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RH-Regularity and Quantitative Approximation for Truncated Poisson Product Matrices on Triple Sequences

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Abstract: We investigate a normalized six-index product matrix formed from truncated Poisson rows, with a common parameter that depends on the output indices. RH-regularity on bounded Pringsheim-convergent triple sequences holds exactly when the parameter tends to infinity. For sublinear power growth, a Poisson lower-tail estimate transfers polynomial input decay to an explicit convergence rate whose exponent is proved to be sharp within the stated class. We distinguish this regularity criterion from diagonal concentration in the superlinear regime. For the associated triple-entire matrix domain, we establish completeness and nonseparability and provide an example showing that the transformation need not preserve the underlying entire-sequence space. Finally, Poisson-weighted Fejér operators yield quantitative periodic approximation estimates, including an exact-order result for the cosine test function.

Keywords: triple sequences, RH-regularity, truncated Poisson product matrix, sharp approximation rate, entire triple sequences, Fejér operators

MSC: 40B05, 40C05, 41A25, 41A36

1. Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$. We work with real triple sequences $x = (x_{mnk})_{m,n,k\in\mathbb{N}}$. Pringsheim convergence, written $P\text{–}\lim x = L$, means that for every $\varepsilon > 0$ there is N such that $|x_{mnk} - L| < \varepsilon$ whenever $m, n, k \geq N$. This extends the classical double-sequence notion introduced by Pringsheim [1] to the triple-sequence setting studied in [2]. Pringsheim convergence does not imply global boundedness; boundedness is imposed explicitly whenever RH-regularity is used. For $A \subseteq \mathbb{N}^3$, its triple natural density, when defined, is

$$\delta_3(A) = P\text{–}\lim_{u,v,w\to\infty} \frac{|A \cap ([1,u] \times [1,v] \times [1,w])|}{uvw}.$$

Statistical convergence requires the exceptional index set to have density zero for every fixed positive error threshold [2–4]; rough statistical convergence replaces the threshold ε by $r + \varepsilon$ [5,6].

The one-index Poisson-like matrix defined by Yaying et al. [7] uses the fixed parameter $\mu > 0$ and the normalized weights

$$p_{n,k}^{\mu} = \frac{\mu^k/k!}{\sum_{j=0}^{n} \mu^j/j!} \quad (0 \leq k \leq n).$$

Our marginal $a_{q,j}(t)$ below is this conditional-Poisson kernel after shifting both the row index $n = q - 1$ and the column index $k = j - 1$, whereas the present *six-index product* and its *common row-dependent* parameter are additional, explicitly specified choices. In particular, the fixed-parameter form of that one-index kernel cannot be cited as a proof of RH-regularity for the three-index product.

The Poisson–Fibonacci–binomial construction of Aiyub, Esi, and Subramanian [8] concerns rough statistical convergence of transformed triple sequences and associated rates. Priya, Esi, and Subramanian [9] subsequently study a band generalized Fibonacci difference operator combined with a Poisson-binomial-like matrix, again in the setting of rough statistical convergence and approximation. These works provide relevant context, but their coefficient structures must be distinguished from the normalized conditional-Poisson product introduced here. The RH criterion in this article is restricted to the coefficients in (3); it does not establish regularity for either of those different constructions. In particular, a common reference to Poisson weights is insufficient to transfer a regularity conclusion between matrices without checking their actual coefficients and parameter dependence. Das [10, Definition 1.6 and Theorem 1.2] provides general RH-regularity criteria for six-dimensional matrices. Theorem 1 specializes the general six-dimensional theory to the coefficients in (3) and gives an explicit growth criterion for their parameter.

The comparison in Table 1 summarizes the roles of these related studies and identifies the coefficient-specific contribution of the present work. The distinction between fixed and row-dependent parameters is essential to interpreting the regularity result correctly.

Table 1. Comparison of related matrix constructions and the scope of the present regularity result.

Work and location	Relevant coefficient, parameter choice, and result
Yaying et al. [7], defining formula	A single truncated, normalized Poisson row with fixed μ; matrix-domain and operator properties on one-index spaces, not a triple RH-regularity theorem.
Aiyub et al. [8]	Poisson–Fibonacci–binomial matrix methods for rough statistical convergence of triple sequences and associated rates. The cited construction is background for comparison; it is not identified with the normalized product in (3).
Priya et al. [9]	A band generalized Fibonacci difference operator combined with a Poisson-binomial-like matrix, studied through rough statistical convergence and approximation. Its difference-operator structure is distinct from the product coefficients used here.
Das [10], Definition 1.6 and Theorem 1.2	General RH-regularity framework for six-dimensional matrices with bounded Pringsheim-convergent inputs.
Present work, (2)–(3), Theorem 1	Three conditional-Poisson marginals with shared λ_{uvw}, exactly normalized for every row; RH-regular *if and only if* $\lambda_{uvw} \to \infty$.

The fixed-parameter obstruction can already be seen on the spike $\xi^{(111)}$, equal to one at $(1,1,1)$ and zero elsewhere. Put $S_q(t) = \sum_{j=0}^{q-1} t^j/j!$. With $\lambda_{uvw} \equiv \lambda_0 > 0$, the matrix defined below gives

$$(\mathcal{P}[\lambda_0]\xi^{(111)})_{uvw} = \frac{1}{S_u(\lambda_0)S_v(\lambda_0)S_w(\lambda_0)} \longrightarrow e^{-3\lambda_0} > 0. \tag{1}$$

Thus the particular fixed-parameter product matrix does not preserve this zero Pringsheim limit. Eq. (1) makes no claim about a different Poisson–Fibonacci construction.

2. Product Matrix and Regularity

For $q \geq 1$, $t > 0$, and $1 \leq j \leq q$, define

$$S_q(t) = \sum_{h=0}^{q-1} \frac{t^h}{h!}, \qquad a_{q,j}(t) = \frac{t^{j-1}}{(j-1)!S_q(t)}; \quad a_{q,j}(t) = 0 \ (j > q). \tag{2}$$

Then $a_{q,j} \geq 0$ and $\sum_{j\geq 1} a_{q,j} = 1$. If $\lambda = (\lambda_{uvw}) > 0$, define a six-index matrix by

$$\begin{aligned} P^{mnk}_{uvw}[\lambda] &= a_{u,m}(\lambda_{uvw})a_{v,n}(\lambda_{uvw})a_{w,k}(\lambda_{uvw}), \\ (\mathcal{P}[\lambda]x)_{uvw} &= \sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P^{mnk}_{uvw}[\lambda]x_{mnk}. \end{aligned} \tag{3}$$

Every row has nonnegative entries, finite support, and total mass equal to one. The incomplete-gamma identity $\Gamma(q,t) = (q-1)!e^{-t}S_q(t)$ identifies each marginal with a Poisson random variable Y_t of mean t:

$$a_{q,j}(t) = \frac{\Pr(Y_t = j-1)}{\Pr(Y_t \le q-1)}. \tag{4}$$

For any fixed $N \ge 2$ and $q \ge N$, the denominator in (2) contains $t^j/j!$ for every $1 \le j < N$; consequently $a_{q,j}(t) \le j/t$ and

$$\sum_{j=1}^{N-1} a_{q,j}(t) \le \frac{N(N-1)}{2t}. \tag{5}$$

A six-dimensional matrix is called *RH-regular* here if it maps each *bounded* Pringsheim-convergent triple sequence to a sequence with the same Pringsheim limit [10].

Theorem 1 (Exact Regularity Criterion)**.** *The matrix $\mathcal{P}[\lambda]$ is RH-regular if and only if $\lambda_{uvw} \to \infty$ in Pringsheim's sense.*

Proof. Suppose $\lambda_{uvw} \to \infty$. For bounded x with P–$\lim x = L$, write $M = \sup_{m,n,k} |x_{mnk} - L|$. Given $\varepsilon > 0$, take $N \ge 2$ with $|x_{mnk} - L| \le \varepsilon$ for all $m,n,k \ge N$. The mass outside this interior box is at most the sum of three marginal lower tails. Hence, for $u,v,w \ge N$,

$$|(\mathcal{P}[\lambda]x)_{uvw} - L| \le \varepsilon + \frac{3MN(N-1)}{2\lambda_{uvw}}.$$

Letting the row indices tend to infinity and then $\varepsilon \downarrow 0$ proves sufficiency. Conversely, if $\lambda_{uvw} \not\to \infty$, some sequence of rows tending to infinity has $\lambda_{uvw} \le C < \infty$. For the zero-limit spike $\xi^{(111)}$,

$$(\mathcal{P}[\lambda]\xi^{(111)})_{uvw} = \frac{1}{S_u(\lambda_{uvw})S_v(\lambda_{uvw})S_w(\lambda_{uvw})} \ge e^{-3C}.$$

Its transform cannot converge to zero; hence RH-regularity fails. □

For $\alpha, \mu > 0$, put $t = \min\{u,v,w\}$, $\lambda_{uvw} = \mu t^\alpha$ and $\mathcal{P}_{\alpha,\mu} = \mathcal{P}[\lambda]$. Every such matrix is RH-regular. The square-root family considered below is $\mathcal{P}(\mu) = \mathcal{P}_{1/2,\mu}$. The following estimate treats $0 < \alpha < 1$, a narrower parameter range required by the particular Poisson-tail calculation used in that estimate. This restriction is not required for RH-regularity itself.

3. Quantitative Estimates and Parameter Regimes

For bounded x with Pringsheim limit L, define

$$M_x = \sup_{m,n,k} |x_{mnk} - L|, \qquad \eta_x(N) = \sup_{m,n,k \ge N} |x_{mnk} - L|.$$

Then $M_x < \infty$ and $\eta_x(N) \to 0$. Decomposing each matrix row into the interior box and its complement yields, for every integer $2 \le N \le \min\{u,v,w\}$, the following elementary bound

$$|(\mathcal{P}_{\alpha,\mu}x)_{uvw} - L| \le \eta_x(N) + \frac{3M_xN(N-1)}{2\lambda_{uvw}}. \tag{6}$$

Theorem 2 (Exponential Exceptional-Mass Estimate)**.** *Fix $\mu > 0$ and $0 < \alpha < 1$, and write $t = \min\{u,v,w\}$, $\lambda = \mu t^\alpha$. For all sufficiently large t, with $N = \lfloor \lambda/2 \rfloor \ge 2$,*

$$|(\mathcal{P}_{\alpha,\mu}x)_{uvw} - L| \le \eta_x(N) + 6M_xe^{-\kappa\lambda}, \qquad \kappa = \frac{1-\log 2}{2} > 0. \tag{7}$$

The threshold beyond which this holds may depend on α and μ, but the estimate holds simultaneously for every row whose minimum coordinate is sufficiently large.

Proof. Eventually $2 \le N \le t$ and $\lambda/t \le 1/2$. For $q \in \{u, v, w\}$, Markov's inequality implies $\Pr(Y_\lambda \le q-1) \ge 1 - \lambda/q \ge 1/2$. With $\theta = \log 2$, Markov's inequality applied to the exponential transform gives

$$\Pr(Y_\lambda \le \lambda/2) \le e^{\theta\lambda/2}\mathbb{E}(e^{-\theta Y_\lambda}) = e^{-\kappa\lambda}.$$

Since $N - 2 \le \lambda/2$, formula (4) yields

$$\sum_{j=1}^{N-1} a_{q,j}(\lambda) = \frac{\Pr(Y_\lambda \le N-2)}{\Pr(Y_\lambda \le q-1)} \le 2e^{-\kappa\lambda}.$$

The mass of the union of the three boundary slabs is at most $6e^{-\kappa\lambda}$. Bounding the input difference by M_x there and by $\eta_x(N)$ on the interior proves (7). □

Corollary 1 (Polynomial Tail Transfer)**.** *Assume all hypotheses of Theorem 2 and suppose $\eta_x(N) = O(N^{-\beta})$ for some $\beta > 0$. Then*

$$|(\mathcal{P}_{\alpha,\mu}x)_{uvw} - L| = O(t^{-\alpha\beta}), \qquad t = \min\{u, v, w\}.$$

For $\alpha = 1/2$, the rate is $O(t^{-\beta/2})$.

Proof. As $t \to \infty$, $N = \lfloor \lambda/2 \rfloor \asymp t^\alpha$, while $e^{-\kappa\lambda} = O(t^{-\alpha\beta})$. Apply (7). □

Remark 1. Choosing $N \asymp \lambda^{1/(\beta+2)}$ in (6) gives only $O(t^{-\alpha\beta/(\beta+2)})$, a valid but weaker elementary estimate. The exponent in Corollary 1 is optimal over the stated polynomial-tail class, as the following proposition shows.

Proposition 1 (Sharpness of the Polynomial Exponent)**.** *Fix $\beta > 0$, $\mu > 0$, and $0 < \alpha < 1$. The bounded sequence $x_{mnk} = 1 + \min\{m, n, k\}^{-\beta}$ has Pringsheim limit 1 and satisfies $\eta_x(N) = N^{-\beta}$. Its transformed error has the exact order*

$$(\mathcal{P}_{\alpha,\mu}x)_{uvw} - 1 = \Theta\big(t^{-\alpha\beta}\big), \qquad t = \min\{u, v, w\} \longrightarrow \infty.$$

Proof. The upper estimate is Corollary 1. Let J_1, J_2, J_3 be independent random variables with marginal distributions $a_{u,\cdot}(\lambda), a_{v,\cdot}(\lambda), a_{w,\cdot}(\lambda)$, where $\lambda = \mu t^\alpha$, and put $H = \min\{J_1, J_2, J_3\}$. Then the error equals $\mathbb{E}H^{-\beta}$. Since $H \le J_1$ and $z \mapsto z^{-\beta}$ is convex on $(0, \infty)$,

$$\mathbb{E}H^{-\beta} \ge \mathbb{E}J_1^{-\beta} \ge (\mathbb{E}J_1)^{-\beta} \ge (1+\lambda)^{-\beta}.$$

For the last inequality, note that $J_1 = Y_\lambda + 1$ conditioned on $Y_\lambda \le u - 1$, and $\mathbb{E}(J_1 - 1) = \lambda \Pr(Y_\lambda \le u-2)/\Pr(Y_\lambda \le u-1) \le \lambda$. Thus the lower bound has order $t^{-\alpha\beta}$. □

Example 1 (Truncation Regime and Oscillation)**.** Consider the alternating input $x_{mnk} = (-1)^{m+n+k-3}$. On the diagonal $u = v = w = s$, its transform equals b_s^3, with

$$b_s = \sum_{j=1}^{s} a_{s,j}(\mu s^\alpha)(-1)^{j-1}.$$

If $0 < \alpha < 1$, the Poisson tail above $s - 1$ tends to zero. Since $\mathbb{E}(-1)^{Y_\lambda} = e^{-2\lambda} \to 0$, also $b_s \to 0$. For $\alpha > 1$, put $\rho_s = (s-1)/(\mu s^\alpha) \to 0$. Then

$$a_{s,s}(\mu s^\alpha)^{-1} = \sum_{h=0}^{s-1} \frac{(s-1)!}{(s-1-h)!(\mu s^\alpha)^h} \le \frac{1}{1-\rho_s},$$

so $a_{s,s} \to 1$. Consequently

$$|(\mathcal{P}_{\alpha,\mu}x)_{sss} - (-1)^{s-1}| \le 2(1 - a_{s,s}(\mu s^\alpha)^3) \longrightarrow 0.$$

The transformed sequence alternates along the diagonal and therefore does not converge, although $\mathcal{P}_{\alpha,\mu}$ is RH-regular. This does not contradict Theorem 1: the oscillating input has no Pringsheim limit. The ratio λ/t describes the truncation regime: when $\alpha > 1$, each marginal along equal-coordinate rows concentrates on its last position and the product matrix becomes asymptotically diagonal along equal-coordinate rows. Regularity still holds because it is tested only on bounded Pringsheim-convergent inputs; the present oscillating sequence is not such an input.

4. Entire-Sequence Matrix Domain and Transferred Rough Limits

The notation Γ^3 below denotes the *triple entire* sequence space, without a factorial. It is distinct from the *triple gai* space χ^3, defined using $\left((m+n+k)!\,|x_{mnk}|\right)^{1/(m+n+k)} \to 0$ [11]. The distinction is substantive, and the two symbols are not interchanged in this article. For $m, n, k \geq 1$, let $p_{mnk} = 1/(m+n+k)$ and define

$$\Gamma^3 = \left\{x : \sup_{m,n,k} |x_{mnk}|^{p_{mnk}} < \infty, |x_{mnk}|^{p_{mnk}} \to 0 \text{ in the Pringsheim sense}\right\},$$

$$d(x,y) = \sup_{m,n,k} |x_{mnk} - y_{mnk}|^{p_{mnk}}.$$

For $\mathcal{P} = \mathcal{P}(\mu)$, define $\Gamma^3(\mathcal{P}) = \{x : \mathcal{P}x \in \Gamma^3\}$ and $d_{\mathcal{P}}(x,y) = d(\mathcal{P}x, \mathcal{P}y)$.

Proposition 2 (Metric-Domain Structure)**.** *The triangular matrix $\mathcal{P}$ defines a linear bijection $\mathcal{P} : \Gamma^3(\mathcal{P}) \to \Gamma^3$ and an isometry for the displayed metrics. Both metric spaces are complete and nonseparable.*

Proof. The diagonal coefficient P^{uvw}_{uvw} is positive. A finite triangular recurrence, with indices ordered by $u+v+w$ therefore gives a unique preimage for every triple array. The isometry assertion follows directly from the definition of the transferred metric. For completeness of Γ^3, a d-Cauchy sequence has coordinatewise limits; passing to these limits in the uniform d-Cauchy inequality yields convergence in d. The inequality $(a+b)^p \leq a^p + b^p$, $0 < p \leq 1$, implies that the limit remains root-bounded and its roots converge to zero in Pringsheim's sense. For nonseparability, for every subset $A \subseteq \mathbb{N}$ set $x^A_{m11} = 1$ when $m \in A$ and zero otherwise. Every x^A lies in Γ^3, while $d(x^A, x^B) = 1$ for distinct A, B. The isometry transfers both properties to $\Gamma^3(\mathcal{P})$. □

Proposition 3 (Failure of Invariance of the Entire-Sequence Space)**.** *For $\mu > 0$ and $0 < \alpha < 1$, $\Gamma^3 \not\subseteq \Gamma^3(\mathcal{P}_{\alpha,\mu})$. In particular, the transform of a finitely supported element of Γ^3 need not belong to Γ^3.*

Proof. Let $x = \xi^{(111)}$. This finite-support array belongs to Γ^3. On the diagonal, with $\lambda = \mu s^\alpha$,

$$y_{sss} = (\mathcal{P}_{\alpha,\mu}x)_{sss} = S_s(\lambda)^{-3}.$$

Since $1 \leq S_s(\lambda) \leq e^\lambda$, we have

$$e^{-\lambda/s} \leq |y_{sss}|^{1/(3s)} \leq 1.$$

As $\lambda/s \to 0$, the diagonal roots tend to 1, rather than 0. Thus $y \notin \Gamma^3$. □

Remark 2. Proposition 3 concerns preservation of the original space, not the invertibility or completeness of the matrix domain. It does not provide a description of the domain's dual spaces.

Remark 3 (Not a Topological Vector Space)**.** The completeness and nonseparability assertions refer to the metrics, not to a topological vector-space structure with the ordinary topology on scalars. The boundary-supported array $x_{m11} = 1$ (zero elsewhere) belongs to Γ^3 because its nonzero coordinates lie outside every sufficiently distant Pringsheim box. For this x,

$$d(cx, 0) = \sup_{m \geq 1} |c|^{1/(m+2)} = 1 \qquad (0 < |c| < 1).$$

Hence scalar multiplication fails to be continuous at zero.

For a real triple sequence z, let $\mathrm{st}\text{-}\mathrm{LIM}^r(z)$ be the set of $\ell \in \mathbb{R}$ satisfying

$$\delta_3\big(\{(u,v,w) : |z_{uvw} - \ell| \geq r + \varepsilon\}\big) = 0 \quad \text{for all } \varepsilon > 0.$$

For any x, let $\mathrm{st}\text{-}\mathrm{LIM}^r_{\mathcal{P}}\, x = \mathrm{st}\text{-}\mathrm{LIM}^r(\mathcal{P}x)$. Unlike Proposition 2, the following is a transfer of elementary rough statistical statements to an arbitrary transformed sequence $z = \mathcal{P}x$, not a property specific to the entire-sequence matrix domain [6].

Proposition 4 (Transferred Rough-Limit Geometry)**.** *For any real triple sequence x and $r \geq 0$, the set $\mathrm{st}\text{-}\mathrm{LIM}^r_{\mathcal{P}}\, x$ is closed and convex, has diameter at most $2r$ whenever it is nonempty, and is nondecreasing with r. If $x \in \Gamma^3(\mathcal{P})$, then*

$$\mathrm{st}\text{-}\mathrm{LIM}^r_{\mathcal{P}}\, x = [-r, r].$$

Proof. For arbitrary z, closedness follows by approximating a limit point of the rough-limit set and shifting the tolerance; convexity follows by taking convex combinations of the two threshold inequalities outside a union of density-zero sets. Applying the triangle inequality to two such inequalities gives the diameter bound $2r$; larger roughness thresholds give set inclusion. If $x \in \Gamma^3(\mathcal{P})$, then $z = \mathcal{P}x$ has $|z_{uvw}|^{1/(u+v+w)} \to 0$. Eventually its root is less than $1/2$, whence $z_{uvw} \to 0$ in Pringsheim's sense. Its rough statistical limit set is therefore exactly $[-r, r]$. □

Remark 4. The geometry above is unchanged if two *transformed* sequences agree outside a density-zero set. No assertion of invariance under arbitrary density-zero alterations of the original input is made.

5. Quantitative Poisson-Weighted Periodic Approximation

Let $C_{2\pi}$ denote the space of real-valued, continuous, 2π-periodic functions, equipped with the supremum norm. Write $\omega(f, \delta)$ for their modulus of continuity and let $e_0 = 1$, $e_1(x) = \cos x$, $e_2(x) = \sin x$. For a family of positive linear operators $L_{mnk} : C_{2\pi} \to C_{2\pi}$, set

$$E^f_{mnk} = \|L_{mnk} f - f\|_\infty, \quad A^{(j)}_{mnk} = \|L_{mnk} e_j - e_j\|_\infty.$$

Their Poisson-weighted average is the well-defined finite-row operator

$$\mathsf{T}_{uvw} f = \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1}^{w} P^{mnk}_{uvw}[\lambda] L_{mnk} f, \qquad \|\mathsf{T}_{uvw} f - f\|_\infty \leq (\mathcal{P}[\lambda] E^f)_{uvw}. \tag{8}$$

This section studies *approximation-error magnitudes*, not merely how often an error exceeds a fixed threshold. Positive-operator test criteria in the periodic setting are classical [12]; what is quantified here is the effect of the specific product matrix.

Lemma 1 (Explicit Periodic Test Inequality)**.** *For $f \in C_{2\pi}$ and $0 < \delta < \pi$, set*

$$B_{f,\delta} = \frac{2\|f\|_\infty}{1 - \cos\delta}, \qquad C_{f,\delta} = \omega(f, \delta) + \|f\|_\infty + B_{f,\delta}.$$

For every positive linear $L : C_{2\pi} \to C_{2\pi}$,

$$\|Lf - f\|_\infty \leq \omega(f, \delta) + C_{f,\delta} \sum_{j=0}^{2} \|Le_j - e_j\|_\infty.$$

Proof. Periodicity gives $|f(t) - f(x)| \leq \omega(f, \delta) + B_{f,\delta}(1 - \cos(t - x))$. Apply the positive functional $L(\cdot)(x)$ and add $|f(x)|\,|L1(x) - 1|$. The identity $1 - \cos(t - x) = 1 - \cos t \cos x - \sin t \sin x$ gives

$$\big|L(1 - \cos(\cdot - x))(x)\big| \leq \sum_{j=0}^{2} \|Le_j - e_j\|_\infty.$$

Also $L1(x) \le 1 + \|Le_0 - e_0\|_\infty$. Collecting terms proves the displayed constant. □

Theorem 3 (Poisson-Transferred Periodic Test Rates). *Fix $\mu > 0$ and $0 < \alpha < 1$, and write $t = \min\{u, v, w\}$. Suppose the three nonnegative test-error arrays $A^{(j)}$ are bounded, converge to zero in Pringsheim's sense, and obey $\eta_{A^{(j)}}(N) = O(N^{-\beta})$ for a common $\beta > 0$. Then $\|\mathsf{T}_{uvw} f - f\|_\infty \to 0$ for every $f \in C_{2\pi}$, where the weights in (8) are those of $\mathcal{P}_{\alpha,\mu}$. Moreover, if f is Hölder continuous with exponent $0 < \gamma \le 1$, then*

$$\|\mathsf{T}_{uvw} f - f\|_\infty = O\big(t^{-\alpha\beta\gamma/(\gamma+2)}\big). \tag{9}$$

Proof. By Corollary 1, $h_t = \sum_{j=0}^{2} (\mathcal{P}_{\alpha,\mu} A^{(j)})_{uvw} = O(t^{-\alpha\beta})$. Positivity and the unit row sums allow Lemma 1 to be averaged, so (8) gives

$$\|\mathsf{T}_{uvw} f - f\|_\infty \le \omega(f, \delta) + C_{f,\delta} h_t.$$

For an arbitrary continuous function f, first fix δ and let $t \to \infty$; then let $\delta \downarrow 0$. For Hölder f, $\omega(f, \delta) = O(\delta^\gamma)$ and $C_{f,\delta} = O(\delta^{-2})$ as $\delta \downarrow 0$. Choose $\delta = t^{-\alpha\beta/(\gamma+2)}$ for large t. The resulting two terms both have the order in (9). □

Theorem 4 (Poisson–Fejér Approximation: A Sharp Cosine Rate). *For $N \ge 1$, let*

$$F_N f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - h) K_N(h)\, dh,$$
$$K_N(h) = \frac{1}{N+1} \left(\frac{\sin((N+1)h/2)}{\sin(h/2)} \right)^2.$$

Take $L_{mnk} = F_{\min\{m,n,k\}}$ in (8) with weights $\mathcal{P}_{\alpha,\mu}$, $\mu > 0$ and $0 < \alpha < 1$. For any periodic Hölder function f of exponent $0 < \gamma < 1$,

$$\|\mathsf{T}_{uvw} f - f\|_\infty = O\big(t^{-\alpha\gamma}\big), \qquad t = \min\{u, v, w\}. \tag{10}$$

For $f(x) = \cos x$ the exact order is

$$\|\mathsf{T}_{uvw} \cos - \cos\|_\infty = \Theta(t^{-\alpha}). \tag{11}$$

Proof. For $0 < \gamma < 1$, the Hölder condition implies $|f(x - h) - f(x)| \le H|h|^\gamma$ on $[-\pi, \pi]$. The standard kernel bounds $K_N(h) \le N + 1$ and $K_N(h) \le \pi^2/((N+1)h^2)$ for $0 < |h| \le \pi$ yield, after splitting the integral at $|h| = (N+1)^{-1}$,

$$\|F_N f - f\|_\infty \le \frac{H}{2\pi} \int_{-\pi}^{\pi} |h|^\gamma K_N(h)\, dh = O((N+1)^{-\gamma}).$$

Consequently E^f_{mnk} is bounded and its tail supremum is $O(N^{-\gamma})$; apply Corollary 1 and (8) to obtain (10).

For the cosine test function, $F_N \cos = \frac{N}{N+1} \cos$. Writing J_1, J_2, J_3 for the three conditionally truncated Poisson index variables and $H = \min(J_1, J_2, J_3)$, linearity yields the *exact* identity

$$\mathsf{T}_{uvw} \cos = \left(1 - \mathbb{E}\frac{1}{H+1}\right) \cos, \quad \|\mathsf{T}_{uvw} \cos - \cos\|_\infty = \mathbb{E}\frac{1}{H+1}.$$

Corollary 1 applied to $z_{mnk} = (\min\{m, n, k\} + 1)^{-1}$ gives the upper bound $O(t^{-\alpha})$. Conversely, $H \le J_1$ and Jensen's inequality give $\mathbb{E}(H+1)^{-1} \ge (\mathbb{E}J_1 + 1)^{-1} \ge (\lambda + 2)^{-1}$, with $\lambda = \mu t^\alpha$. This proves (11). □

The finite-row evaluations in Table 2 compare the polynomial-tail error with the exponential upper estimate and the cosine error with its Jensen lower bound. All five rows use the same parameter choices, so the decreasing errors can be compared directly as the common row index increases.

For each row in Table 2, the exact error for the polynomial-tail input lies below the exponential estimate (7) and above the Jensen lower bound from Proposition 1. The cosine column is the exact Poisson-weighted Fejér error, with its own Jensen lower bound. These finite evaluations illustrate the proved inequalities; they do not replace their asymptotic proofs.

Table 2. Finite-row evaluations of transformed errors and their explicit bounds. Here $u = v = w = t$, $\alpha = 0.7$, $\mu = 2$, and the polynomial input has $\beta = 0.75$. Values are rounded to five decimal places and are obtained by evaluating the finite conditional-Poisson sums.

t	$\mathbb{E}H^{-0.75}$	upper bound	$\mathbb{E}(H+1)^{-1}$	Jensen lower
120	0.05223	0.08310	0.01917	0.01693
260	0.03416	0.05400	0.01097	0.00999
600	0.02168	0.03480	0.00601	0.00562
1400	0.01373	0.02233	0.00328	0.00312
3000	0.00913	0.01497	0.00190	0.00183

Remark 5 (Density Versus Magnitude and Prior Korovkin Work)**.** If a transformed error converges in Pringsheim's sense, then its exceptional-index proportion at any fixed threshold is $O(u^{-1} + v^{-1} + w^{-1})$: all exceptional indices lie in three finite-width coordinate slabs. This is a generic fact of ordinary Pringsheim convergence, not a special Poisson rate; therefore we do not promote it to a separate approximation theorem. Theorem 3 uses the classical periodic test inequality, while Theorem 4 quantifies the specific product kernel's smoothing of a Fejér family. Demirci–Dirik [12] study a different one-index statistical σ-convergence framework, including perturbed Fejér operators. No equivalence or superiority between their method and the present quantitative product weighting is claimed.

6. Conclusion

The normalized truncated-Poisson product is RH-regular exactly when its row parameter diverges. Its polynomial-tail transfer bound has a matching lower-bound example, while the superlinear regime concentrates mass near the diagonal even though RH-regularity persists. The triple-entire matrix domain is isometric to the original entire-sequence space under the specified metrics, but the product transformation need not preserve the original space. Finally, Poisson-weighted positive operators admit a quantitative periodic test criterion, and Poisson–Fejér means satisfy a sharp $t^{-\alpha}$ cosine rate. These are coefficient-specific calculations and applications of classical summability and positive-operator principles, not new general RH or Korovkin characterizations.

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Data Availability: No external dataset was used. The article contains analytical constructions, and a reproducibility script for finite numerical illustrations is available from the corresponding author upon reasonable request.

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