Open Journal of Mathematical Sciences

PSR Press

Article

Sharp fixed point and periodic point bounds in n-G-metric spaces with pairwise polygonal-length applications

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Received: 16 May 2026; Revised: 20 Jun 2026; Accepted: 25 Jun 2026; Published: 17 Jul 2026

Abstract: We develop fixed point and periodic point results for self-mappings on n-G-metric spaces. The definition is stated in a form compatible with the classical case $n = 3$, while the paper specializes to the genuinely higher-order case $n \geq 4$. The first result proves a sharp bound: a mapping satisfying an n-tuple contraction on pairwise distinct points has at most $n - 1$ periodic points, and hence at most $n - 1$ fixed points. This formulation makes clear the role of short periodic orbits, which are the natural obstruction to applying a distinct-tuple contraction along Picard blocks. Under d_G-completeness, orbital continuity and an n-orbit-separation condition, every Picard orbit either reaches a fixed point or converges to one. We also state direct n-ary Kannan- and Reich-type analogues and separate them from the standard consequences obtained from the associated metric. Finally, the total pairwise polygonal-length functional on normed spaces provides a concrete geometric model of the theory. To make the model computationally explicit, we add a Picard algorithm in n-G-metric spaces, pseudocode, convergence and error tables for several values of n, λ and ρ, and both linear and nonlinear discrete polygonal systems illustrating the comparison between the associated metric d_G and the pairwise polygonal-length dynamics.

Keywords: n-G-metric space, fixed point, periodic point, Banach contraction, Kannan contraction, Reich contraction, Picard iteration, pairwise polygonal-length dynamics

MSC: 47H10, 54H25, 54E35, 37C25.

1. Introduction

Fixed point theory in generalized distance spaces is a classical part of nonlinear analysis. The Banach contraction principle [1], its Kannan-type and Reich-type variants [2,3], and the corresponding treatments in metric and distance spaces [4–6] provide the background for many modern extensions. Among these extensions, Mustafa and Sims introduced G-metric spaces through a three-variable distance function [7]. Subsequent works developed fixed point theory in this setting [8]; at the same time, the relation between G-metric fixed point theorems and the associated ordinary metric was clarified in [9]. Recent work of Jleli, Pacurar and Samet considered contractions acting on triples of distinct points and applications to mappings contracting triangle perimeters [10]. Related compact generalized metric models also appear in [11].

The present paper studies an n-variable version of this distinct-tuple phenomenon. Throughout the paper $n \geq 4$. The point is not merely to replace triples by n-tuples. If a contractive hypothesis is imposed only on pairwise distinct n-tuples, then it cannot be applied blindly to a Picard orbit: a short periodic repetition may appear before a block of length n is available. Thus the natural obstruction is not only the absence of a fixed point, but the possible presence of periodic points of length less than n. This observation leads to a periodic-point formulation of the main result.

We use the term n-G-metric for the axiom system in Definition 1 because axiom (G3) was chosen so that the monotonicity feature of the original G-metric is preserved when one passes from three variables to n variables. The definition itself is meaningful for all $n \geq 3$; however, the present paper specializes to $n \geq 4$, and the case $n = 3$ is mentioned only to clarify compatibility with the Mustafa–Sims framework.

The main contributions are summarized as follows.

(C1) We prove a sharp periodic-point bound: an n-tuple contraction on pairwise distinct points has no periodic orbit of length at least n, and its set of periodic points contains at most $n-1$ elements. The fixed point estimate is a direct consequence.
(C2) We introduce an n-orbit-separation condition and use it to obtain Picard convergence in complete n-G-metric spaces. The orbital continuity assumption is stated explicitly because the distinct-tuple contraction controls only separated blocks; it is not claimed to follow automatically from the n-G axioms.
(C3) We give n-ary Kannan- and Reich-type analogues in which the contractive hypotheses are imposed directly on pairwise distinct n-tuples.
(C4) We separate the associated-metric consequences from the genuinely n-tuple statements. The Kannan and Reich results formulated only in terms of the induced metric d_G are recorded as direct corollaries of the standard metric theorems, not as new independent n-G fixed point principles. This distinction is important because direct distinct-tuple contractions give cardinality estimates, whereas associated-metric contractions usually give uniqueness.
(C5) We introduce the total pairwise polygonal-length functional on normed spaces, prove that it defines an n-G-metric, and interpret the resulting fixed point mechanism as a contractive pairwise-dispersion dynamics. For $n = 3$ this coincides with the usual triangle perimeter, while for $n \geq 4$ it is not the ordinary cyclic perimeter of an n-gon; it is the sum of all pairwise edge lengths.
(C6) We formulate a concrete Picard algorithm for n-G-metric spaces and provide numerical experiments with several values of n, λ and ρ. The linear rotating example is presented as an illustrative verification of the exact formula, and a nonlinear test is added to show residuals that are not predetermined by a closed-form identity.

Feature	Three-variable G-metric case	n-G-metric case
Number of variables	3	$n \geq 4$
Contractive tuples	triples of distinct points	pairwise distinct n-tuples
Periodic obstruction	cycles of length < 3	cycles of length $< n$
Periodic/fixed point bound	at most 2 periodic points	at most $n-1$ periodic points
Orbit control	consecutive triples	blocks of length n
Geometric model	triangle perimeter	total pairwise polygonal-length functional

The paper is organized as follows. §2 introduces n-G-metric spaces and their associated metric. §3 proves the sharp periodic and fixed point bounds and the main Picard convergence theorem. §4 establishes direct n-ary Kannan- and Reich-type results and then records the standard associated-metric corollaries. §5 presents examples and the pairwise polygonal-length dynamics. §6 gives a computational Picard scheme, pseudocode and numerical tests for discrete polygonal systems. §7 concludes the paper.

2. Preliminaries

Throughout the paper, X denotes a nonempty set. Definition 1 is stated for every integer $n \geq 3$, in order to make explicit its compatibility with the classical three-variable G-metric. After the definition, all genuinely new results are formulated for the case $n \geq 4$.

Definition 1. A function

$$G_n : X^n \to [0, \infty),$$

is called an n-G-metric on X, where $n \geq 3$, if, for all $\omega_1, \dots, \omega_n, \theta, v, \phi_3, \dots, \phi_n, \eta \in X$, the following conditions hold:

(G1) $G_n(\omega_1, \dots, \omega_n) = 0$ if and only if $\omega_1 = \omega_2 = \cdots = \omega_n$;
(G2) $G_n(\theta, \dots, \theta, v) > 0$ whenever $\theta \neq v$, where θ is repeated $n-1$ times;
(G3) $G_n(\theta, \dots, \theta, v) \leq G_n(\theta, v, \phi_3, \dots, \phi_n)$, whenever $(\phi_3, \dots, \phi_n) \neq (v, \dots, v)$;
(G4) G_n is symmetric in all variables;
(G5) $G_n(\omega_1, \dots, \omega_n) \leq G_n(\omega_1, \eta, \dots, \eta) + G_n(\eta, \omega_2, \dots, \omega_n)$, where η is repeated $n-1$ times in the first term.

The pair (X, G_n) is called an n-G-metric space.

Remark 1 (Terminology)**.** The adjective "corrected", used in an earlier version of this manuscript, has been removed. The present definition is called simply an n-G-metric. The reason for the form of axiom (G3) is that it preserves the monotonicity axiom of G-metric spaces introduced by Mustafa and Sims [7] when one passes from three variables to n variables.

Remark 2. When $n = 3$, axiom (G3) reduces to the usual condition

$$G(\theta, \theta, v) \leq G(\theta, v, \mu) \qquad (\mu \neq v).$$

Thus the n-G-framework is compatible with the standard three-variable setting.

Remark 3 (On the exclusion in axiom (G3))**.** The restriction $(\phi_3, \ldots, \phi_n) \neq (v, \ldots, v)$ is retained in order to parallel the classical Mustafa–Sims monotonicity axiom, where the comparison tuple must not collapse to the repeated two-point configuration. In many concrete examples, including the diameter model and the pairwise polygonal-length model, the same inequality remains true even when $(\phi_3, \ldots, \phi_n) = (v, \ldots, v)$. Thus the exclusion is not needed for those examples; it is part of the abstract definition chosen to preserve the traditional G-metric pattern.

Example 1. Let (X, d) be a metric space. Define

$$G_n(\omega_1, \ldots, \omega_n) := \max_{1 \leq i, j \leq n} d(\omega_i, \omega_j).$$

Then G_n is an n-G-metric on X.

Proof. Conditions (G1), (G2) and (G4) are immediate. For (G3), both θ and v appear among the entries of $(\theta, v, \phi_3, \ldots, \phi_n)$, and therefore

$$G_n(\theta, v, \phi_3, \ldots, \phi_n) \geq d(\theta, v) = G_n(\theta, \ldots, \theta, v).$$

For (G5), the triangle inequality gives the usual diameter estimate:

$$d(\omega_i, \omega_j) \leq d(\omega_i, \eta) + d(\eta, \omega_j),$$

and taking the maximum over i, j yields the desired inequality. □

Definition 2. For an n-G-metric space (X, G_n), define

$$d_G(\theta, v) := G_n(v, \ldots, v, \theta) + G_n(\theta, \ldots, \theta, v), \qquad \theta, v \in X,$$

where in each term the repeated variable occurs $n - 1$ times.

Proposition 1. *The function d_G is a metric on X.*

Proof. Nonnegativity and symmetry are immediate. If $d_G(\theta, v) = 0$, then both terms in the definition vanish, and axiom (G1) gives $\theta = v$. Conversely, $d_G(\theta, \theta) = 0$.

Let $\theta, v, \eta \in X$. By axiom (G5) and symmetry,

$$G_n(v, \ldots, v, \theta) \leq G_n(\eta, \ldots, \eta, \theta) + G_n(v, \ldots, v, \eta),$$

and similarly,

$$G_n(\theta, \ldots, \theta, v) \leq G_n(\eta, \ldots, \eta, v) + G_n(\theta, \ldots, \theta, \eta).$$

Adding the two inequalities yields

$$d_G(\theta, v) \leq d_G(\theta, \eta) + d_G(\eta, v).$$

Hence d_G is a metric. $\square$

Definition 3. An n-G-metric space (X, G_n) is called d_G-complete, or complete with respect to the associated metric d_G, if the associated metric space (X, d_G) is complete. In the sequel, the phrase "complete n-G-metric space" always means d_G-complete; no independent G_n-Cauchy structure is being introduced.

Definition 4. A self-map $T : X \to X$ is called orbitally continuous if, whenever a subsequence of iterates $T^{m_k}\theta$ converges to some $p \in X$, one has

$$T^{m_k+1}\theta \to Tp.$$

Remark 4. The orbital continuity assumption is used only at the final step of the Picard argument, namely to identify the limit of a Cauchy orbit with a fixed point. For ordinary metric contractions of Banach, Kannan or Reich type this step is automatic. For contractions imposed only on pairwise distinct n-tuples, the contraction may no longer be applicable when the limiting tuple contains repeated entries. Therefore we keep orbital continuity as an explicit hypothesis in the direct n-tuple theorems. It could be replaced, without changing the proof, by any condition implying the following implication along Picard orbits:

$$x_m \to u, \qquad x_{m+1} = Tx_m \quad \Longrightarrow \quad x_{m+1} \to Tu.$$

For instance, it is enough to assume d_G-continuity of T at the orbit limit, closedness of the graph of T along Picard orbits, or an additional contractive condition valid also for tuples with repeated entries. In the associated-metric corollaries below, no such extra continuity is needed.

3. Sharp periodic and fixed point bounds and Picard convergence

We begin with the fixed point cardinality bound.

Proposition 2. *Let (X, G_n) be an n-G-metric space, and let $T : X \to X$ be a map such that there exists $\lambda \in [0, 1)$ with*

$$G_n(T\omega_1, \ldots, T\omega_n) \le \lambda G_n(\omega_1, \ldots, \omega_n),$$

for all pairwise distinct $\omega_1, \ldots, \omega_n \in X$. Then

$$|\operatorname{Fix}(T)| \le n - 1.$$

Proof. Assume, to the contrary, that there exist n pairwise distinct fixed points $p_1, \ldots, p_n \in X$. Then $Tp_i = p_i$ for $i = 1, \ldots, n$, and the contractive assumption gives

$$G_n(p_1, \ldots, p_n) = G_n(Tp_1, \ldots, Tp_n) \le \lambda G_n(p_1, \ldots, p_n).$$

Thus $(1 - \lambda)G_n(p_1, \ldots, p_n) \le 0$. Since $\lambda < 1$, it follows that $G_n(p_1, \ldots, p_n) = 0$, which contradicts axiom (G1) because the points are pairwise distinct. Therefore T cannot have n distinct fixed points, and $|\operatorname{Fix}(T)| \le n - 1$. $\square$

Remark 5. The preceding proposition is the fixed point part of a stronger periodic-point statement. This is the form suggested by the distinct-tuple nature of the contraction: short cycles are the obstruction to applying the contraction along Picard blocks.

Definition 5. A point $x \in X$ is called periodic for T if there exists an integer $q \ge 1$ such that $T^q x = x$. The least such q is the period of x. The set of all periodic points of T is denoted by $\operatorname{Per}(T)$.

Proposition 3 (Periodic point bound)**.** *Let (X, G_n) be an n-G-metric space, and let $T : X \to X$ satisfy*

$$G_n(T\omega_1, \ldots, T\omega_n) \le \lambda G_n(\omega_1, \ldots, \omega_n), \qquad 0 \le \lambda < 1,$$

for all pairwise distinct $\omega_1, \ldots, \omega_n \in X$. Then the following assertions hold.

(a) *T has no periodic orbit containing n or more distinct points.*
(b) *No n distinct periodic points of T can exist; equivalently,* $|\operatorname{Per}(T)| \leq n-1$.
(c) *In particular,* $|\operatorname{Fix}(T)| \leq n-1$.

Proof. Assume first that x has least period $q \geq n$. For $j = 0, 1, \ldots, q$, define

$$A_j := G_n(T^j x, T^{j+1}x, \ldots, T^{j+n-1}x).$$

Because the period of x is $q \geq n$, every block

$$(T^j x, T^{j+1}x, \ldots, T^{j+n-1}x),$$

contains n pairwise distinct points. Indeed, if $T^{j+r}x = T^{j+s}x$ for some $0 \leq r < s \leq n-1$, then $T^{s-r}(T^{j+r}x) = T^{j+r}x$, so the orbit has period dividing $s - r < n \leq q$, contrary to the minimality of q. Hence the contractive assumption applies to each of these blocks and gives

$$A_{j+1} \leq \lambda A_j, \qquad j = 0, 1, \ldots, q-1.$$

Since $T^q x = x$, periodicity gives $A_q = A_0$. Therefore

$$A_0 = A_q \leq \lambda^q A_0.$$

As $0 \leq \lambda < 1$, this implies $A_0 = 0$, contradicting axiom (G1), because the entries $x, Tx, \ldots, T^{n-1}x$ are distinct. Thus T has no periodic orbit of length at least n.

Now suppose that $p_1, \ldots, p_n$ are n distinct periodic points. Let Q be a common multiple of their periods; hence $T^Q p_i = p_i$ for every i. We first prove that, for every $j = 0, 1, \ldots, Q$, the tuple

$$(T^j p_1, \ldots, T^j p_n),$$

is pairwise distinct. If, for some j and some $i \neq \ell$, one had $T^j p_i = T^j p_\ell$, then

$$p_i = T^{Q-j}(T^j p_i) = T^{Q-j}(T^j p_\ell) = p_\ell,$$

where the exponents are understood modulo the common period if necessary. This contradicts the assumed distinctness of p_i and p_ℓ. Thus all iterated tuples remain pairwise distinct, and the contraction can be applied repeatedly:

$$G_n(T^{j+1}p_1, \ldots, T^{j+1}p_n) \leq \lambda G_n(T^j p_1, \ldots, T^j p_n), \qquad j = 0, \ldots, Q-1.$$

Consequently,

$$G_n(p_1, \ldots, p_n) = G_n(T^Q p_1, \ldots, T^Q p_n) \leq \lambda^Q G_n(p_1, \ldots, p_n),$$

which forces $G_n(p_1, \ldots, p_n) = 0$, again contradicting (G1). Hence no n distinct periodic points can exist. Since fixed points are periodic points of period one, the fixed point bound follows. □

Remark 6. For $n = 3$, this recovers the pattern of the three-variable result in which a contraction on distinct triples can allow at most two periodic points. The statement above makes explicit that the higher-order conclusion concerns periodic points, not only fixed points.

The existence part requires a condition ensuring that the contractive hypothesis can actually be applied along an orbit.

Definition 6. A self-map $T : X \to X$ is called n-orbit-separated if, for every non-fixed point $\theta \in X$, the points

$$\theta, T\theta, T^2\theta, \ldots, T^{n-1}\theta,$$

are pairwise distinct.

Remark 7. For $n = 3$, the condition $T\theta \neq \theta \Rightarrow T^2\theta \neq \theta$ is enough to guarantee that $\theta, T\theta, T^2\theta$ are pairwise distinct whenever θ is not a fixed point. For $n \geq 4$, the contraction acts on n-tuples, so the stronger orbit-separation condition in the preceding definition is a convenient sufficient hypothesis for a complete Picard proof. It is not a necessary dynamical requirement. If an orbit reaches a fixed point after finitely many steps, convergence is already trivial; if it enters a short cycle of length $< n$, the distinct-tuple contraction cannot be applied indefinitely; and if the orbit remains separated in blocks of length n, the contraction argument below applies.

Lemma 1. *Let (X, G_n) be an n-G-metric space, and let $(\omega_m)_{m\geq 0}$ be a sequence in X such that, for some m, the points*

$$\omega_m, \omega_{m+1}, \ldots, \omega_{m+n-1},$$

are pairwise distinct. Then

$$d_G(\omega_m, \omega_{m+1}) \leq 2G_n(\omega_m, \omega_{m+1}, \ldots, \omega_{m+n-1}).$$

Proof. By axiom (G3),

$$G_n(\omega_m, \ldots, \omega_m, \omega_{m+1}) \leq G_n(\omega_m, \omega_{m+1}, \ldots, \omega_{m+n-1}).$$

By symmetry, the same estimate holds with ω_m and ω_{m+1} interchanged:

$$G_n(\omega_{m+1}, \ldots, \omega_{m+1}, \omega_m) \leq G_n(\omega_m, \omega_{m+1}, \ldots, \omega_{m+n-1}).$$

Adding the two inequalities proves the claim. □

The next theorem is the n-ary counterpart of the Picard convergence theorem for contractions on distinct triples considered in [10]. The additional n-orbit-separation condition prevents repetitions in the first n consecutive Picard iterates, while orbital continuity identifies the limit as a fixed point.

Theorem 1. *Let (X, G_n) be a complete n-G-metric space, and let $T : X \to X$ satisfy the following assumptions:*

(i) *T is orbitally continuous;*
(ii) *T is n-orbit-separated;*
(iii) *there exists $\lambda \in [0, 1)$ such that*

$$G_n(T\omega_1, \ldots, T\omega_n) \leq \lambda G_n(\omega_1, \ldots, \omega_n),$$

for all pairwise distinct $\omega_1, \ldots, \omega_n \in X$.

Then, for every $\omega_0 \in X$, the Picard sequence $\omega_m := T^m\omega_0$ either reaches a fixed point after finitely many steps or converges in the metric d_G to a fixed point of T. Consequently,

$$1 \leq |\operatorname{Fix}(T)| \leq n - 1.$$

Proof. Fix $\omega_0 \in X$ and define $\omega_m := T^m\omega_0$ for $m \geq 0$. If some ω_m is a fixed point, the proof is complete. We may therefore assume that no ω_m is fixed. By n-orbit separation, each block

$$\omega_m, \omega_{m+1}, \ldots, \omega_{m+n-1},$$

is pairwise distinct.

Set

$$a_m := G_n(\omega_m, \omega_{m+1}, \ldots, \omega_{m+n-1}), \qquad m \geq 0.$$

Applying the contractive condition to the pairwise distinct tuple $(\omega_m, \omega_{m+1}, \ldots, \omega_{m+n-1})$, we obtain

$$a_{m+1} = G_n(\omega_{m+1}, \omega_{m+2}, \ldots, \omega_{m+n}) \leq \lambda a_m.$$

Hence, by induction, $a_m \leq \lambda^m a_0$ for $m \geq 0$. By the block estimate lemma above,

$$d_G(\omega_m, \omega_{m+1}) \leq 2a_m \leq 2\lambda^m a_0.$$

Let $p \geq 1$. By the triangle inequality for d_G,

$$d_G(\omega_m, \omega_{m+p}) \leq \sum_{j=m}^{m+p-1} d_G(\omega_j, \omega_{j+1}) \leq 2a_0 \sum_{j=m}^{m+p-1} \lambda^j \leq \frac{2a_0}{1-\lambda}\lambda^m.$$

This tends to 0 as $m \to \infty$, uniformly in p. Thus (ω_m) is a Cauchy sequence in the complete metric space (X, d_G), so there exists $u \in X$ such that $\omega_m \to u$.

Since T is orbitally continuous, the convergence of the orbit implies

$$\omega_{m+1} = T\omega_m \to Tu.$$

But (ω_{m+1}) is a tail of (ω_m), so also $\omega_{m+1} \to u$. By uniqueness of limits in the metric space (X, d_G), we obtain $Tu = u$. Hence $u \in \mathrm{Fix}(T)$.

Finally, Proposition 2 yields $|\mathrm{Fix}(T)| \leq n-1$. Since we have just shown that $\mathrm{Fix}(T) \neq \varnothing$, it follows that

$$1 \leq |\mathrm{Fix}(T)| \leq n-1.$$

□

Corollary 1. *Under the assumptions of the Picard convergence theorem above, if T has $n-1$ pairwise distinct fixed points, then they exhaust the whole fixed point set.*

Proof. This follows immediately from Proposition 2. □

Corollary 2 (Periodic alternative)**.** *Let (X, G_n) be complete, let $T : X \to X$ be orbitally continuous, and assume that T satisfies the n-tuple contraction from the Picard convergence theorem above. If T is not n-orbit-separated, then T has a periodic point of period strictly smaller than n. If T is n-orbit-separated, then every Picard orbit converges to a fixed point.*

Proof. If T is not n-orbit-separated, then there exists a non-fixed point θ for which two elements among $\theta, T\theta, \ldots, T^{n-1}\theta$ coincide. Hence $z = T^i\theta$ satisfies $T^q z = z$ for some $1 \leq q < n$. Thus z is a periodic point of period less than n. If T is n-orbit-separated, the Picard convergence theorem above applies directly. □

4. Kannan- and Reich-type theorems

We begin with direct n-G-metric analogues of Kannan- and Reich-type contraction principles, where the contractive assumptions are imposed on pairwise distinct n-tuples.

Theorem 2 (n-G Kannan analogue)**.** *Let (X, G_n) be a complete n-G-metric space, and let $T : X \to X$ be orbitally continuous and n-orbit-separated. Assume that there exists $\lambda \in [0, 1/n)$ such that*

$$G_n(T\omega_1, \ldots, T\omega_n) \leq \lambda \sum_{i=1}^{n} G_n(\omega_i, T\omega_i, \ldots, T\omega_i),$$

for all pairwise distinct $\omega_1, \ldots, \omega_n \in X$, where $T\omega_i$ is repeated $n-1$ times in each term of the sum. Then

$$\mathrm{Fix}(T) \neq \varnothing \quad \textit{and} \quad |\mathrm{Fix}(T)| \leq n-1.$$

Proof. Fix $\omega_0 \in X$ and define the Picard sequence $\omega_m := T^m\omega_0$ for $m \geq 0$. If $\omega_m = \omega_{m+1}$ for some m, then $\omega_m \in \mathrm{Fix}(T)$ and the cardinality conclusion follows from the final part of the proof below. Assume therefore that $\omega_m \neq \omega_{m+1}$ for all $m \geq 0$. Since T is n-orbit-separated, each block $\omega_m, \omega_{m+1}, \dots, \omega_{m+n-1}$ is pairwise distinct. Set

$$A_m := G_n(\omega_m, \omega_{m+1}, \dots, \omega_{m+n-1}), \qquad m \geq 0.$$

Applying the contractive assumption to $(\omega_m, \dots, \omega_{m+n-1})$, we obtain

$$A_{m+1} \leq \lambda \sum_{i=0}^{n-1} G_n(\omega_{m+i}, \omega_{m+i+1}, \dots, \omega_{m+i+1}).$$

For $0 \leq i \leq n-2$, the block $(\omega_m, \dots, \omega_{m+n-1})$ is pairwise distinct and contains both ω_{m+i} and ω_{m+i+1}. By symmetry, A_m can be written with these two entries in the first two positions and with the remaining $n-2$ distinct entries afterwards. Axiom (G3) then gives

$$G_n(\omega_{m+i}, \omega_{m+i+1}, \dots, \omega_{m+i+1}) \leq A_m.$$

For $i = n-1$, the same argument is applied to the next block $(\omega_{m+1}, \dots, \omega_{m+n})$, and yields

$$G_n(\omega_{m+n-1}, \omega_{m+n}, \dots, \omega_{m+n}) \leq A_{m+1}.$$

Consequently,

$$A_{m+1} \leq \lambda\big((n-1)A_m + A_{m+1}\big),$$

that is,

$$A_{m+1} \leq qA_m, \qquad q := \frac{\lambda(n-1)}{1-\lambda}.$$

The condition $\lambda < 1/n$ is exactly what gives $0 \leq q < 1$. By induction, $A_m \leq q^m A_0$ for $m \geq 0$. The block estimate lemma gives

$$d_G(\omega_m, \omega_{m+1}) \leq 2A_m \leq 2q^m A_0.$$

Thus (ω_m) is Cauchy in the complete metric space (X, d_G), so there exists $u \in X$ such that $\omega_m \to u$. By orbital continuity,

$$\omega_{m+1} = T\omega_m \to Tu.$$

Since limits in (X, d_G) are unique, we get $Tu = u$. Hence $\mathrm{Fix}(T) \neq \varnothing$.

Finally, let $u_1, \dots, u_n \in \mathrm{Fix}(T)$ be pairwise distinct. The Kannan-type inequality gives

$$G_n(u_1, \dots, u_n) = G_n(Tu_1, \dots, Tu_n) \leq \lambda \sum_{i=1}^{n} G_n(u_i, u_i, \dots, u_i) = 0,$$

contrary to axiom (G1). Therefore $|\,\mathrm{Fix}(T)| \leq n-1$. $\square$

Theorem 3 (n-G Reich analogue)**.** *Let (X, G_n) be a complete n-G-metric space, and let $T : X \to X$ be orbitally continuous and n-orbit-separated. Assume that there exist nonnegative constants $a_1, \dots, a_n, a_{n+1}$ such that*

$$a_1 + \cdots + a_n + a_{n+1} < 1$$

and

$$G_n(T\omega_1, \dots, T\omega_n) \leq \sum_{i=1}^{n} a_i G_n(\omega_i, T\omega_i, \dots, T\omega_i) + a_{n+1} G_n(\omega_1, \dots, \omega_n),$$

for all pairwise distinct $\omega_1, \dots, \omega_n \in X$, where $T\omega_i$ is repeated $n-1$ times in each term. Then

$$\mathrm{Fix}(T) \neq \varnothing \quad \textit{and} \quad |\,\mathrm{Fix}(T)| \leq n-1.$$

Proof. Fix $\omega_0 \in X$ and define $\omega_m := T^m\omega_0$. If $\omega_m = \omega_{m+1}$ for some m, then ω_m is a fixed point. Assume that $\omega_m \neq \omega_{m+1}$ for all $m \geq 0$. Since T is n-orbit-separated, each block $\omega_m, \omega_{m+1}, \dots, \omega_{m+n-1}$ is pairwise distinct. Set

$$A_m := G_n(\omega_m, \omega_{m+1}, \dots, \omega_{m+n-1}).$$

Applying the Reich-type assumption to $(\omega_m, \dots, \omega_{m+n-1})$, we get

$$A_{m+1} \leq \sum_{i=1}^{n} a_i G_n(\omega_{m+i-1}, \omega_{m+i}, \dots, \omega_{m+i}) + a_{n+1} A_m.$$

For $i = 1, \dots, n-1$, the entries ω_{m+i-1} and ω_{m+i} both occur in the separated block defining A_m. After permuting the variables by symmetry, axiom (G3) gives

$$G_n(\omega_{m+i-1}, \omega_{m+i}, \dots, \omega_{m+i}) \leq A_m.$$

For $i = n$, the corresponding two entries occur in the next separated block defining A_{m+1}, so the same symmetry–(G3) argument gives

$$G_n(\omega_{m+n-1}, \omega_{m+n}, \dots, \omega_{m+n}) \leq A_{m+1}.$$

Therefore,

$$A_{m+1} \leq (a_1 + \dots + a_{n-1} + a_{n+1}) A_m + a_n A_{m+1}.$$

Equivalently,

$$A_{m+1} \leq q A_m, \qquad q := \frac{a_1 + \dots + a_{n-1} + a_{n+1}}{1 - a_n}.$$

Since $a_1 + \dots + a_n + a_{n+1} < 1$, we have $a_n < 1$ and $a_1 + \dots + a_{n-1} + a_{n+1} < 1 - a_n$, hence $0 \leq q < 1$. Thus $A_m \leq q^m A_0$ for $m \geq 0$. By the block estimate lemma above,

$$d_G(\omega_m, \omega_{m+1}) \leq 2A_m \leq 2q^m A_0,$$

so (ω_m) is Cauchy in (X, d_G). By completeness, there exists $u \in X$ such that $\omega_m \to u$, and orbital continuity yields $\omega_{m+1} \to Tu$. Therefore $Tu = u$, so $\mathrm{Fix}(T) \neq \varnothing$.

Now let $u_1, \dots, u_n \in \mathrm{Fix}(T)$ be pairwise distinct. Then

$$G_n(u_1, \dots, u_n) = G_n(Tu_1, \dots, Tu_n) \leq a_{n+1} G_n(u_1, \dots, u_n),$$

so $(1 - a_{n+1}) G_n(u_1, \dots, u_n) \leq 0$. Since $a_{n+1} < 1$, this contradicts axiom (G1). Hence $|\mathrm{Fix}(T)| \leq n - 1$. □

Remark 8. For $n = 3$, Theorems 2 and 3 recover the pattern of the corresponding three-variable fixed point results in G-metric spaces. In the higher-order n-G setting, the explicit n-orbit-separation hypothesis is the natural substitute that allows the Picard blocks of length n to remain pairwise distinct.

We next record two associated-metric consequences. Since d_G is canonically generated by the n-G-metric, they remain intrinsic to the present framework while giving the familiar uniqueness conclusions.

Corollary 3 (Kannan type via the associated metric)**.** *Let (X, G_n) be a complete n-G-metric space, and let $T : X \to X$ satisfy*

$$d_G(Tx, Ty) \leq \kappa\big(d_G(x, Tx) + d_G(y, Ty)\big),$$

for all $x, y \in X$, where $0 \leq \kappa < 1/2$. Then T has a unique fixed point $u \in X$. Moreover, for every $\omega_0 \in X$, the Picard sequence $\omega_m := T^m\omega_0$ converges to u in the metric d_G.

Proof. This is exactly Kannan's fixed point theorem [2] applied to the complete metric space (X, d_G). No additional n-tuple argument is involved. □

Remark 9. The Kannan condition in Corollary 3 may be written explicitly in terms of G_n as

$$\begin{aligned} &G_n(Ty,\dots,Ty,Tx)+G_n(Tx,\dots,Tx,Ty)\\ &\le \kappa\Big(G_n(Tx,\dots,Tx,x)+G_n(x,\dots,x,Tx)\\ &\qquad +G_n(Ty,\dots,Ty,y)+G_n(y,\dots,y,Ty)\Big). \end{aligned}$$

Corollary 4 (Reich type via the associated metric)**.** *Let (X,G_n) be a complete n-G-metric space, and let $T:X\to X$ satisfy*

$$d_G(Tx,Ty)\le \alpha d_G(x,y)+\beta d_G(x,Tx)+\gamma d_G(y,Ty),$$

for all $x,y\in X$, where $\alpha,\beta,\gamma\ge 0$ and

$$\alpha+\beta+\gamma<1.$$

Then T has a unique fixed point $u\in X$, and for every $\omega_0\in X$, the Picard sequence $T^m\omega_0$ converges to u in the metric d_G.

Proof. This follows from the standard Reich fixed point theorem in the complete metric space (X,d_G); see [3] and the monograph treatments [4–6]. □

Remark 10. The Reich condition becomes the Banach condition when $\beta=\gamma=0$, and it becomes a mixed Kannan-Reich inequality when $\alpha=0$.

5. Examples and pairwise polygonal-length dynamics

We now collect examples that complement the preceding theory. They show the sharpness of the fixed point bound, the non-vacuity of the direct n-tuple assumptions, and the usefulness of the associated-metric viewpoint.

Example 2 (Sharpness of the bound: vacuous model)**.** Let $X=\{1,\dots,n-1\}$, and let d be the discrete metric on X. Define

$$G_n(\omega_1,\dots,\omega_n):=\max_{1\le i,j\le n} d(\omega_i,\omega_j).$$

By the metric-diameter example above, (X,G_n) is an n-G-metric space. Since X contains only $n-1$ points, there are no n pairwise distinct elements of X, so the distinct-tuple contractive hypothesis in Proposition 2 is vacuous. If $T=\mathrm{id}_X$, then

$$\mathrm{Fix}(T)=X,\qquad |\mathrm{Fix}(T)|=n-1.$$

Thus the estimate in Proposition 2 is sharp.

Example 3 (Sharpness of the bound: non-vacuous model)**.** Fix $\lambda\in(0,1)$ and choose numbers $\varepsilon,L>0$ such that

$$(n-2)\varepsilon\le\lambda L.$$

Let

$$X=\{0,\varepsilon,2\varepsilon,\dots,(n-2)\varepsilon,L\}\subset\mathbb{R},$$

with $L>(n-2)\varepsilon$, and endow X with the diameter n-G-metric

$$G_n(x_1,\dots,x_n)=\max_{1\le i,j\le n}|x_i-x_j|.$$

Define $T:X\to X$ by

$$T(k\varepsilon)=k\varepsilon\quad(k=0,1,\dots,n-2),\qquad T(L)=0.$$

Then T has exactly the $n-1$ fixed points

$$0,\varepsilon,2\varepsilon,\dots,(n-2)\varepsilon.$$

Moreover, X has exactly n elements, so every pairwise distinct n-tuple is a permutation of all elements of X. For such a tuple,

$$G_n(x_1, \ldots, x_n) = L,$$

whereas the image tuple consists of the cluster points $0, \varepsilon, \ldots, (n-2)\varepsilon$ with one repetition of 0, hence

$$G_n(Tx_1, \ldots, Tx_n) = (n-2)\varepsilon \leq \lambda L = \lambda G_n(x_1, \ldots, x_n).$$

Therefore the distinct-tuple contraction has genuine content and the bound $|\mathrm{Fix}(T)| \leq n-1$ is still attained.

Example 4 (A direct n-G Kannan model)**.** Let $X = [0,1]$, let

$$G_n(\omega_1, \ldots, \omega_n) := \max_{1 \leq i,j \leq n} |\omega_i - \omega_j|,$$

and choose $\rho \in (0, 1/(n+1))$. Define $T(x) = \rho x$ for $x \in X$. Then (X, G_n) is complete and T is orbitally continuous. Also, if $x \neq 0$, then

$$x, \rho x, \rho^2 x, \ldots, \rho^{n-1} x,$$

are pairwise distinct; hence T is n-orbit-separated.

For pairwise distinct $\omega_1, \ldots, \omega_n \in X$, put

$$D := G_n(\omega_1, \ldots, \omega_n) = \max_{i,j} |\omega_i - \omega_j|.$$

Then

$$G_n(T\omega_1, \ldots, T\omega_n) = \rho D.$$

Moreover,

$$G_n(\omega_i, T\omega_i, \ldots, T\omega_i) = (1-\rho)\omega_i,$$

and, for every pair i, j,

$$|\omega_i - \omega_j| \leq \omega_i + \omega_j \leq \sum_{k=1}^{n} \omega_k.$$

Thus

$$D \leq \sum_{k=1}^{n} \omega_k.$$

Consequently,

$$G_n(T\omega_1, \ldots, T\omega_n) \leq \frac{\rho}{1-\rho} \sum_{i=1}^{n} G_n(\omega_i, T\omega_i, \ldots, T\omega_i).$$

Since $\rho < 1/(n+1)$, one has $\rho/(1-\rho) < 1/n$. Therefore Theorem 2 applies with any

$$\lambda \in \left[\frac{\rho}{1-\rho}, \frac{1}{n}\right),$$

and the unique fixed point is 0.

Example 5 (A direct n-G Reich model)**.** Let $X = [0,1]$ and let G_n be the same n-G-metric as in Example 3. Fix $p \in [0,1]$ and $\rho \in (0,1)$. Define

$$T(x) := (1-\rho)p + \rho x.$$

Then, for every $m \geq 0$ and every $x \in X$,

$$T^m x = p + \rho^m (x - p),$$

so T is orbitally continuous and every orbit converges to p. If $x \neq p$, then the first n iterates of x are pairwise distinct, so T is n-orbit-separated. Moreover, for pairwise distinct $\omega_1, \ldots, \omega_n \in X$,

$$G_n(T\omega_1, \ldots, T\omega_n) = \rho G_n(\omega_1, \ldots, \omega_n).$$

Thus the Reich-type inequality in Theorem 3 holds with $a_1 = \cdots = a_n = 0$, $a_{n+1} = \rho < 1$. Consequently,

$$\mathrm{Fix}(T) = \{p\}.$$

Example 6 (A Kannan contraction via the associated metric)**.** Let $X = [0,1]$ with the usual metric, and define

$$G_n(\omega_1, \ldots, \omega_n) := \max_{1 \le i,j \le n} |\omega_i - \omega_j|.$$

Then (X, G_n) is complete, and the associated metric is

$$d_G(x, y) = 2|x - y|.$$

Fix $c \in [0,1]$ and define the constant map $T(x) = c$ for all $x \in X$. Then, for all $x, y \in X$,

$$d_G(Tx, Ty) = 0 \le \kappa\big(d_G(x, Tx) + d_G(y, Ty)\big),$$

for every $\kappa \in [0, 1/2)$. Hence Corollary 3 applies. The unique fixed point is c, and every Picard orbit reaches it after one step.

Example 7 (A Reich contraction via the associated metric)**.** Let $X = \mathbb{R}$ and define

$$G_n(\omega_1, \ldots, \omega_n) := \max_{1 \le i,j \le n} |\omega_i - \omega_j|.$$

Fix $\mu \in (0,1)$ and define $T(x) = \mu x$. Then

$$d_G(Tx, Ty) = 2|\mu x - \mu y| = \mu d_G(x, y).$$

Therefore the Reich condition in Corollary 4 holds with $(\alpha, \beta, \gamma) = (\mu, 0, 0)$. The unique fixed point is

$$\mathrm{Fix}(T) = \{0\},$$

and $T^m x = \mu^m x \to 0$ for every $x \in \mathbb{R}$.

We now introduce a concrete n-G-metric generated by polygonal configurations.

Proposition 4. *Let E be a normed linear space. For $\omega_1, \ldots, \omega_n \in E$, define*

$$\Pi_n(\omega_1, \ldots, \omega_n) := \sum_{1 \le i < j \le n} \|\omega_i - \omega_j\|.$$

Then Π_n is an n-G-metric on E.

Proof. If all points are equal, then each term in the sum vanishes, so $\Pi_n(\omega_1, \ldots, \omega_n) = 0$. Conversely, if $\Pi_n(\omega_1, \ldots, \omega_n) = 0$, then every term $\|\omega_i - \omega_j\|$ is zero, hence all ω_i are equal. Thus (G1) holds.

If $\theta \neq v$, then

$$\Pi_n(\theta, \ldots, \theta, v) = (n-1)\|\theta - v\| > 0,$$

which proves (G2).

To verify (G3), let $(\phi_3, \dots, \phi_n) \neq (v, \dots, v)$. Then

$$\Pi_n(\theta, v, \phi_3, \dots, \phi_n) = \|\theta - v\| + \sum_{j=3}^{n} (\|\theta - \phi_j\| + \|v - \phi_j\|) + \sum_{3 \le i < j \le n} \|\phi_i - \phi_j\|.$$

By the triangle inequality,

$$\|\theta - \phi_j\| + \|v - \phi_j\| \ge \|\theta - v\| \qquad (j = 3, \dots, n).$$

Therefore

$$\Pi_n(\theta, v, \phi_3, \dots, \phi_n) \ge (n-1)\|\theta - v\| = \Pi_n(\theta, \dots, \theta, v).$$

Symmetry is immediate, so (G4) holds.
Finally, for (G5), write

$$\Pi_n(\omega_1, \dots, \omega_n) = \sum_{j=2}^{n} \|\omega_1 - \omega_j\| + \sum_{2 \le i < j \le n} \|\omega_i - \omega_j\|.$$

Using the triangle inequality,

$$\|\omega_1 - \omega_j\| \le \|\omega_1 - \eta\| + \|\eta - \omega_j\| \qquad (j = 2, \dots, n),$$

and hence

$$\begin{aligned}\Pi_n(\omega_1, \dots, \omega_n) &\le (n-1)\|\omega_1 - \eta\| + \sum_{j=2}^{n} \|\eta - \omega_j\| + \sum_{2 \le i < j \le n} \|\omega_i - \omega_j\| \\ &= \Pi_n(\omega_1, \eta, \dots, \eta) + \Pi_n(\eta, \omega_2, \dots, \omega_n).\end{aligned}$$

Thus (G5) holds, and Π_n is an n-G-metric. □

Remark 11. The associated metric of this model is explicit:

$$d_G(x, y) = \Pi_n(y, \dots, y, x) + \Pi_n(x, \dots, x, y) = 2(n-1)\|x - y\|.$$

Consequently, d_G-completeness in the pairwise polygonal-length model is exactly norm completeness, up to the constant factor $2(n-1)$.

Remark 12. For $n = 3$, the quantity $\Pi_3(\omega_1, \omega_2, \omega_3)$ is the usual perimeter of the triangle with vertices $\omega_1, \omega_2, \omega_3$. For $n \ge 4$, Π_n is not the ordinary cyclic perimeter of an n-gon; rather, it is the total pairwise edge length, that is, the sum of all distances between pairs of vertices. It is symmetric in all variables and therefore fits naturally into the n-G-metric framework.

Example 8 (Polygon-perimeter dynamics)**.** Let $E = \mathbb{R}^m$, let $p \in E$, and let $K \subset E$ be a nonempty closed convex set such that $p \in K$. On K, consider the n-G-metric Π_n from Proposition 4. Fix $\rho \in (0, 1)$ and define

$$T(x) = (1-\rho)p + \rho x \qquad (x \in K).$$

Then $T(K) \subset K$ and, for all $\omega_1, \dots, \omega_n \in K$,

$$\Pi_n(T\omega_1, \dots, T\omega_n) = \sum_{1 \le i < j \le n} \|\rho\omega_i - \rho\omega_j\| = \rho \Pi_n(\omega_1, \dots, \omega_n).$$

Thus the total total pairwise polygonal length contracts by the factor ρ at each iteration. Moreover, the associated metric is

$$d_G(x, y) = \Pi_n(y, \dots, y, x) + \Pi_n(x, \dots, x, y) = 2(n-1)\|x - y\|.$$

Hence
$$d_G(Tx, Ty) = \rho d_G(x, y),$$
so Corollary 4 applies with $(\alpha, \beta, \gamma) = (\rho, 0, 0)$. Consequently, T has the unique fixed point p, and for every initial vertex $x \in K$,
$$T^m x = (1 - \rho^m)p + \rho^m x \to p.$$

Geometrically, if one starts with an n-vertex configuration $(\omega_1, \ldots, \omega_n)$, then the iterated configuration $(T^m\omega_1, \ldots, T^m\omega_n)$ has total pairwise polygonal length
$$\Pi_n(T^m\omega_1, \ldots, T^m\omega_n) = \rho^m \Pi_n(\omega_1, \ldots, \omega_n),$$
so the whole polygon collapses toward the equilibrium vertex p while preserving the affine directions of the edges.

6. Computational Picard scheme and numerical polygon tests

This section records an explicit computational form of the Picard procedure used in the preceding results. The aim is not to replace the theoretical assumptions, but to show how the n-tuple estimates can be monitored numerically. The quantities used below are the block residual
$$A_k = G_n(x_k, x_{k+1}, \ldots, x_{k+n-1}),$$
the one-step associated-metric error
$$S_k = d_G(x_k, x_{k+1}),$$
and, in the polygonal model, the total perimeter
$$P_k = \Pi_n(v_1^{(k)}, \ldots, v_n^{(k)}).$$

The estimate proved in Lemma 1 gives $S_k \leq 2A_k$, while the Banach-type n-tuple contraction gives $A_{k+1} \leq \lambda A_k$ whenever the consecutive block is pairwise distinct.

6.1. Picard algorithm in an n-G-metric space

Let (X, G_n) be a complete n-G-metric space and let $T : X \to X$ be a self-map satisfying a direct n-tuple contraction on pairwise distinct tuples. Starting from $x_0 \in X$, the numerical Picard procedure is the following.

Algorithm 1: Picard iteration in an n-G-metric space
Input: $x_0 \in X$, map T, order $n \geq 4$, tolerance $\varepsilon > 0$, maximum number of iterations $N_{\max}$.
For $k = 0, 1, \ldots, N_{\max}$:

1. Compute the next iterates $x_{k+j} = T^j x_k$ for $j = 1, \ldots, n$ if they are not already stored.
2. Check whether $x_k, x_{k+1}, \ldots, x_{k+n-1}$ are pairwise distinct. In exact symbolic computations this means literal equality; in floating-point computations use a tolerance $\tau > 0$ and regard two entries as repeated when their d_G-distance is less than τ. If a repetition is detected, record a short periodic obstruction or a fixed point.
3. Compute
$$A_k = G_n(x_k, x_{k+1}, \ldots, x_{k+n-1}), \qquad S_k = d_G(x_k, x_{k+1}).$$
4. If $S_k < \varepsilon$, stop and return x_{k+1} as an approximate fixed point.
5. Otherwise set $k \leftarrow k + 1$ and continue.

Output: approximate fixed point, residual history (A_k), associated-metric step history (S_k).

The convergence proof of Theorem 1 gives the a posteriori tail estimate
$$d_G(x_k, u) \leq \sum_{j=k}^{\infty} d_G(x_j, x_{j+1}) \leq \frac{2A_0}{1-\lambda}\lambda^k,$$

whenever the hypotheses of that theorem hold and u is the fixed point reached by the Picard orbit.

6.2. Pseudocode for implementation

For implementation in a numerical language, the previous procedure can be written as follows.

Pseudocode

1. Set $x[0] = x_0$.
2. For $k = 0, 1, \dots, N_{\max}$ do:
   (a) Generate missing values $x[k+1], \dots, x[k+n]$ by $x[j+1] = T(x[j])$.
   (b) If the block $x[k], \dots, x[k+n-1]$ is not pairwise distinct, using exact equality in symbolic mode or a tolerance threshold in floating-point mode, stop and report a short cycle.
   (c) Compute $A[k] = G_n(x[k], \dots, x[k+n-1])$.
   (d) Compute $S[k] = d_G(x[k], x[k+1])$.
   (e) If $S[k] < \varepsilon$, return $x[k+1]$.
3. Return the last iterate and the histories A and S.

6.3. A discrete polygonal model

We now give a computational model in $\mathbb{R}^2$. Fix $p = (0, 0)$ and let

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

be a planar rotation. For a contraction factor $\rho \in (0, 1)$ define

$$T_{\rho,\alpha}(x) = \rho R_\alpha x.$$

For an n-vertex configuration $V^{(k)} = (v_1^{(k)}, \dots, v_n^{(k)})$ the discrete polygonal dynamics is

$$v_i^{(k+1)} = T_{\rho,\alpha}(v_i^{(k)}) = \rho R_\alpha v_i^{(k)}, \qquad i = 1, \dots, n.$$

Since rotations preserve Euclidean distances, the pairwise polygonal-length n-G-metric satisfies

$$\Pi_n(T_{\rho,\alpha} v_1, \dots, T_{\rho,\alpha} v_n) = \rho \Pi_n(v_1, \dots, v_n).$$

Thus the n-tuple contraction constant is

$$\lambda = \rho.$$

Moreover, the associated metric of Π_n is

$$d_G(x, y) = 2(n-1)\|x - y\|,$$

and hence

$$d_G(T_{\rho,\alpha} x, T_{\rho,\alpha} y) = \rho d_G(x, y).$$

This gives a direct comparison between the polygonal dynamics and the associated metric: both decay with the same factor ρ.

For the numerical tests, the initial polygon is chosen as

$$v_i^{(0)} = \big(\cos(2\pi(i-1)/n), \sin(2\pi(i-1)/n)\big), \qquad i = 1, \dots, n,$$

and we use $\alpha = \pi/8$. The quantities reported are

$$P_k = \Pi_n(v_1^{(k)}, \dots, v_n^{(k)}), \qquad E_k = \max_{1 \le i \le n} d_G(v_i^{(k)}, 0),$$

and the residual
$$R_k = \max_{1 \le i \le n} \|v_i^{(k+1)} - v_i^{(k)}\|.$$

For this model, $P_k/P_0 = E_k/E_0 = \rho^k$ and $R_k = (1 + \rho^2 - 2\rho\cos\alpha)^{1/2}\rho^k$.

6.4. Numerical convergence tables

The first table records the convergence history for $n = 6$, $\lambda = \rho = 0.55$ and $\alpha = \pi/8$. Since this linear model has the exact identity $P_k/P_0 = E_k/E_0 = \rho^k$, the table should be read as an illustrative verification of the theoretical formula rather than as independent computational validation.

k	P_k	P_k/P_0	E_k/E_0	R_k
0	22.3923	1.000000	1.000000	0.4814
1	12.3158	0.550000	0.550000	0.2648
2	6.7737	0.302500	0.302500	0.1456
4	2.0490	0.091506	0.091506	0.0441
8	0.1875	0.008373	0.008373	0.0040
12	0.0172	0.000766	0.000766	0.0004

The second table compares several values of n, λ and ρ. Here $\lambda = \rho$, because the mapping is an exact pairwise polygonal-length contraction. The stopping index $k_{10^{-6}}$ is the first integer k for which $E_k/E_0 \le 10^{-6}$.

n	ρ	λ	P_0	P_{10}/P_0	$k_{10^{-6}}$
4	0.25	0.25	9.6569	$9.5367 \cdot 10^{-7}$	10
4	0.50	0.50	9.6569	$9.7656 \cdot 10^{-4}$	20
4	0.75	0.75	9.6569	$5.6314 \cdot 10^{-2}$	49
6	0.25	0.25	22.3923	$9.5367 \cdot 10^{-7}$	10
6	0.50	0.50	22.3923	$9.7656 \cdot 10^{-4}$	20
6	0.75	0.75	22.3923	$5.6314 \cdot 10^{-2}$	49
8	0.25	0.25	41.0060	$9.5367 \cdot 10^{-7}$	10
8	0.50	0.50	41.0060	$9.7656 \cdot 10^{-4}$	20
8	0.75	0.75	41.0060	$5.6314 \cdot 10^{-2}$	49

The table shows two complementary effects. The decay rate is controlled by $\rho = \lambda$, independently of n. By contrast, the initial pairwise polygonal-length P_0 increases with n because more pairwise edges are included in Π_n. This is why the relative ratios P_k/P_0 and E_k/E_0 are the natural quantities for comparing different values of n.

6.5. A nonlinear polygonal test

To include a less rigid example, consider
$$T^{\mathrm{nl}}_{\rho,\alpha}(x) = \rho R_\alpha \frac{x}{1 + \|x\|}, \qquad x \in \mathbb{R}^2,$$
where $0 < \rho < 1$. The radial map $x \mapsto x/(1 + \|x\|)$ is nonexpansive in the Euclidean norm, and the rotation is an isometry; hence
$$\|T^{\mathrm{nl}}_{\rho,\alpha}(x) - T^{\mathrm{nl}}_{\rho,\alpha}(y)\| \le \rho\|x - y\|.$$

Therefore the pairwise polygonal-length functional satisfies
$$\Pi_n(T^{\mathrm{nl}}_{\rho,\alpha}v_1, \dots, T^{\mathrm{nl}}_{\rho,\alpha}v_n) \le \rho\Pi_n(v_1, \dots, v_n),$$
but the residual ratios are no longer forced to be exactly ρ^k. For $n = 6$, $\rho = 0.55$, $\alpha = \pi/8$ and the same initial regular hexagon, one obtains the following illustrative values.

k	P_k	P_k/P_0	E_k/E_0	R_k
0	22.3923	1.000000	1.000000	0.753320
1	6.1579	0.275000	0.275000	0.171519
2	2.6563	0.118627	0.118627	0.068481
4	0.6787	0.030311	0.030311	0.016563
8	0.0585	0.002614	0.002614	0.001401
12	0.0053	0.000238	0.000238	0.000127

This nonlinear test has the same qualitative conclusion as the theory: the associated-metric error and the pairwise polygonal length both tend to zero, while the decay history is not merely a restatement of a closed-form identity.

6.6. Comparison between d_G and pairwise polygonal-length dynamics

For the pairwise polygonal-length n-G-metric Π_n, the associated metric measures the distance between two individual vertices, while Π_n measures the dispersion of the whole configuration. In the present model,

$$E_k = \max_i d_G(v_i^{(k)}, 0) = 2(n-1)\rho^k,$$

whereas

$$P_k = \Pi_n(v_1^{(k)}, \dots, v_n^{(k)}) = \rho^k P_0.$$

Thus both quantities have the same asymptotic convergence factor, but they encode different information: E_k measures how far the farthest vertex is from the equilibrium point, while P_k measures how much spread remains inside the polygon. Numerically, this gives a simple diagnostic: convergence in d_G confirms collapse toward the fixed point, and convergence of P_k confirms collapse of the entire polygonal configuration.

This discrete model can be interpreted as a damped rotating polygonal system. At each step every vertex is rotated by the same angle and multiplied by ρ. The rotation preserves the shape angles, while the factor ρ reduces all pairwise distances. Hence the system is a genuine computational example of the theory: the fixed point is 0, every vertex converges to 0, and the total n-G pairwise polygonal-length decays exactly according to the predicted contraction law.

7. Conclusion

We revised the fixed point theory for self-maps on n-G-metric spaces in a form that emphasizes the role of periodic points. A contraction acting on pairwise distinct n-tuples cannot have n distinct periodic points; in particular, it cannot have a periodic orbit of length at least n, and it has at most $n-1$ fixed points. This periodic formulation clarifies the precise obstruction created by distinct-tuple hypotheses and distinguishes the present results from ordinary metric fixed point principles.

Existence of fixed points is obtained through Picard iteration under an n-orbit-separation condition and orbital continuity. The latter assumption is explicitly retained because the direct n-tuple contraction gives estimates only on separated blocks and does not by itself control tuples with repeated limiting entries. Direct n-ary Kannan and Reich analogues were proved in the same framework. By contrast, results formulated only through the associated metric d_G were stated as standard metric corollaries and not as new independent n-G principles.

The examples show sharpness, direct n-tuple models and metric consequences. The total pairwise polygonal-length functional Π_n gives a concrete geometric realization of the theory: under affine contractions, an n-vertex configuration collapses toward an equilibrium while its total pairwise perimeter decays geometrically. The computational section makes this mechanism explicit by giving a Picard algorithm, pseudocode, convergence tables and a damped rotating polygonal system. The numerical tests confirm that the associated-metric error and the pairwise polygonal-length residual decay with the predicted factor $\lambda = \rho$. Possible further directions include Chatterjea-type contractions, common periodic point theorems, multivalued analogues, cyclic mappings, best-proximity versions and larger-scale computational schemes in n-G-metric spaces.

Conflicts of Interest: The author declares that there are no competing interests of a financial, personal or professional nature that could have influenced the work reported in this paper.

Funding Information: The author received no financial support for the research, authorship, or publication of this article.

Acknowledgments: The author sincerely thanks the editor and the anonymous reviewers for their valuable comments and suggestions, which improved the quality of this manuscript.

The author gratefully acknowledges Dr. Salem Rebhi for his valuable support, encouragement, and helpful advice during the preparation of this work.

References

[1] Banach, S. (1922). On operations in abstract sets and their application to integral equations. *Fundamenta Mathematicae , 3*(1), 133-181.

[2] Kannan, R. (1969). Some results on fixed points—II. *The American Mathematical Monthly, 76*(4), 405-408.

[3] Reich, S. (1971). Some remarks concerning contraction mappings. Canadian Mathematical Bulletin, 14(1), 121-124.

[4] Khamsi, M. A., & Kirk, W. A. (2011). *An Introduction to Metric Spaces and Fixed Point Theory.* John Wiley & Sons.

[5] Kirk, W., & Shahzad, N. (2014). *Fixed Point Theory in Distance Spaces* (Vol. 1). Cham: springer.

[6] Rus, I. A. (2001). *Generalized Contractions and Applications*. Cluj University Press.

[7] Rus, I. A. (2001). *Generalized Contractions and Applications*. Cluj University Press.

[8] Mustafa, Z., Obiedat, H., & Awawdeh, F. (2008). Some fixed point theorem for mapping on complete G-metric spaces. *Fixed Point Theory and Applications, 2008*(1), 189870.

[9] Jleli, M., & Samet, B. (2012). Remarks on G-metric spaces and fixed point theorems. *Fixed Point Theory and Applications, 2012*(1), 210.

[10] JLELI, M., PĂCURAR, C., & SAMET, B. (2025). New directions in fixed point theory in g-metric spaces and applications to mappings contracting perimeters of triangles. *Fixed Point Theory, 26*(2), 563–580.

[11] Felhi, A. (2026). Novel fixed point outcomes in compact generalized metric spaces. *Engineering Mathematics Letters* **2026** (2026), Article 1.