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# Solvability of discrete second-order two-point boundary value problems

**Jagan Mohan Jonnalagadda**

Department of Mathematics, Birla Institute of Technology & Science Pilani, Hyderabad, Telangana, India - 500078; j.jaganmohan@hotmail.com

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**Abstract:** This article considers a second-order difference equation with constant coefficients in its standard form and two different classes of two-point homogeneous boundary conditions. First, we construct the corresponding Green functions and derive some important properties for further analysis. Next, we propose adequate conditions for the existence of solutions to the considered boundary value problems. Finally, we offer two examples to show the applicability of the main results.

**Keywords:** second-order difference equation, constant coefficients, standard form, boundary value problem, Green's function, fixed point, existence

**MSC:** 39A12, 39A23, 39A27.

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## 1. Introduction

Merdivenci [1] considered the following second-order boundary value problem for difference equations, and established criteria for the existence of solutions by an application of a fixed point theorem:

$$\left(\Delta^2 y\right)(\zeta-1) = f(\zeta, y(\zeta)), \quad \zeta \in \mathbb{N}_1^T, \tag{1}$$

$$Ay(0) - B\left(\Delta y\right)(0) = 0, \quad Cy(T) + D\left(\Delta y\right)(T) = 0, \tag{2}$$

where $A, B, C \geq 0$, $A^2 + B^2 > 0$, $D > 0$ with $ACT + AD + BC > 0$; $T \in \mathbb{N}_3$; $y : \mathbb{N}_0^{T+1} \to \mathbb{R}$, and $f : \mathbb{N}_1^T \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to its second argument. Also, Atici [2] proved the existence of solutions for the nonlinear discrete Sturm–Liouville equation

$$\left(\Delta^2 y\right)(\zeta-1) - \alpha y(\zeta) = f(\zeta, y(\zeta)), \quad \zeta \in \mathbb{N}_1^T, \tag{3}$$

with $\alpha \geq 0$ subject to two-point homogeneous separated boundary conditions (2) using some results of differentiable operators. Following these works, Aykut et al. [3] investigated the existence and uniqueness of solutions for the boundary value problem (3) - (2) using fixed point theory under the following assumptions: $\alpha \geq 0$; $A, B, C, D \geq 0$, $A + B > 0$, $C + D > 0$, $A + C > 0$ (if $\alpha = 0$). Further, in [4–6], Atici et al. proved the existence of solutions of the second-order nonlinear difference Eq. (3) for $\alpha > 0$ with periodic boundary conditions

$$y(0) = y(T), \quad \left(\Delta y\right)(0) = \left(\Delta y\right)(T), \tag{4}$$

using suitable fixed point theorems.

In 2015, Lyons and Neugebauer [7] studied the second-order nonlinear difference Eq. (1) satisfying the anti-periodic boundary conditions

$$y(0) + y(T) = 0, \quad \left(\Delta y\right)(0) + \left(\Delta y\right)(T) = 0. \tag{5}$$

It is obvious that the second-order difference Eq. (1) associated with (4) or any of the following pairs of two-point boundary conditions is at resonance.

$$(\Delta y)(0) = 0, \quad (\Delta y)(T) = 0, \tag{6}$$

$$y(0) = 0, \quad (\Delta y)(0) = (\Delta y)(T). \tag{7}$$

At the same time, the boundary value problems (3) - (4) and (3) - (6) are nonresonant for $\alpha > 0$. Recently, the authors of [8–10] observed that the second-order difference Eq. (1) associated with the following pairs of two-point boundary conditions is also at resonance.

$$y(0) = y(T), \qquad (\Delta y)(0) + (\Delta y)(T) = 0, \tag{8}$$

$$y(0) + y(T) = 0, \qquad (\Delta y)(0) = (\Delta y)(T). \tag{9}$$

Inspired by these studies, in this article, we examine the second order difference equation

$$\left(\Delta^2 y\right)(\zeta - 1) - \alpha (\Delta y)(\zeta - 1) - \beta y(\zeta) = f(\zeta, y(\zeta)), \quad \zeta \in \mathbb{N}_1^T, \tag{10}$$

associated with the boundary conditions either (2) or

$$Ay(0) + By(T) = 0, \quad C(\Delta y)(0) + D(\Delta y)(T) = 0, \tag{11}$$

where $\alpha, \beta \geq 0$; $A, B, C, D \in \mathbb{R}$ with $A^2 + B^2, C^2 + D^2 > 0$; $T \in \mathbb{N}_3$; $y : \mathbb{N}_0^{T+1} \to \mathbb{R}$, and $f : \mathbb{N}_1^T \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to its second argument.

Clearly, the boundary conditions (4), (5), (7), (8), and (9) are particular cases of (11). Moreover, a suitable choice of $\alpha$ and $\beta$ yield the boundary value problems (10) - (7), (10) - (8), and (10) - (9) non-resonant.

In particular, the following Tables 1 and 2 illustrate that the boundary value problems considered in [1–10] as special cases of the boundary value problems (10) - (2) and (10) - (11):

**Table 1.** Special cases of the boundary value problem (10) - (2) studied in the literature

| $\alpha$ | $\beta$ | $A$ | $B$ | $C$ | $D$ | Reference |
|---|---|---|---|---|---|---|
| 0 | 0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $> 0$ | [1] |
| $\geq 0$ | 0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $> 0$ | [2] |
| $\geq 0$ | 0 | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ | [3] |

**Table 2.** Special cases of the boundary value problem (10) - (11) studied in the literature

| $\alpha$ | $\beta$ | $A$ | $B$ | $C$ | $D$ | Reference |
|---|---|---|---|---|---|---|
| $> 0$ | 0 | 1 | $-1$ | 1 | $-1$ | [4–6] |
| 0 | 0 | 1 | 1 | 1 | 1 | [7] |
| 0 | 0 | 1 | $-1$ | 1 | 1 | [8–10] |
| 0 | 0 | 1 | 1 | 1 | $-1$ | [8–10] |

Consequently, the results established in this article not only encompass the works of [1–10] as special cases, but also extend to a broad range of second-order difference equations through appropriate choices of the parameters $\alpha$ and $\beta$. Moreover, they accommodate a wide variety of boundary conditions via suitable selections of the constants $A$, $B$, $C$, and $D$.

We organize the present article as follows: §2 contains preliminaries on discrete calculus [11] and fixed point theory [12]. In §3, we obtain the expressions for the Green functions associated with the boundary value problem (10) - (2) and (10) - (11). We also derive a few important properties of the Green functions, which we will use to deduce the main results. In §4, we establish sufficient conditions for the existence of solutions to the boundary value problems (10) - (2) and (10) - (11). §5 presents two examples to show the applicability of the main results.

## 2. Preliminaries

In this section, we present the following fundamentals of discrete calculus [11] and fixed point theory [12], which we will use throughout the article. Denote by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for any real numbers $a$ and $b$ such that $b - a \in \mathbb{N}_1$.

**Definition 1.** [11] The forward jump operator $\sigma : \mathbb{N}_a \to \mathbb{N}_{a+1}$ is defined by

$$\sigma(\zeta) = \zeta + 1, \quad \zeta \in \mathbb{N}_a.$$

**Definition 2.** [11] Let $y : \mathbb{N}_a^b \to \mathbb{R}$ and $M \in \mathbb{N}_1$. The first-order forward difference of $y$ is defined by

$$(\Delta y)(\zeta) = y(\sigma(\zeta)) - y(\zeta), \quad \zeta \in \mathbb{N}_a^{b-1},$$

and the $M^{th}$-order forward difference of $y$ is defined recursively by

$$\left(\Delta^M y\right)(\zeta) = \left(\Delta\left(\Delta^{M-1} y\right)\right)(\zeta), \quad \zeta \in \mathbb{N}_a^{b-M}.$$

Consider

$$\mathcal{B}_1 = \left\{y : \mathbb{N}_0^{T+1} \to \mathbb{R} \mid Ay(0) - B(\Delta y)(0) = 0,\ Cy(T) + D(\Delta y)(T) = 0\right\} \subseteq \mathbb{R}^{T+2},$$

and

$$\mathcal{B}_2 = \left\{y : \mathbb{N}_0^{T+1} \to \mathbb{R} \mid Ay(0) + By(T) = 0,\ C(\Delta y)(0) + D(\Delta y)(T) = 0\right\} \subseteq \mathbb{R}^{T+2}.$$

Clearly, $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces equipped with the maximum norm defined by

$$\|y\| = \max_{\zeta \in \mathbb{N}_0^{T+1}} |y(\zeta)|,$$

for any $y \in \mathcal{B}_1$ (or $\mathcal{B}_2$).

We apply the Leray–Schauder nonlinear alternative to establish sufficient conditions for the existence of solutions to the boundary value problem (10) - (11). We state this theorem as follows for convenience:

**Theorem 1.** [12] *(Leray–Schauder Nonlinear Alternative) Let $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$ be a Banach space, $C$ a closed, convex subset of $\mathcal{B}$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $T : \bar{U} \to C$ is a completely continuous map. Then, either*

1. *$T$ has a fixed point in $\bar{U}$; or*
2. *there exist a $y \in \partial U$ and $\lambda \in (0, 1)$ such that $y = \lambda T y$.*

## 3. Green functions & their properties

In this section, we obtain the expressions for the Green functions associated with the boundary value problems (10) - (2) and (10) - (11). We also derive a few important properties of the Green functions, which we will use to deduce the main results.

Denote by

$$\begin{aligned}
\lambda &= \frac{(2 + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\beta}}{2}, \\
\Lambda &= \lambda - \frac{1 + \alpha}{\lambda} = \frac{\lambda^2 - \alpha - 1}{\lambda}, \\
g(x) &= \lambda^x - \left(\frac{1 + \alpha}{\lambda}\right)^x, \quad x \in \mathbb{N}_0, \\
h(x) &= \lambda^{-x} - \left(\frac{1 + \alpha}{\lambda}\right)^{-x}, \quad x \in \mathbb{N}_0, \\
\omega(x) &= g(x + 1) - (1 + \alpha) g(x), \quad x \in \mathbb{N}_0, \\
w(x) &= (1 + \alpha)^{-1} h(x) - h(x + 1), \quad x \in \mathbb{N}_0.
\end{aligned}$$

**Lemma 1.** *We have that*

(1) $\lambda \geq 1+\alpha+\beta \geq 1+\alpha$;
(2) $\lambda^2-(2+\alpha+\beta)\lambda+(1+\alpha)=0$;
(3) $\lambda \geq 1$;
(4) $\Lambda \geq 0$;
(5) $g : \mathbb{N}_0 \to \mathbb{R}$ *is a nonnegative nondecreasing function;*
(6) $h : \mathbb{N}_0 \to \mathbb{R}$ *is a nonpositive nonincreasing function;*
(7) $\omega : \mathbb{N}_0 \to \mathbb{R}$ *is a nonnegative nondecreasing function;*
(8) $w : \mathbb{N}_0 \to \mathbb{R}$ *is a nonnegative nondecreasing function.*

**Proof.** First, we prove (1). Consider

$$\lambda = \frac{(2+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\beta}}{2} \geq \frac{(2+\alpha+\beta)+(\alpha+\beta)}{2} = 1+\alpha+\beta \geq 1+\alpha.$$

The proof of (1) is complete. Next, we prove (2). The roots of the quadratic equation $x^2-(2+\alpha+\beta)x+(1+\alpha)=0$ are

$$x = \frac{(2+\alpha+\beta)\pm\sqrt{(2+\alpha+\beta)^2-4(1+\alpha)}}{2} = \frac{(2+\alpha+\beta)\pm\sqrt{(\alpha+\beta)^2+4\beta}}{2}.$$

Since

$$\lambda = \frac{(2+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\beta}}{2},$$

it satisfies the quadratic equation $x^2-(2+\alpha+\beta)x+(1+\alpha)=0$. The proof of (2) is complete. Now, we prove (3). From (1), we have $\lambda \geq 1+\alpha \geq 1$. The proof of (3) is complete. Next, we prove (4). From (1) and (3), we have $\frac{1+\alpha}{\lambda} \leq 1$ and $\lambda \geq 1$ implying that

$$\Lambda = \lambda - \frac{1+\alpha}{\lambda} \geq 0.$$

The proof of (4) is complete. Now, we prove (5). Clearly, $g(0)=0$, $g(1)=\Lambda \geq 0$, and

$$g(2) = \lambda^2 - \left(\frac{1+\alpha}{\lambda}\right)^2 = \Lambda\left(\lambda+\frac{1+\alpha}{\lambda}\right) \geq 0.$$

For $x \in \mathbb{N}_0$, consider

$$\begin{aligned}
(\Delta g)(x) &= g(x+1)-g(x) \\
&= \left[\lambda^{x+1}-\left(\frac{1+\alpha}{\lambda}\right)^{x+1}\right]-\left[\lambda^x-\left(\frac{1+\alpha}{\lambda}\right)^x\right] \\
&= \left[\lambda^{x+1}-\lambda^x\right]-\left[\left(\frac{1+\alpha}{\lambda}\right)^{x+1}-\left(\frac{1+\alpha}{\lambda}\right)^x\right] \\
&= (\lambda-1)\lambda^x-\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^x \\
&= (\lambda-1)\lambda^x+\left(1-\frac{1+\alpha}{\lambda}\right)\left(\frac{1+\alpha}{\lambda}\right)^x.
\end{aligned}$$

Clearly, $\lambda-1 \geq 0$ and $1-\dfrac{1+\alpha}{\lambda} \geq 0$. Since $\lambda^x > 0$ and $\left(\frac{1+\alpha}{\lambda}\right)^x > 0$ for all $x \in \mathbb{N}_0$, we obtain that

$$(\Delta g)(x) \geq 0, \quad x \in \mathbb{N}_0,$$

implying that $g$ is a nonnegative nondecreasing function. Next, we prove (6). Clearly, $h(0)=0$, and

$$h(1) = \frac{1}{\lambda}-\frac{\lambda}{1+\alpha} = \frac{1+\alpha-\lambda}{\lambda(1+\alpha)} \leq 0.$$

For $x \in \mathbb{N}_0$, consider

$$\begin{aligned}
(\Delta h)(x) &= h(x+1) - h(x) \\
&= \left[\lambda^{-x-1} - \left(\frac{1+\alpha}{\lambda}\right)^{-x-1}\right] - \left[\lambda^{-x} - \left(\frac{1+\alpha}{\lambda}\right)^{-x}\right] \\
&= \left[\lambda^{-x-1} - \lambda^{-x}\right] - \left[\left(\frac{1+\alpha}{\lambda}\right)^{-x-1} - \left(\frac{1+\alpha}{\lambda}\right)^{-x}\right] \\
&= (1-\lambda)\lambda^{-x-1} - \left(1 - \frac{1+\alpha}{\lambda}\right)\left(\frac{1+\alpha}{\lambda}\right)^{-x-1} \\
&= -(\lambda-1)\lambda^{-x-1} - \left(1 - \frac{1+\alpha}{\lambda}\right)\left(\frac{1+\alpha}{\lambda}\right)^{-x-1}.
\end{aligned}$$

Clearly, $\lambda - 1 \geq 0$ and $1 - \dfrac{1+\alpha}{\lambda} \geq 0$. Since $\lambda^{-x-1} > 0$ and $\left(\frac{1+\alpha}{\lambda}\right)^{-x-1} > 0$ for all $x \in \mathbb{N}_0$, we obtain that

$$(\Delta h)(x) \leq 0, \quad x \in \mathbb{N}_0,$$

implying that $h$ is a nonpositive nonincreasing function. Now, we prove (7). Clearly,

$$\omega(0) = g(1) - (1+\alpha)g(0) = \Lambda \geq 0.$$

For $x \in \mathbb{N}_0$, consider

$$\begin{aligned}
(\Delta\omega)(x) &= \omega(x+1) - \omega(x) \\
&= [g(x+2) - (1+\alpha)g(x+1)] - [g(x+1) - (1+\alpha)g(x)] \\
&= g(x+2) - (2+\alpha)g(x+1) + (1+\alpha)g(x) \\
&= \left[\lambda^{x+2} - \left(\frac{1+\alpha}{\lambda}\right)^{x+2}\right] - (2+\alpha)\left[\lambda^{x+1} - \left(\frac{1+\alpha}{\lambda}\right)^{x+1}\right] \\
&\quad + (1+\alpha)\left[\lambda^{x} - \left(\frac{1+\alpha}{\lambda}\right)^{x}\right] \\
&= \left[\lambda^{x+2} - (2+\alpha)\lambda^{x+1} + (1+\alpha)\lambda^{x}\right] \\
&\quad - \left[\left(\frac{1+\alpha}{\lambda}\right)^{x+2} - (2+\alpha)\left(\frac{1+\alpha}{\lambda}\right)^{x+1} + (1+\alpha)\left(\frac{1+\alpha}{\lambda}\right)^{x}\right] \\
&= \lambda^{x}\left[\lambda^2 - (2+\alpha)\lambda + (1+\alpha)\right] \\
&\quad - \frac{1}{\lambda}\left(\frac{1+\alpha}{\lambda}\right)^{x+1}\left[\lambda^2 - (2+\alpha)\lambda + (1+\alpha)\right] \\
&= \frac{1}{\lambda}\left[\lambda^2 - (2+\alpha)\lambda + (1+\alpha)\right] g(x+1) \\
&\geq \frac{1}{\lambda}\left[\lambda^2 - (2+\alpha+\beta)\lambda + (1+\alpha)\right] g(x+1) \\
&= 0,
\end{aligned}$$

implying that $\omega$ is a nonnegative nondecreasing function. Finally, we prove (8). Clearly,

$$w(0) = (1+\alpha)^{-1}h(0) - h(1) = \frac{\lambda - \alpha - 1}{\lambda(1+\alpha)} \geq 0.$$

For $x \in \mathbb{N}_0$, consider

$$(\Delta w)(x) = w(x+1) - w(x)$$

$$
\begin{aligned}
&= \Big[(1+\alpha)^{-1}h(x+1) - h(x+2)\Big] - \Big[(1+\alpha)^{-1}h(x) - h(x+1)\Big] \\
&= -h(x+2) + \left(1 + \frac{1}{1+\alpha}\right) h(x+1) - \frac{h(x)}{(1+\alpha)} \\
&= -\left[\lambda^{-x-2} - \left(\frac{1+\alpha}{\lambda}\right)^{-x-2}\right] + \left(1+\frac{1}{1+\alpha}\right)\left[\lambda^{-x-1} - \left(\frac{1+\alpha}{\lambda}\right)^{-x-1}\right] \\
&\quad - \frac{1}{(1+\alpha)}\left[\lambda^{-x} - \left(\frac{1+\alpha}{\lambda}\right)^{-x}\right] \\
&= -\left[\lambda^{-x-2} - \left(1+\frac{1}{1+\alpha}\right)\lambda^{-x-1} + \frac{1}{(1+\alpha)}\lambda^{-x}\right] \\
&\quad + \left[\left(\frac{1+\alpha}{\lambda}\right)^{-x-2} - \left(1+\frac{1}{1+\alpha}\right)\left(\frac{1+\alpha}{\lambda}\right)^{-x-1} + \frac{1}{(1+\alpha)}\left(\frac{1+\alpha}{\lambda}\right)^{-x}\right] \\
&= -\frac{1}{\lambda(1+\alpha)}\Big[\lambda^2 - (2+\alpha)\lambda + (1+\alpha)\Big] h(x+1) \\
&\geq -\frac{1}{\lambda(1+\alpha)}\Big[\lambda^2 - (2+\alpha+\beta)\lambda + (1+\alpha)\Big] h(x+1) = 0,
\end{aligned}
$$

implying that $w$ is a nonnegative nondecreasing function. The proof is complete. □

**Remark 1.** To the end of this article, we assume that $\Lambda \neq 0$.

**Lemma 2.** *A general solution of the homogeneous second-order difference equation*

$$\left(\Delta^2 y\right)(\zeta-1) - \alpha\left(\Delta y\right)(\zeta-1) - \beta y(\zeta) = 0, \quad \zeta \in \mathbb{N}_1, \tag{12}$$

*is given by*

$$y(\zeta) = C_1\lambda^{\zeta} + C_2\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}, \quad \zeta \in \mathbb{N}_0, \tag{13}$$

*where $C_1$ and $C_2$ are arbitrary constants.*

**Proof.** In order to prove that (13) is a general solution of (12), it is enough to show that $y_1(\zeta) = \lambda^{\zeta}$ and $y_2(\zeta) = \left(\frac{1+\alpha}{\lambda}\right)^{\zeta}$ are two linearly independent solutions of (12) on $\mathbb{N}_0$. For $\zeta \in \mathbb{N}_1$, consider

$$
\begin{aligned}
\left(\Delta^2 y_1\right)(\zeta-1) - \alpha\left(\Delta y_1\right)(\zeta-1) - \beta y_1(\zeta) &= y_1(\zeta+1) - (2+\alpha+\beta)y_1(\zeta) + (1+\alpha)y_1(\zeta-1) \\
&= \lambda^{\zeta+1} - (2+\alpha+\beta)\lambda^{\zeta} + (1+\alpha)\lambda^{\zeta-1} \\
&= \lambda^{\zeta-1}\Big[\lambda^2 - (2+\alpha+\beta)\lambda + (1+\alpha)\Big] \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\Delta^2 y_2\right)(\zeta-1) - \alpha\left(\Delta y_2\right)(\zeta-1) - \beta y_2(\zeta) &= y_2(\zeta+1) - (2+\alpha+\beta)y_2(\zeta) + (1+\alpha)y_2(\zeta-1) \\
&= \left(\frac{1+\alpha}{\lambda}\right)^{\zeta+1} - (2+\alpha+\beta)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta} + (1+\alpha)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta-1} \\
&= \left(\frac{1+\alpha}{\lambda}\right)^{\zeta-1}\left[\left(\frac{1+\alpha}{\lambda}\right)^2 - (2+\alpha+\beta)\left(\frac{1+\alpha}{\lambda}\right) + (1+\alpha)\right] \\
&= \frac{1}{\lambda}\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}\Big[\lambda^2 - (2+\alpha+\beta)\lambda + (1+\alpha)\Big] \\
&= 0,
\end{aligned}
$$

implying that $y_1(\zeta) = \lambda^\zeta$ and $y_2(\zeta) = \left(\frac{1+\alpha}{\lambda}\right)^\zeta$ are solutions of (12) on $\mathbb{N}_0$. Further, the Wronskian of $y_1$ and $y_2$ is given by

$$\begin{aligned}
W(y_1, y_2)(\zeta) &= \begin{vmatrix} y_1(\zeta) & y_2(\zeta) \\ (\Delta y_1)(\zeta) & (\Delta y_2)(\zeta) \end{vmatrix} \\
&= \begin{vmatrix} \lambda^\zeta & \left(\frac{1+\alpha}{\lambda}\right)^\zeta \\ (\lambda - 1)\lambda^\zeta & \left(\frac{1+\alpha}{\lambda} - 1\right)\left(\frac{1+\alpha}{\lambda}\right)^\zeta \end{vmatrix} \\
&= \left(\frac{1+\alpha}{\lambda} - 1\right)\lambda^\zeta \left(\frac{1+\alpha}{\lambda}\right)^\zeta - (\lambda - 1)\lambda^\zeta \left(\frac{1+\alpha}{\lambda}\right)^\zeta \\
&= -\Lambda(1+\alpha)^\zeta \neq 0, \quad \zeta \in \mathbb{N}_1,
\end{aligned}$$

implying that $y_1(\zeta) = \lambda^\zeta$ and $y_2(\zeta) = \left(\frac{1+\alpha}{\lambda}\right)^\zeta$ are linearly independent on $\mathbb{N}_0$. The proof is complete. □

**Lemma 3.** *Assume k is a real-valued function defined on a discrete set $\mathbb{N}_1$. A general solution of the nonhomogeneous second-order difference equation*

$$\left(\Delta^2 y\right)(\zeta - 1) - \alpha\,(\Delta y)(\zeta - 1) - \beta y(\zeta) = k(\zeta), \quad \zeta \in \mathbb{N}_1, \tag{14}$$

*is given by*

$$\begin{aligned}
y(\zeta) &= C_1\lambda^\zeta + C_2\left(\frac{1+\alpha}{\lambda}\right)^\zeta + \frac{1}{\Lambda}\sum_{s=1}^{\zeta-1}\left[\lambda^{\zeta-s} - \left(\frac{1+\alpha}{\lambda}\right)^{\zeta-s}\right]k(s) &(15)\\
&= C_1\lambda^\zeta + C_2\left(\frac{1+\alpha}{\lambda}\right)^\zeta + \frac{1}{\Lambda}\sum_{s=1}^{\zeta-1} g(\zeta - s)k(s), &(16)
\end{aligned}$$

*for $\zeta \in \mathbb{N}_0$. Here $C_1$, $C_2$ are arbitrary constants and*

$$g(\zeta - s) = \lambda^{\zeta-s} - \left(\frac{1+\alpha}{\lambda}\right)^{\zeta-s}, \quad s \in \mathbb{N}_1^{\zeta-1}.$$

**Proof.** In view of Lemma 2, it is enough to show that

$$v(\zeta) = \frac{1}{\Lambda}\sum_{s=1}^{\zeta-1} g(\zeta - s)k(s), \quad \zeta \in \mathbb{N}_0,$$

is a particular solution of (14). For this purpose, we claim that

$$\left(\Delta^2 v\right)(\zeta - 1) - \alpha\,(\Delta v)(\zeta - 1) - \beta v(\zeta) = k(\zeta), \quad \zeta \in \mathbb{N}_1. \tag{17}$$

To see this, for $\zeta \in \mathbb{N}_1$, consider

$$\begin{aligned}
&\left(\Delta^2 v\right)(\zeta - 1) - \alpha\,(\Delta v)(\zeta - 1) - \beta v(\zeta) \\
&= v(\zeta + 1) - (2 + \alpha + \beta)v(\zeta) + (1 + \alpha)v(\zeta - 1) \\
&= \frac{1}{\Lambda}\sum_{s=1}^{\zeta} g(\zeta - s + 1)k(s) - \left(\frac{2+\alpha+\beta}{\Lambda}\right)\sum_{s=1}^{\zeta-1} g(\zeta - s)k(s) + \left(\frac{1+\alpha}{\Lambda}\right)\sum_{s=1}^{\zeta-2} g(\zeta - s - 1)k(s) \\
&= \frac{1}{\Lambda}\left[g(1)k(\zeta) + g(2)k(\zeta - 1)\right] - \left(\frac{2+\alpha+\beta}{\Lambda}\right) g(1)k(\zeta - 1)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Lambda}\sum_{s=1}^{\zeta-2}\left[\lambda^{\zeta-s+1}-(2+\alpha+\beta)\lambda^{\zeta-s}+(1+\alpha)\lambda^{\zeta-s-1}\right]k(s)\\
&-\frac{1}{\Lambda}\sum_{s=1}^{\zeta-2}\left[\left(\frac{1+\alpha}{\lambda}\right)^{\zeta-s+1}-(2+\alpha+\beta)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta-s}+(1+\alpha)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta-s-1}\right]k(s)\\
=\,& k(\zeta)+\left(\frac{\lambda^2+\alpha+1}{\lambda}\right)k(\zeta-1)-(2+\alpha+\beta)k(\zeta-1)\\
&+\frac{1}{\Lambda}\lambda^{\zeta-s-1}\sum_{s=1}^{\zeta-2}\left[\lambda^2-(2+\alpha+\beta)\lambda+(1+\alpha)\right]k(s)\\
&-\frac{1}{\lambda\Lambda}\left(\frac{1+\alpha}{\lambda}\right)^{\zeta-s}\sum_{s=1}^{\zeta-2}\left[\lambda^2-(2+\alpha+\beta)\lambda+(1+\alpha)\right]k(s)\\
=\,& k(\zeta)+\frac{1}{\lambda}\left[\lambda^2-(2+\alpha+\beta)\lambda+(1+\alpha)\right]k(\zeta-1)\\
=\,& k(\zeta),
\end{aligned}
$$

implying that (17) holds. The proof is complete. □

**Remark 2.** Consider (15). Then, for $\zeta\in\mathbb{N}_0$,

$$
\begin{aligned}
(\Delta y)(\zeta) &= y(\zeta+1)-y(\zeta)\\
&= C_1\lambda^{\zeta+1}+C_2\left(\frac{1+\alpha}{\lambda}\right)^{\zeta+1}+\frac{1}{\Lambda}\sum_{s=1}^{\zeta}g(\zeta-s+1)k(s)-\left[C_1\lambda^{\zeta}+C_2\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}+\frac{1}{\Lambda}\sum_{s=1}^{\zeta-1}g(\zeta-s)k(s)\right]\\
&= C_1(\lambda-1)\lambda^{\zeta}+C_2\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}+\frac{1}{\Lambda}g(1)k(\zeta)+\frac{1}{\Lambda}\sum_{s=1}^{\zeta-1}\left[g(\zeta-s+1)-g(\zeta-s)\right]k(s)\\
&= C_1(\lambda-1)\lambda^{\zeta}+C_2\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}+\frac{1}{\Lambda}\sum_{s=1}^{\zeta}\left[g(\zeta-s+1)-g(\zeta-s)\right]k(s)\\
&= C_1(\lambda-1)\lambda^{\zeta}+C_2\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}+\frac{1}{\Lambda}\sum_{s=1}^{\zeta}\left[(\lambda-1)\lambda^{\zeta-s}-\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta-s}\right]k(s).
\end{aligned}
$$

Using Lemmas 2 and 3, we obtain the expressions for the Green functions associated with the linear boundary value problems

$$
\begin{cases}
\left(\Delta^2 y\right)(\zeta-1)-\alpha\left(\Delta y\right)(\zeta-1)-\beta y(\zeta)=k(\zeta), & \zeta\in\mathbb{N}_1^T,\\
Ay(0)-B\left(\Delta y\right)(0)=0, \quad Cy(T)+D\left(\Delta y\right)(T)=0,
\end{cases}
\tag{18}
$$

and

$$
\begin{cases}
\left(\Delta^2 y\right)(\zeta-1)-\alpha\left(\Delta y\right)(\zeta-1)-\beta y(\zeta)=k(\zeta), & \zeta\in\mathbb{N}_1^T,\\
Ay(0)+By(T)=0, \quad C\left(\Delta y\right)(0)+D\left(\Delta y\right)(T)=0.
\end{cases}
\tag{19}
$$

For convenience, we use the following notations:

$$
\begin{aligned}
&E_1=A+B\lambda^T,\\
&E_2=A+B\left(\frac{1+\alpha}{\lambda}\right)^T,\\
&E_3=C+D\lambda^T,\\
&E_4=C+D\left(\frac{1+\alpha}{\lambda}\right)^T,\\
&F_1=A-B(\lambda-1),\\
&F_2=A-B\left(\frac{1+\alpha}{\lambda}-1\right),
\end{aligned}
$$

$$
\begin{aligned}
&F_3 = C\lambda^T + D(\lambda-1)\lambda^T,\\
&F_4 = C\left(\frac{1+\alpha}{\lambda}\right)^T + D\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^T,\\
&-\xi = \Lambda\left[AC + BD(1+\lambda)^T\right] + AD\left[g(T+1)-g(T)\right] + BC\omega(T-1),\\
&-\chi = ACg(T) + AD\left[g(T+1)-g(T)\right] + BC\omega(T-1) + BD(\lambda-1)\left(1-\frac{1+\alpha}{\lambda}\right)g(T).
\end{aligned}
$$

**Remark 3.** To the end of this article, we assume that $\xi \neq 0$ and $\chi \neq 0$.

**Lemma 4.** *Assume $k$ is a real-valued function defined on a discrete finite set $\mathbb{N}_1^T$. Then, the linear boundary value problem (19) has a unique solution given in the form*

$$
y(\zeta) = \sum_{s=1}^{T} \mathcal{H}(\zeta,s)k(s), \quad \zeta \in \mathbb{N}_0^{T+1}, \tag{20}
$$

*where*

$$
\mathcal{H}(\zeta,s) = \frac{1}{\Lambda}\begin{cases}\mathcal{H}_1(\zeta,s), & s\in\mathbb{N}_1^{\zeta},\\ \mathcal{H}_2(\zeta,s), & s\in\mathbb{N}_{\zeta}^{T},\end{cases} \tag{21}
$$

$$
\begin{aligned}
\xi\mathcal{H}_2(\zeta,s) =& ADg(\zeta)\left[g(T-s+1)-g(T-s)\right]\\
&+ BCg(T-s)\omega(\zeta-1) + BD\Lambda(1+\alpha)^T h(s-\zeta), \quad (\zeta,s)\in\mathbb{N}_0^{T+1}\times\mathbb{N}_1^{\zeta},
\end{aligned}
$$

*and*

$$
\begin{aligned}
\xi\mathcal{H}_1(\zeta,s) =& ADg(\zeta)\left[g(T-s+1)-g(T-s)\right] + BCg(T-s)\omega(\zeta-1) + BD\Lambda(1+\alpha)^T g(\zeta-s)\\
&+ \xi g(\zeta-s), \quad (\zeta,s)\in\mathbb{N}_0^{T+1}\times\mathbb{N}_{\zeta}^{T}.
\end{aligned}
$$

**Proof.** From Lemma 3, a general solution of the second-order difference equation in (19) is given by

$$
y(\zeta) = C_1\lambda^{\zeta} + C_2\left(\frac{1+\alpha}{\lambda}\right)^{\zeta} + \frac{1}{\Lambda}\sum_{s=1}^{\zeta-1} g(\zeta-s)k(s), \quad \zeta\in\mathbb{N}_0^{T+1}, \tag{22}
$$

where $C_1$ and $C_2$ are arbitrary constants. It follows from Remark 2 that

$$
(\Delta y)(\zeta) = C_1(\lambda-1)\lambda^{\zeta} + C_2\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta} + \frac{1}{\Lambda}\sum_{s=1}^{\zeta}\left[g(\zeta-s+1)-g(\zeta-s)\right]k(s), \quad \zeta\in\mathbb{N}_0^{T}. \tag{23}
$$

Using $Ay(0)+By(T)=0$ in (22), we obtain

$$
C_1E_1 + C_2E_2 = -\frac{B}{\Lambda}\sum_{s=1}^{T-1} g(T-s)k(s). \tag{24}
$$

Using $C(\Delta y)(0) + D(\Delta y)(T) = 0$ in (23), we obtain

$$
(\lambda-1)C_1E_3 + \left(\frac{1+\alpha}{\lambda}-1\right)C_2E_4 = -\frac{D}{\Lambda}\sum_{s=1}^{T}\left[g(T-s+1)-g(T-s)\right]k(s). \tag{25}
$$

From (24) and (25), we have

$$
C_1 = -\frac{1}{\Lambda\xi}\sum_{s=1}^{T}\left[\left(\frac{1+\alpha}{\lambda}-1\right)BE_4 g(T-s) - DE_2\left[g(T-s+1)-g(T-s)\right]\right]k(s), \tag{26}
$$

and

$$C_2 = -\frac{1}{\Lambda\xi}\sum_{s=1}^{T}\left[-B(\lambda-1)E_3 g(T-s) + DE_1\left[g(T-s+1)-g(T-s)\right]\right]k(s). \tag{27}$$

Substituting the equalities (26) and (27) in (22), we obtain (20). The proof is complete. □

**Remark 4.** Imposing the boundary conditions $Ay(0)+By(T)=0$ and $C\,(\Delta y)\,(0)+D\,(\Delta y)\,(T)=0$ on the general solution $y(\zeta)=C_1\lambda^{\zeta}+C_2\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}$ of the associated homogeneous second-order difference equation

$$\left(\Delta^2 y\right)(\zeta-1)-\alpha\,(\Delta y)\,(\zeta-1)-\beta y(\zeta)=0, \quad \zeta\in\mathbb{N}_1^T,$$

yields a linear algebraic system for the constants $C_1$ and $C_2$:

$$\begin{pmatrix} E_1 & E_2 \\ (\lambda-1)E_3 & \left(\frac{1+\alpha}{\lambda}-1\right)E_4 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the co-efficient matrix of this system is given by $-\xi$. The condition $\xi\neq 0$ guarantees that the homogeneous boundary value problem

$$\begin{cases} \left(\Delta^2 y\right)(\zeta-1)-\alpha\,(\Delta y)\,(\zeta-1)-\beta y(\zeta)=0, & \zeta\in\mathbb{N}_1^T, \\ Ay(0)+By(T)=0, \quad C\,(\Delta y)\,(0)+D\,(\Delta y)\,(T)=0. \end{cases}$$

has only the trivial solution. Consequently, the linear boundary value problem (19) has a unique solution given in the form (20).

**Lemma 5.** *Assume $k$ is a real-valued function defined on a discrete finite set $\mathbb{N}_1^T$. Then, the linear boundary value problem (18) has a unique solution given in the form*

$$y(\zeta)=\sum_{s=1}^{T}\mathcal{G}(\zeta,s)k(s), \quad \zeta\in\mathbb{N}_0^{T+1}, \tag{28}$$

*where*

$$\mathcal{G}(\zeta,s)=\frac{1}{\Lambda}\begin{cases}\mathcal{G}_1(\zeta,s), & s\in\mathbb{N}_1^{\zeta}, \\ \mathcal{G}_2(\zeta,s), & s\in\mathbb{N}_{\zeta}^{T},\end{cases} \tag{29}$$

$$\chi\mathcal{G}_2(\zeta,s)=\left[Cg(T-s)+D\left(g(T-s+1)-g(T-s)\right)\right]\times\left[Ag(\zeta)+B\omega(\zeta-1)\right], \quad (\zeta,s)\in\mathbb{N}_0^{T+1}\times\mathbb{N}_1^{\zeta}, \tag{30}$$

*and*

$$\mathcal{G}_1(\zeta,s)=\mathcal{G}_2(\zeta,s)+g(\zeta-s), \quad (\zeta,s)\in\mathbb{N}_0^{T+1}\times\mathbb{N}_{\zeta}^{T}. \tag{31}$$

**Proof.** From Lemma 3, a general solution of the second-order difference equation in (18) is given by

$$y(\zeta)=C_1\lambda^{\zeta}+C_2\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}+\frac{1}{\Lambda}\sum_{s=1}^{\zeta-1}g(\zeta-s)k(s), \quad \zeta\in\mathbb{N}_0^{T+1}, \tag{32}$$

where $C_1$ and $C_2$ are arbitrary constants. It follows from Remark 2 that

$$(\Delta y)\,(\zeta)=C_1(\lambda-1)\lambda^{\zeta}+C_2\left(\frac{1+\alpha}{\lambda}-1\right)\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}+\frac{1}{\Lambda}\sum_{s=1}^{\zeta}\left[g(\zeta-s+1)-g(\zeta-s)\right]k(s), \quad \zeta\in\mathbb{N}_0^T. \tag{33}$$

Using $Ay(0)-B\,(\Delta y)\,(0)=0$ in (32) - (33), we obtain

$$C_1F_1+C_2F_2=0. \tag{34}$$

Using $Cy(T) + D\,(\Delta y)\,(T) = 0$ in (32) - (33), we obtain

$$C_1F_3 + C_2F_4 = -\frac{1}{\Lambda}\sum_{s=1}^{T}\left[Cg(T-s) + D\left(g(T-s+1) - g(T-s)\right)\right]k(s). \tag{35}$$

From (34) and (35), we have

$$C_1 = \frac{F_2}{\Lambda\chi}\sum_{s=1}^{T}\left[Cg(T-s) + D\left(g(T-s+1) - g(T-s)\right)\right]k(s), \tag{36}$$

and

$$C_2 = -\frac{F_1}{\Lambda\chi}\sum_{s=1}^{T}\left[Cg(T-s) + D\left(g(T-s+1) - g(T-s)\right)\right]k(s). \tag{37}$$

Substituting the equalities (36) and (37) in (32), we obtain (28). The proof is complete. □

**Remark 5.** Imposing the boundary conditions $Ay(0) - B\,(\Delta y)\,(0) = 0$ and $Cy(T) + D\,(\Delta y)\,(T) = 0$ on the general solution $y(\zeta) = C_1\lambda^{\zeta} + C_2\left(\frac{1+\alpha}{\lambda}\right)^{\zeta}$ of the associated homogeneous second-order difference equation

$$\left(\Delta^2 y\right)(\zeta-1) - \alpha\,(\Delta y)\,(\zeta-1) - \beta y(\zeta) = 0, \quad \zeta \in \mathbb{N}_1^T,$$

yields a linear algebraic system for the constants $C_1$ and $C_2$:

$$\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the co-efficient matrix of this system is given by $-\chi$. The condition $\chi \neq 0$ guarantees that the homogeneous boundary value problem

$$\begin{cases} \left(\Delta^2 y\right)(\zeta-1) - \alpha\,(\Delta y)\,(\zeta-1) - \beta y(\zeta) = 0, & \zeta \in \mathbb{N}_1^T, \\ Ay(0) - B\,(\Delta y)\,(0) = 0, \quad Cy(T) + D\,(\Delta y)\,(T) = 0, \end{cases}$$

has only the trivial solution. Consequently, the linear boundary value problem (18) has a unique solution given in the form (28).

**Lemma 6.** *The Green's function $\mathcal{H}(\zeta, s)$ given by (21) satisfies the following property:*

$$\sum_{s=1}^{T}|\mathcal{H}(\zeta,s)| \le \Upsilon, \quad \zeta \in \mathbb{N}_0^{T+1},$$

*where*

$$\begin{aligned}\Upsilon = \frac{1}{|\xi|}\max_{\zeta\in\mathbb{N}_0^{T+1}}\Bigg(\Bigg[&|A||D|g(\zeta)\sum_{s=1}^{T}\left[g(T-s+1) - g(T-s)\right] \\ &+ |B||C|\omega(\zeta-1)\sum_{s=1}^{T}g(T-s) + |B||D|\Lambda(1+\alpha)^T\sum_{s=1}^{\zeta}g(\zeta-s) \\ &+ |B||D|\Lambda(1+\alpha)^T\sum_{s=\zeta}^{T}|h(s-\zeta)|\Bigg] + \sum_{s=1}^{\zeta}g(\zeta-s)\Bigg).\end{aligned}$$

**Proof.** For $\zeta \in \mathbb{N}_0^{T+1}$, consider

$$\sum_{s=1}^{T}|\mathcal{H}(\zeta,s)| = \sum_{s=\zeta}^{T}|\mathcal{H}_2(\zeta,s)| + \sum_{s=1}^{\zeta}|\mathcal{H}_1(\zeta,s)|$$

$$
\begin{aligned}
&= \frac{1}{|\xi|}\sum_{s=\zeta}^{T}|\xi\mathcal{H}_2(\zeta,s)| + \frac{1}{|\xi|}\sum_{s=1}^{\zeta}|\xi\mathcal{H}_1(\zeta,s)| \\
&\leq \frac{1}{|\xi|}\sum_{s=1}^{T}\left|ADg(\zeta)\left[g(T-s+1)-g(T-s)\right]+BCg(T-s)\omega(\zeta-1)\right| \\
&\quad + \frac{1}{|\xi|}\sum_{s=1}^{\zeta}\left|BD\Lambda(1+\alpha)^T g(\zeta-s)\right| + \frac{1}{|\xi|}\sum_{s=\zeta}^{T}\left|BD\Lambda(1+\alpha)^T h(s-\zeta)\right| + \sum_{s=1}^{\zeta} g(\zeta-s) \\
&\leq \frac{1}{|\xi|}\Bigg[|A||D|g(\zeta)\sum_{s=1}^{T}\left[g(T-s+1)-g(T-s)\right] \\
&\quad + |B||C|\omega(\zeta-1)\sum_{s=1}^{T} g(T-s) + |B||D|\Lambda(1+\alpha)^T\sum_{s=1}^{\zeta} g(\zeta-s) \\
&\quad + |B||D|\Lambda(1+\alpha)^T\sum_{s=\zeta}^{T}|h(s-\zeta)|\Bigg] + \sum_{s=1}^{\zeta} g(\zeta-s) \\
&\leq \Upsilon.
\end{aligned}
$$

The proof is complete. □

**Lemma 7.** *The Green's function $\mathcal{G}(\zeta,s)$ given by (29) satisfies the following property:*

$$
\sum_{s=1}^{T}|\mathcal{G}(\zeta,s)| \leq \Theta, \quad \zeta\in\mathbb{N}_0^{T+1},
$$

*where*

$$
\Theta = \max_{\zeta\in\mathbb{N}_0^{T+1}}\left(\frac{\left[|A|\,g(\zeta)+|B|\,\omega(\zeta-1)\right]}{|\chi|}\times\left[|C|\sum_{s=1}^{T} g(T-s)+|D|\sum_{s=1}^{T}\left[g(T-s+1)-g(T-s)\right]\right]+\sum_{s=1}^{\zeta} g(\zeta-s)\right).
$$

**Proof.** For $\zeta\in\mathbb{N}_0^{T+1}$, consider

$$
\begin{aligned}
\sum_{s=1}^{T}|\mathcal{G}(\zeta,s)| &= \sum_{s=\zeta}^{T}|\mathcal{G}_2(\zeta,s)| + \sum_{s=1}^{\zeta}|\mathcal{G}_1(\zeta,s)| \\
&= \frac{1}{|\chi|}\sum_{s=\zeta}^{T}|\chi\mathcal{G}_2(\zeta,s)| + \frac{1}{|\chi|}\sum_{s=1}^{\zeta}|\chi\mathcal{G}_1(\zeta,s)| \\
&\leq \frac{1}{|\chi|}\sum_{s=1}^{T}\left|\left[Cg(T-s)+D\left(g(T-s+1)-g(T-s)\right)\right]\left[Ag(\zeta)+B\omega(\zeta-1)\right]\right| + \sum_{s=1}^{\zeta} g(\zeta-s) \\
&\leq \frac{\left[|A|\,g(\zeta)+|B|\,\omega(\zeta-1)\right]}{|\chi|}\times\left[|C|\sum_{s=1}^{T} g(T-s)+|D|\sum_{s=1}^{T}\left[g(T-s+1)-g(T-s)\right]\right]+\sum_{s=1}^{\zeta} g(\zeta-s) \\
&\leq \Theta.
\end{aligned}
$$

The proof is complete. □

## 4. Main Results

This section establishes sufficient conditions on the existence of solutions to the boundary value problems (10) - (2) and (10) - (11). Lemma 5 implies the equivalence between the solutions of (10) - (2) and the solutions of the summation equation

$$
y(\zeta) = \sum_{s=1}^{T}\mathcal{G}(\zeta,s)f(s,y(s)), \quad \zeta\in\mathbb{N}_0^{T+1}.
$$

Similarly, Lemma 4 implies the equivalence between the solutions of (10) - (11) and the solutions of the summation equation

$$y(\zeta)=\sum_{s=1}^{T}\mathcal{H}(\zeta,s)f(s,y(s)),\quad \zeta\in\mathbb{N}_0^{T+1}.$$

Define the operators $\mathfrak{R}:\mathcal{B}_1\to\mathcal{B}_1$ and $\mathfrak{S}:\mathcal{B}_2\to\mathcal{B}_2$ by

$$(\mathfrak{R}y)\,(\zeta)=\sum_{s=1}^{T}\mathcal{G}(\zeta,s)f(s,y(s)),\quad \zeta\in\mathbb{N}_0^{T+1},$$

and

$$(\mathfrak{S}y)\,(\zeta)=\sum_{s=1}^{T}\mathcal{H}(\zeta,s)f(s,y(s)),\quad \zeta\in\mathbb{N}_0^{T+1}.$$

It follows from Lemma 5 that $y$ is a fixed point of $\mathfrak{R}$ if, and only if, $y$ is a solution of (10) - (2). Also, from Lemma 4, $y$ is a fixed point of $\mathfrak{S}$ if, and only if, $y$ is a solution of (10) - (11). Let

$$L=\{y\in\mathcal{B}_1:\|y\|\leq n\}\,,$$

and

$$K=\{y\in\mathcal{B}_2:\|y\|\leq r\}\,.$$

Clearly, $L$ and $K$ are nonempty bounded closed convex subsets of the finite dimensional normed space $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively.

Now, we apply Theorem 1 to discuss the existence of solutions to (10) - (2) and (10) - (11).

**Theorem 2.** *Assume the following conditions hold:*

(C1) *There exist* $p:\mathbb{N}_1^T\to[0,\infty)$ *and a nondecreasing function* $q:[0,\infty)\to[0,\infty)$ *such that*

$$|f(\zeta,y)|\leq p(\zeta)q\left(\|y\|\right),\quad (\zeta,y)\in\mathbb{N}_1^T\times\mathbb{R}.$$

(C2) *There exists* $N>0$ *such that*

$$\frac{N}{\Upsilon\Omega q\,(N)}>1,$$

*where*

$$\Omega=\max_{\zeta\in\mathbb{N}_1^T}p(\zeta).$$

*Then,* (10) - (11) *has a solution in* $\mathcal{B}_2$.

**Proof.** Since $\mathbb{N}_0^{T+1}$ is a discrete set, $\mathcal{B}_2$ is finite dimensional normed space and $\mathfrak{S}$ is a continuous operator, it follows immediately that $\mathfrak{S}$ is completely continuous.

Next, we suppose $y\in\mathcal{B}_2$ and that for some $0<\nu<1$, $y=\nu\mathfrak{S}y$. Then, for $\zeta\in\mathbb{N}_0^{T+1}$, and again by (C1),

$$\begin{aligned}|y(\zeta)|&=|\nu(\mathfrak{S}y)(\zeta)|\\&\leq\sum_{s=1}^{T}|\mathcal{H}(\zeta,s)|\,|f(s,y(s))|\\&\leq\sum_{s=1}^{T}|\mathcal{H}(\zeta,s)|\,p(s)q\left(|y(s)|\right)\\&\leq q\left(\|y\|\right)\sum_{s=1}^{T}|\mathcal{H}(\zeta,s)|\,p(s)\\&\leq\Upsilon\Omega q\left(\|y\|\right),\end{aligned}$$

implying that

$$\frac{\|y\|}{\Upsilon\Omega q\left(\|y\|\right)} \leq 1.$$

It follows from (C2) that $\|y\| \neq N$. If we set

$$U = \left\{ y \in \mathcal{B}_2 : \|y\| < N \right\},$$

then the operator $\mathfrak{S} : \bar{U} \to \mathcal{B}_2$ is completely continuous. From the choice of $U$, there is no $y \in \partial U$ such that $y = \nu \mathfrak{S} y$ for some $0 < \nu < 1$. It follows from Theorem 1 that $\mathfrak{S}$ has a fixed point $y_0 \in \bar{U}$, which is a desired solution of (10) - (11). □

**Theorem 3.** *Assume the following conditions hold:*

(C1) *There exist* $p : \mathbb{N}_1^T \to [0, \infty)$ *and a nondecreasing function* $q : [0, \infty) \to [0, \infty)$ *such that*

$$|f(\zeta, y)| \leq p(\zeta) q\left(\|y\|\right), \quad (\zeta, y) \in \mathbb{N}_1^T \times \mathbb{R}.$$

(C3) *There exists* $M > 0$ *such that*

$$\frac{M}{\Theta\Omega q\left(M\right)} > 1,$$

*where*

$$\Omega = \max_{\zeta \in \mathbb{N}_1^T} p(\zeta).$$

*Then,* (10) - (2) *has a solution in* $\mathcal{B}_1$.

**Proof.** Since $\mathbb{N}_0^{T+1}$ is a discrete set, $\mathcal{B}_1$ is finite dimensional normed space and $\mathfrak{R}$ is a continuous operator, it follows immediately that $\mathfrak{R}$ is completely continuous.

Next, we suppose $y \in \mathcal{B}_1$ and that for some $0 < \nu < 1$, $y = \nu \mathfrak{R} y$. Then, for $\zeta \in \mathbb{N}_0^{T+1}$, and again by (C1),

$$\begin{aligned}
|y(\zeta)| &= |\nu(\mathfrak{R}y)(\zeta)| \\
&\leq \sum_{s=1}^{T} |\mathcal{G}(\zeta, s)| \, |f(s, y(s))| \\
&\leq \sum_{s=1}^{T} |\mathcal{G}(\zeta, s)| \, p(s) q\left(|y(s)|\right) \\
&\leq q\left(\|y\|\right) \sum_{s=1}^{T} |\mathcal{G}(\zeta, s)| \, p(s) \\
&\leq \Theta\Omega q\left(\|y\|\right),
\end{aligned}$$

implying that

$$\frac{\|y\|}{\Theta\Omega q\left(\|y\|\right)} \leq 1.$$

It follows from (C2) that $\|y\| \neq M$. If we set

$$V = \left\{ y \in \mathcal{B}_1 : \|y\| < M \right\},$$

then the operator $\mathfrak{R} : \bar{V} \to \mathcal{B}_1$ is completely continuous. From the choice of $V$, there is no $y \in \partial V$ such that $y = \nu \mathfrak{R} y$ for some $0 < \nu < 1$. It follows from Theorem 1 that $\mathfrak{R}$ has a fixed point $y_0 \in \bar{V}$, which is a desired solution of (10) - (2). □

## 5. Examples

In this section, we provide two examples to demonstrate the applicability of established results.

**Example 1.** Consider (10) - (11) with $A = B = C = D = 1$, $T = 5$, $\alpha = 1$, $\beta = 0$ and $f(\zeta, s) = \zeta s^2$. Clearly,

$$|f(\zeta,s)| \le p(\zeta) q(|s|), \quad (\zeta,s) \in \mathbb{N}_1^5 \times \mathbb{R},$$

where

$$p(\zeta) = \zeta, \quad \zeta \in \mathbb{N}_1^5,$$

and

$$q(|s|) = |s|^2 = s^2, \quad s \in \mathbb{R}.$$

Also, $p : \mathbb{N}_1^5 \to [0,\infty)$ and $q : [0,\infty) \to [0,\infty)$ is a nondecreasing function. Thus, the assumption (C1) of Theorem 2 holds. Further, we have

$$\Omega = \max_{\zeta \in \mathbb{N}_1^5} p(\zeta) = 5.$$

Now, we calculate $\Upsilon$. We have $\lambda = 2$, $\Lambda = 1$,

$$\begin{aligned} g(x) &= 2^x - 1, \quad x \in \mathbb{N}_0, \\ h(x) &= 2^{-x} - 1, \quad x \in \mathbb{N}_0, \\ \omega(x) &= 1, \quad x \in \mathbb{N}_0, \\ w(x) &= \frac{1}{2}, \quad x \in \mathbb{N}_0. \end{aligned}$$

Also,

$$\begin{aligned} -\xi &= \Lambda \left[ AC + BD(1+\lambda)^T \right] + AD\left[g(T+1) - g(T)\right] + BC\omega(T-1) \\ &= 1 + 3^5 + 2^6 - 2^5 + 1 \\ &= 277, \end{aligned}$$

implying that $\xi = -277$. The corresponding Green's function is given by

$$\mathcal{H}(\zeta,s) = \begin{cases} \mathcal{H}_1(\zeta,s), & s \in \mathbb{N}_1^\zeta, \\ \mathcal{H}_2(\zeta,s), & s \in \mathbb{N}_\zeta^5, \end{cases}$$

where

$$\mathcal{H}_2(\zeta,s) = -\frac{1}{277}\left(2^{\zeta-s+6} - 33\right),$$

and

$$\mathcal{H}_1(\zeta,s) = \frac{213}{277} 2^{\zeta-s} - \frac{244}{277}.$$

Consequently,

$$\begin{aligned} \Upsilon = \frac{1}{|\xi|} \max_{\zeta \in \mathbb{N}_0^{T+1}} &\Bigg( \Bigg[ |A||D| g(\zeta) \sum_{s=1}^{T} \left[ g(T-s+1) - g(T-s) \right] \\ &+ |B||C|\omega(\zeta-1) \sum_{s=1}^{T} g(T-s) + |B||D|\Lambda(1+\alpha)^T \sum_{s=1}^{\zeta} g(\zeta-s) \\ &+ |B||D|\Lambda(1+\alpha)^T \sum_{s=\zeta}^{T} |h(s-\zeta)| \Bigg] + \sum_{s=1}^{\zeta} g(\zeta-s) \Bigg) \\ = \frac{1}{277} \max_{\zeta \in \mathbb{N}_0^6} &\Bigg( \Bigg[ g(\zeta) \sum_{s=1}^{5} \left[ g(6-s) - g(5-s) \right] + \omega(\zeta-1) \sum_{s=1}^{5} g(5-s) + (32) \sum_{s=1}^{\zeta} g(\zeta-s) \\ &+ (32) \sum_{s=\zeta}^{5} |h(s-\zeta)| \Bigg] + \sum_{s=1}^{\zeta} g(\zeta-s) \Bigg). \end{aligned}$$

We have

$$\begin{aligned}
\sum_{s=1}^{\zeta} g(\zeta - s) &= \sum_{s=1}^{\zeta} 2^{\zeta - s} - \sum_{s=1}^{\zeta} 1 \\
&= \left[\frac{2^{\zeta} - 1}{2 - 1}\right] - \zeta \\
&= 2^{\zeta} - \zeta - 1, \\
\sum_{s=\zeta}^{T} |h(s - \zeta)| &= \sum_{s=\zeta}^{5} \left[1 - 2^{\zeta - s}\right] \\
&= \sum_{s=\zeta}^{5} 1 - \sum_{s=\zeta}^{5} 2^{\zeta - s} \\
&= 6 - \zeta - \sum_{s=0}^{5-\zeta} \left(\frac{1}{2}\right)^{s} \\
&= 6 - \zeta - \left[\frac{1 - \left(\frac{1}{2}\right)^{6-\zeta}}{1 - \left(\frac{1}{2}\right)}\right] \\
&= 4 - \zeta + 2^{\zeta - 5}, \\
\sum_{s=1}^{5} g(5 - s) &= 2^{5} - 5 - 1 \\
&= 26,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{s=1}^{5} [g(6 - s) - g(5 - s)] &= \sum_{s=1}^{5} g(6 - s) - \sum_{s=1}^{5} g(5 - s) \\
&= \sum_{s=1}^{5} \left[2^{6-s} - 1\right] - 26 \\
&= \sum_{s=1}^{5} 2^{6-s} - \sum_{s=1}^{5} 1 - 26 \\
&= 2\left[\frac{2^{5} - 1}{2 - 1}\right] - 5 - 26 \\
&= 62 - 31 = 31.
\end{aligned}$$

Then,

$$\begin{aligned}
\Upsilon = \frac{1}{277} \max_{\zeta \in \mathbb{N}_0^6} \Bigg( \Bigg[ & 31\left(2^{\zeta} - 1\right) + 26 + (32)\left(2^{\zeta} - \zeta - 1\right) \\
&+ (32)\left(4 - \zeta + 2^{\zeta - 5}\right) \Bigg] + \left(2^{\zeta} - \zeta - 1\right) \Bigg) = \frac{3860}{277}.
\end{aligned}$$

Then, there exists $0 < N < \frac{277}{19300}$ such that

$$\frac{N}{\Upsilon \Omega q(N)} > 1,$$

implying that the assumption (C2) of Theorem 2 holds. Therefore, by Theorem 2, the boundary value problem (10) - (11) has a solution in $\mathcal{B}_2$.

**Example 2.** Consider (10) - (11) with $A = B = C = D = 1$, $T = 5$, $\alpha = 0$, $\beta = \frac{1}{2}$ and $f(\zeta, s) = \zeta s^2$. Clearly,

$$|f(\zeta,s)| \leq p(\zeta)q(|s|), \quad (\zeta,s) \in \mathbb{N}_1^5 \times \mathbb{R},$$

where

$$p(\zeta) = \zeta, \quad \zeta \in \mathbb{N}_1^5,$$

and

$$q(|s|) = |s|^2 = s^2, \quad s \in \mathbb{R}.$$

Also, $p : \mathbb{N}_1^5 \to [0, \infty)$ and $q : [0, \infty) \to [0, \infty)$ is a nondecreasing function. Thus, the assumption (C1) of Theorem 2 holds. Further, we have

$$\Omega = \max_{\zeta \in \mathbb{N}_1^5} p(\zeta) = 5.$$

Now, we calculate $\Upsilon$. We have $\lambda = 2$, $\Lambda = \frac{3}{2}$,

$$\begin{aligned}
g(x) &= 2^x - 2^{-x}, \quad x \in \mathbb{N}_0, \\
h(x) &= 2^{-x} - 2^x, \quad x \in \mathbb{N}_0, \\
\omega(x) &= 2^x + 2^{-x-1}, \quad x \in \mathbb{N}_0, \\
w(x) &= 2^x + 2^{-x-1}, \quad x \in \mathbb{N}_0.
\end{aligned}$$

Also,

$$\begin{aligned}
-\xi &= \Lambda \left[AC + BD(1+\lambda)^T\right] + AD\left[g(T+1) - g(T)\right] + BC\omega(T-1) \\
&= \frac{3}{2}\left[1 + 3^5\right] + \left[2^6 - 2^{-6}\right] - \left[2^5 - 2^{-5}\right] + 2^4 + 2^{-5} \\
&= \frac{26499}{64},
\end{aligned}$$

implying that $\xi = -\frac{26499}{64}$. The corresponding Green's function is given by

$$\mathcal{H}(\zeta, s) = \frac{2}{3}\begin{cases} \mathcal{H}_1(\zeta,s), & s \in \mathbb{N}_1^\zeta, \\ \mathcal{H}_2(\zeta,s), & s \in \mathbb{N}_\zeta^5, \end{cases}$$

where

$$\xi\mathcal{H}_2(\zeta,s) = \left(2^\zeta - 2^{-\zeta}\right)\left(2^{\zeta-s} + 2^{s-\zeta-1}\right) + \left(2^{5-s} - 2^{s-5}\right)\left(2^{\zeta-1} + 2^{-\zeta}\right) + 3\left(2^{\zeta-s-1} - 2^{s-\zeta-1}\right),$$

and

$$\begin{aligned}
\xi\mathcal{H}_1(\zeta,s) &= \left(2^\zeta - 2^{-\zeta}\right)\left(2^{\zeta-s} + 2^{s-\zeta-1}\right) \\
&\quad + \left(2^{5-s} - 2^{s-5}\right)\left(2^{\zeta-1} + 2^{-\zeta}\right) + 3\left(2^{\zeta-s-1} - 2^{s-\zeta-1}\right) + \xi\left(2^{\zeta-s} - 2^{s-\zeta}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Upsilon = \frac{1}{|\xi|}\max_{\zeta \in \mathbb{N}_0^{T+1}} \Bigg( \Bigg[ &|A||D|g(\zeta)\sum_{s=1}^{T}\left[g(T-s+1) - g(T-s)\right] \\
&+ |B||C|\omega(\zeta-1)\sum_{s=1}^{T} g(T-s) + |B||D|\Lambda(1+\alpha)^T \sum_{s=1}^{\zeta} g(\zeta - s) \\
&+ |B||D|\Lambda(1+\alpha)^T \sum_{s=\zeta}^{T} |h(s-\zeta)| \Bigg] + \sum_{s=1}^{\zeta} g(\zeta - s) \Bigg)
\end{aligned}$$

$$= \frac{64}{26499}\max_{\zeta\in\mathbb{N}_0^6}\left(\left[g(\zeta)\sum_{s=1}^{5}[g(6-s)-g(5-s)]+\omega(\zeta-1)\sum_{s=1}^{5}g(5-s)+(32)\sum_{s=1}^{\zeta}g(\zeta-s)\right.\right.$$
$$\left.\left.+(32)\sum_{s=\zeta}^{5}|h(s-\zeta)|\right]+\sum_{s=1}^{\zeta}g(\zeta-s)\right).$$

We have

$$\begin{aligned}\sum_{s=1}^{\zeta}g(\zeta-s) &= \sum_{s=1}^{\zeta}2^{\zeta-s}-\sum_{s=1}^{\zeta}\left(\frac{1}{2}\right)^{\zeta-s}\\ &= \left[\frac{2^{\zeta}-1}{2-1}\right]-\left[\frac{1-\left(\frac{1}{2}\right)^{\zeta}}{1-\left(\frac{1}{2}\right)}\right]\\ &= 2^{\zeta}+2^{1-\zeta}-3,\end{aligned}$$

$$\begin{aligned}\sum_{s=\zeta}^{T}|h(s-\zeta)| &= \sum_{s=\zeta}^{5}\left[\left(\frac{1}{2}\right)^{\zeta-s}-2^{\zeta-s}\right]\\ &= \sum_{s=\zeta}^{5}\left(\frac{1}{2}\right)^{\zeta-s}-\sum_{s=\zeta}^{5}2^{\zeta-s}\\ &= \sum_{s=0}^{5-\zeta}2^{s}-\sum_{s=0}^{5-\zeta}\left(\frac{1}{2}\right)^{s}\\ &= \left[\frac{2^{6-\zeta}-1}{2-1}\right]-\left[\frac{1-\left(\frac{1}{2}\right)^{6-\zeta}}{1-\left(\frac{1}{2}\right)}\right]\\ &= 2^{6-\zeta}+2^{\zeta-5}-3,\\ \sum_{s=1}^{5}g(5-s) &= 2^{5}+2^{-4}-3=\frac{465}{16},\end{aligned}$$

and

$$\begin{aligned}\sum_{s=1}^{5}[g(6-s)-g(5-s)] &= \sum_{s=1}^{5}g(6-s)-\sum_{s=1}^{5}g(5-s)\\ &= \sum_{s=1}^{5}\left[2^{6-s}-\left(\frac{1}{2}\right)^{6-s}\right]-\frac{465}{16}\\ &= \sum_{s=1}^{5}2^{6-s}-\sum_{s=1}^{5}\left(\frac{1}{2}\right)^{6-s}-\frac{465}{16}\\ &= 2\left[\frac{2^{5}-1}{2-1}\right]-\frac{1}{2}\left[\frac{1-\left(\frac{1}{2}\right)^{5}}{1-\left(\frac{1}{2}\right)}\right]-\frac{465}{16}\\ &= \frac{1023}{32}.\end{aligned}$$

Then,

$$Y = \frac{64}{26499}\max_{\zeta\in\mathbb{N}_0^6}\left(\left[\frac{1023}{32}\left(2^{\zeta}-2^{-\zeta}\right)+\frac{465}{16}\left(2^{\zeta-1}+2^{-\zeta}\right)+(32)\left(2^{\zeta}+2^{1-\zeta}-3\right)\right.\right.$$
$$\left.\left.+(32)\left(2^{6-\zeta}+2^{\zeta-5}-3\right)\right]+\left(2^{\zeta}+2^{1-\zeta}-3\right)\right)=\frac{319360}{26499}.$$

Then, there exists $0 < N < \frac{85}{5122}$ such that

$$\frac{N}{\Upsilon\Omega q\left(N\right)} > 1,$$

implying that the assumption (C2) of Theorem 2 holds. Therefore, by Theorem 2, the boundary value problem (10) - (11) has a solution in $\mathcal{B}_2$.

## 6. Conclusion

In this article, we established sufficient conditions for the existence of solutions to the boundary value problems (10) - (2) and (10) - (11) using Leray–Schauder nonlinear alternative. The results established in this article not only subsume the works of [1–10] as particular cases, but also extend to a broad class of second-order difference equations through appropriate choices of the parameters $\alpha$ and $\beta$. Moreover, they accommodate a wide variety of boundary conditions via suitable selections of the constants $A$, $B$, $C$, and $D$. To the best of our knowledge, no existing studies in the literature address the existence of solutions for the discrete boundary value problems (10) - (2) and (10) - (11) by employing nonlinear analytical techniques such as fixed point theory, fixed point index theory, coincidence degree theory, critical point theory, or variational methods.

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