ and, since Y has no non-empty rare sets, rare continuity is ordinary continuity, so the theorem says nothing new either. By Theorem 10 and Proposition 8, σ_{rn} is neither indiscrete nor equal to σ exactly when (Y, σ) is non-discrete and has a clopen singleton; only for such codomains can Theorem 12 say more than in the two cases just described.

Remark 14. The converse of Theorem 12 does not hold in general: continuity into (Y, σ_{rn}) does not imply rare continuity, even when σ_{rn} is non-trivial.

Let $X = \mathbb{R}$ with the standard topology, and let $Y = \mathbb{R} \sqcup \{p\}$ be the topological sum of $\mathbb{R}$ and a point p, with topology σ. Then $\mathrm{Iso}(Y) = \{p\}$, which is clopen, so $\sigma_{rn} = \{\emptyset, \mathbb{R}, Y\}$ by Proposition 6; this topology lies strictly between the indiscrete topology and σ. Define $f : \mathbb{R} \to Y$ by $f(x) = x$ for $x < 0$ and $f(x) = x + 2$ for $x \geq 0$. Since $f^{-1}(\mathbb{R}) = \mathbb{R}$, the map f is continuous into (Y, σ_{rn}).

Suppose that f is rarely continuous at 0, and let $G = (1.5, 2.5)$, an open neighbourhood of $f(0) = 2$ in Y. Then there are a rare set R_G of Y with $G \cap \mathrm{cl}(R_G) = \emptyset$ and an open set $U \ni 0$ with $f(U) \subseteq G \cup R_G$. Choose $0 < \delta \leq \frac{1}{2}$ with $(-\delta, \delta) \subseteq U$. Then $f((-\delta, \delta)) = (-\delta, 0) \cup [2, 2 + \delta)$, and since $(-\delta, 0) \cap G = \emptyset$ we get $(-\delta, 0) \subseteq R_G$. As $(-\delta, 0)$ is open in Y, this contradicts $\mathrm{int}(R_G) = \emptyset$. Hence f is not rarely continuous.

Theorem 13. *Let (X, τ) be a topological space. If τ is discrete, then $(\tau_{rn})_{rn} = \tau_{rn} = \tau$. If τ is not discrete, then (X, τ_{rn}) has no isolated points and $(\tau_{rn})_{rn} = \{\emptyset, X\}$.*

Proof. The discrete case follows from Proposition 7(i). Suppose τ is not discrete, so that $X \setminus \mathrm{Iso}(X) \neq \emptyset$, and suppose $\{x\} \in \tau_{rn}$ for some x. By Theorem 9, $\{x\} \in \tau$ and $\{x\} \supseteq X \setminus \mathrm{Iso}(X)$, so $X \setminus \mathrm{Iso}(X) = \{x\}$; thus x is not isolated in τ, contradicting $\{x\} \in \tau$. Hence (X, τ_{rn}) has no isolated points, and Corollary 3, applied to (X, τ_{rn}), gives $(\tau_{rn})_{rn} = \{\emptyset, X\}$. □

Theorem 13 is the net counterpart of [14, Theorems 33, 34 and Corollary 35], where the corresponding statement for $(\tau_{rs})_{rs}$ is proved under the assumption that X has at least two rare elements. That assumption is used in [14] because a rare element of τ may become isolated in τ_{rs} [14, Remark 32]; for nets this cannot happen, since $\tau_{rn} \subseteq \tau$. In the sequential case as well, the assumption can be weakened to non-discreteness (Corollary 5).

5. Relationship Between τ_{rn} and τ_{rs}

The rarely sequential topology τ_{rs} of [14] (Definition 2) is defined in terms of rarely convergent sequences. In this section we compare it with τ_{rn}. Theorem 14 establishes the inclusion $\tau_{rn} \subseteq \tau_{rs}$ in every topological space, and Theorem 15 gives a necessary and sufficient condition for equality. In particular, equality holds in every sequential space (Corollary 4), and also in some non-sequential spaces (Remark 15), while the inclusion is strict, for instance, in the Arens–Fort space (Example 16). Using the description of τ_{rs} in Theorem 15, we also

show that $(\tau_{rs})_{rs}$ is indiscrete for every non-discrete space (Corollary 5), which weakens the hypothesis of [14, Corollary 35] to non-discreteness.

Theorem 14. *For any topological space (X, τ), $\tau_{rn} \subseteq \tau_{rs}$.*

Proof. Clearly $\emptyset \in \tau_{rs}$. Let $\emptyset \neq A \in \tau_{rn}$. By Theorem 9, $A \in \tau$ and A contains all rare sets. Since every open set is sequentially open ($\tau \subseteq \tau_s$), A is sequentially open. By Theorem 2, $A \in \tau_{rs}$. □

Theorem 15. *Let (X, τ) be a topological space. Then*

$$\tau_{rs} = \{\emptyset\} \cup \{X \setminus F : F \subseteq \mathrm{Iso}(X) \text{ and } F \text{ is sequentially closed in } X\}.$$

Consequently, $\tau_{rn} = \tau_{rs}$ if and only if every subset of $\mathrm{Iso}(X)$ that is sequentially closed in X is closed in X.

Proof. *Description of τ_{rs}.* By Theorem 2 and Remark 1, a non-empty set A belongs to τ_{rs} if and only if A is sequentially open and $A \supseteq X \setminus \mathrm{Iso}(X)$, that is, if and only if $F := X \setminus A$ is sequentially closed in X and $F \subseteq \mathrm{Iso}(X)$. The case $F = X$, which can occur only when X is discrete, gives $A = \emptyset$. This proves the displayed formula.

($\Leftarrow$) Suppose that every subset of $\mathrm{Iso}(X)$ that is sequentially closed in X is closed in X. Since every closed set is sequentially closed, the subsets of $\mathrm{Iso}(X)$ that are sequentially closed in X are then exactly those that are closed in X. Hence the formula for τ_{rs} coincides with the formula for τ_{rn} in Proposition 6, and $\tau_{rn} = \tau_{rs}$.

($\Rightarrow$) Suppose that $\tau_{rn} = \tau_{rs}$, and let $F \subseteq \mathrm{Iso}(X)$ be sequentially closed in X. If $F = X$, then F is closed. Otherwise $X \setminus F$ is a non-empty member of $\tau_{rs} = \tau_{rn}$, hence open in X (Remark 10), so F is closed in X. □

Corollary 4. *If (X, τ) is a sequential space, in particular if X is first countable or metrizable, then $\tau_{rn} = \tau_{rs}$.*

Proof. In a sequential space every sequentially closed set is closed [16]; apply Theorem 15. □

The description of τ_{rs} in Theorem 15 also yields the sequential counterpart of Theorem 13 without the hypothesis of [14, Corollary 35] that X has at least two rare elements.

Corollary 5. *If (X, τ) is not discrete, then $(\tau_{rs})_{rs} = \{\emptyset, X\}$.*

Proof. Put $\sigma = \tau_{rs}$ and $U = X \setminus \mathrm{Iso}(X)$, which is non-empty because τ is not discrete. By Theorem 15, every non-empty member of σ contains U. If v is an isolated point of (X, σ), then $\{v\} \supseteq U$, so $U = \{v\}$. Hence (X, σ) has at most one isolated point, and if it has one, say u, then every non-empty member of σ contains u. By Theorem 15 applied to (X, σ), the non-empty members of σ_{rs} are the sets $X \setminus F$ with $F \neq X$ a set of isolated points of (X, σ) that is sequentially closed in (X, σ). If (X, σ) has no isolated points, only $F = \emptyset$ occurs. Otherwise the only candidates are $F = \emptyset$ and $F = \{u\}$. The constant sequence at u converges in (X, σ) to every point of X, because every non-empty member of σ contains u. Since a non-discrete space has at least two points, $\{u\}$ is therefore not sequentially closed in (X, σ). In both cases $\sigma_{rs} = \{\emptyset, X\}$. □

For spaces with at least two points the hypothesis is also necessary: if X is discrete and has at least two points, then $\tau_{rs} = \tau$ [14, Example 26], so $(\tau_{rs})_{rs} = \tau$ is not indiscrete.

Remark 15. Sequentiality of (X, τ) is sufficient but not necessary for $\tau_{rn} = \tau_{rs}$; by Theorem 15 only the sequentially closed sets of isolated points matter.

(a) If X has no isolated points, then $\mathrm{Iso}(X) = \emptyset$ and both topologies are indiscrete. For instance $\mathbb{R}^J$ with J uncountable and the product topology is not sequential (the set of points with countable support is sequentially closed, dense and proper), yet $\tau_{rn} = \tau_{rs} = \{\emptyset, \mathbb{R}^J\}$.

(b) If $X = \mathbb{R}^J \sqcup \{p\}$ is the topological sum of such a space with a point, with topology τ, then $\mathrm{Iso}(X) = \{p\}$ is clopen, every subset of $\mathrm{Iso}(X)$ is closed, and $\tau_{rn} = \tau_{rs} = \{\emptyset, \mathbb{R}^J, X\}$ although (X, τ) is not sequential (its open subspace $\mathbb{R}^J$ is not).

Example 16. (a) Let X be the Arens–Fort space of Example 1. Its rare sets are $\emptyset$ and $\{(0,0)\}$, and $\mathrm{Iso}(X) = X \setminus \{(0,0)\}$.

The rarely sequential topology τ_{rs}. By Example 1, no sequence in $X \setminus \{(0,0)\}$ converges to $(0,0)$; hence a sequence converging to $(0,0)$ is eventually equal to $(0,0)$ (otherwise its terms different from $(0,0)$ would form such a sequence), and a sequence converging to an isolated point is eventually constant. Hence every subset of X is sequentially open (τ_s is discrete). By Theorem 2, a non-empty set A belongs to τ_{rs} if and only if it is sequentially open and contains all rare sets; hence $\tau_{rs} = \{\emptyset\} \cup \{A \subseteq X : (0,0) \in A\}$.

The rarely net-open topology τ_{rn}. By Theorem 9, a non-empty set A belongs to τ_{rn} if and only if $A \in \tau$ and $(0,0) \in A$. Hence $\tau_{rn} = \{A \in \tau : (0,0) \in A\} \cup \{\emptyset\}$.

Strict inclusion. Let $A = \{(0,0)\} \cup \{(m,1) : m \in \mathbb{N}^+\}$. Since $(0,0) \in A$, we have $A \in \tau_{rs}$. However, $C_m \setminus A = \{(m,n) : n \geq 2\}$ is infinite for every m, so A does not satisfy the neighbourhood condition for $(0,0)$, giving $A \notin \tau$. By Theorem 9, $A \notin \tau_{rn}$.

Therefore $A \in \tau_{rs} \setminus \tau_{rn}$, proving $\tau_{rn} \subsetneq \tau_{rs}$. In the terms of Theorem 15, the set $F = X \setminus A = \{(m,n) : m \geq 1,\ n \geq 2\} \subseteq \mathrm{Iso}(X)$ is sequentially closed but not closed.

(b) Another instance is the space of [14, Remark 32]. Let $X = \mathbb{R}$ with the topology whose open sets are all subsets of $\mathbb{R} \setminus \{0\}$ together with all sets containing 0 with countable complement (the Fortissimo space [24, Counterexample 25]). Here $\mathrm{Iso}(X) = \mathbb{R} \setminus \{0\}$. A sequence of non-zero points cannot converge to 0, because removing its countably many terms from $\mathbb{R}$ leaves an open neighbourhood of 0; hence $F = \mathbb{R} \setminus \{0\}$ is sequentially closed, but it is not closed. By Theorem 15, $\{0\} = X \setminus F \in \tau_{rs} \setminus \tau_{rn}$, in accordance with [14, Remark 32].

6. Conclusion

We have extended rare convergence from sequences to nets. The tail characterisation (Lemma 1) shows that rare convergence is convergence up to sets with empty interior; consequently, the set of rare limits of a net is closed and, in a Hausdorff space, it is empty, a singleton or the whole space (Theorem 3). In particular, unlike sequences, nets in $\mathbb{R}$ can have a unique rare limit. Since the union of all rare sets is the set of non-isolated points, the rarely net-open topology τ_{rn} and the rarely sequential topology τ_{rs} of [14] admit explicit descriptions (Proposition 6, Theorem 15). These descriptions determine when τ_{rn} is non-trivial, which of the separation axioms T_0, T_1 and T_2 it satisfies and what its iterates are, and they show that $\tau_{rn} = \tau_{rs}$ if and only if every sequentially closed set of isolated points is closed.

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