

Article

Statistically Bornological Convergence

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Abstract: In this paper, we introduce and investigate the concept of *statistically bornological convergence* for sequences of subsets in metric spaces. This notion combines the localization principle of bornological convergence with the asymptotic flexibility of statistical convergence. A sequence of sets is said to be statistically bornologically convergent if the bornological inclusion conditions hold for a set of indices with natural density one. We provide examples distinguishing this concept from classical bornological and Hausdorff convergence. Under appropriate boundedness assumptions, we establish a functional characterization using excess functionals. We prove stability under bi-Lipschitz embeddings using a direct inclusion-based approach with properly defined pushforward ideals, and establish a subsequence theorem via the diagonal density lemma. The relationship with Wijsman statistical convergence is clarified.

Keywords: bornological convergence, statistical convergence, statistically bornological convergence, wijsman convergence

MSC: Primary 40B05, 40A35; Secondary 54A20.

1. Introduction

Set convergence notions play a central role in variational analysis, set-valued analysis, and hyperspace topology. Among them, *bornological convergence* provides a localization principle: instead of controlling the global behavior of sets, one tests convergence on members of a fixed ideal (typically the bornology of bounded subsets) [1–5]. On the other hand, *statistical convergence* weakens the usual tail condition by allowing exceptional indices of natural density zero [6,7] and has been developed in many contexts, including sequences of sets [8].

A natural limitation of classical bornological convergence is that the inclusion conditions are required to hold eventually for all indices. In applications involving perturbations, noisy approximations, or irregular sampling processes, such uniform tail control may be too restrictive. Even if the local behavior of the sets is stable on bounded regions, sporadic large deviations may prevent classical bornological convergence.

On the other hand, statistical convergence allows exceptional indices provided they form a set of natural density zero. However, statistical convergence alone does not incorporate any localization principle; it treats the space globally and does not distinguish between bounded and unbounded regions.

These observations suggest a natural question: *Can one localize convergence through an ideal, while simultaneously relaxing the index requirement through density?*

The notion introduced in this paper provides a positive answer. By combining ideal-based localization with statistical prevalence, statistically bornological convergence captures stable local behavior on bounded test sets, while tolerating sparse global irregularities. This hybrid perspective fills a structural gap between bornological and Wijsman-type statistical convergences and yields a flexible framework for the study of set convergence in noncompact or perturbative environments.

The purpose of this paper is to combine these mechanisms and introduce *statistically bornological convergence* for sequences of subsets in metric spaces. Informally, the usual bornological inclusion conditions are required to hold not eventually for all indices, but on a density-one set of indices. This hybrid viewpoint yields a framework that is both localized (through the ideal) and asymptotic (through density).

The main contributions of this paper are:

- We define statistically bornological convergence with explicit conventions for the empty limit set.
- We provide examples separating it from classical bornological and Hausdorff-type convergences.
- Under a boundedness assumption, we obtain a functional characterization via excess functionals.
- We prove stability under bi-Lipschitz embeddings using a pushforward ideal on the image space.
- Assuming a countable base of the ideal, we obtain a density-one subsequence theorem yielding classical bornological convergence.
- We clarify the relation to Wijsman statistical convergence, and recover Wijsman statistical convergence under a uniform boundedness hypothesis.

This paper is organized as follows: §2 collects the background and fixes notation. §3 introduces the notion and basic examples. §4 establishes the functional characterization. §5 proves bi-Lipschitz stability. §6 contains the subsequence theorem. §7 discusses the relationship with Wijsman statistical convergence.

2. Preliminaries

We recall the basic notions used throughout the paper and fix notation.

2.1. Bornological convergence

Let (X, d) be a metric space and let $\mathcal{S} \subseteq \mathcal{P}(X)$ be an ideal. For a net $\{A_\lambda\}_{\lambda \in \Lambda}$ of nonempty subsets of X and a set $A \subseteq X$, we say that A_λ *bornologically converges* to A with respect to \mathcal{S} , written

$$A = \mathcal{S}\text{-}\lim_{\lambda} A_{\lambda},$$

if for every $S \in \mathcal{S}$ and every $\varepsilon > 0$ there exists λ_0 such that for all $\lambda \succeq \lambda_0$,

$$A \cap S \subseteq (A_\lambda)^\varepsilon \quad \text{and} \quad A_\lambda \cap S \subseteq A^\varepsilon,$$

where $A^\varepsilon := \{x \in X : d(x, A) < \varepsilon\}$. Equivalently,

$$\lim_{\lambda} \sup_{x \in A \cap S} d(x, A_\lambda) = 0, \quad \lim_{\lambda} \sup_{x \in A_\lambda \cap S} d(x, A) = 0.$$

We refer to [3,5,9] for further background and related hyperspace topologies (see also [10,11]).

2.2. Statistical convergence

A set $E \subseteq \mathbb{N}$ has *natural density* $\delta(E)$ if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{|E \cap \{1, \dots, n\}|}{n},$$

exists. A sequence (x_n) of real numbers is *statistically convergent* to L if for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0,$$

equivalently,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

See [6,7,12] for general theory and [8] for statistical convergence of sequences of sets.

3. Standing assumptions and conventions

Throughout the paper, the following standing assumptions and conventions are in force.

- (X, d) denotes a metric space.
- $\mathcal{S} \subseteq \mathcal{P}(X)$ is an ideal that *covers* X , that is, for every $x \in X$ there exists $S \in \mathcal{S}$ such that $x \in S$.
- We assume that \mathcal{S} is *ball-admissible*, meaning

$$\forall x \in X, \forall r > 0, \quad B(x, r) \in \mathcal{S}.$$

In particular, this assumption is satisfied when \mathcal{S} is the ideal of all bounded subsets of X .

- All sequences $\{A_n\}_{n \in \mathbb{N}}$ consist of nonempty subsets of X . The limit set A may be empty or nonempty.
- When $A = \emptyset$, we adopt the following conventions:
 - $d(x, \emptyset) := \infty$ for all $x \in X$,
 - $\emptyset^\varepsilon := \emptyset$ for all $\varepsilon > 0$,
 - for nonempty $B \subseteq X$, $B^\varepsilon := \{x \in X : d(x, B) < \varepsilon\}$.
- Throughout, \mathcal{S} is an ideal of subsets of X (closed under taking subsets and finite unions).
- For excess-type quantities, we use the convention

$$\sup \emptyset := 0.$$

In particular, all excess functionals defined in the sequel are nonnegative.

- By \mathbb{N} we denote the set of positive integers.

Whenever Cesàro averages of distance-type quantities are considered, we will explicitly assume boundedness of the relevant sets in order to ensure finiteness of the corresponding suprema.

4. Statistical bornological convergence: Definition

Definition 1. Let $E \subseteq \mathbb{N}$. The *natural density* of E , if it exists, is

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{|E \cap \{1, \dots, n\}|}{n}.$$

If the limit does not exist, we may consider the upper density

$$\bar{\delta}(E) = \limsup_{n \rightarrow \infty} \frac{|E \cap \{1, \dots, n\}|}{n}.$$

In this paper, when we say “density one”, we mean that the natural density exists and equals one; equivalently, the complement has upper density zero.

Definition 2. Let (X, d) be a metric space and $\mathcal{S} \subseteq \mathcal{P}(X)$ an ideal covering X . A sequence $\{A_n\}_{n \in \mathbb{N}}$ of nonempty subsets of X is said to *statistically bornologically converge* to a set $A \subseteq X$ (possibly empty) with respect to \mathcal{S} if for every $S \in \mathcal{S}$ and every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : A_n \cap S \subseteq A^\varepsilon \text{ and } A \cap S \subseteq A_n^\varepsilon\}) = 1.$$

We denote this by $A = \mathcal{S}\text{-st-lim } A_n$.

Remark 1. If a sequence converges bornologically to A , then it trivially converges statistically bornologically, since the inclusion conditions hold for all n , hence on a set of density one. The converse need not hold; statistically bornological convergence allows a sparse set of indices where the conditions fail.

Example 1 (Nonempty limit). Let $X = \mathbb{R}$ with the usual metric, and let \mathcal{S} be the family of bounded subsets of \mathbb{R} (which forms an ideal covering \mathbb{R}). Define

$$A_n = \begin{cases} [-1/n, 1/n], & \text{if } n \text{ is not a power of two,} \\ [-n, n], & \text{if } n = 2^k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Let $A = \{0\}$. For n not a power of two, A_n shrinks to $\{0\}$; for $n = 2^k$, A_n is large. The set of powers of two has density zero, so for any $S \in \mathcal{S}$ (bounded) and $\varepsilon > 0$, the inclusion conditions hold for all n outside this sparse set. Thus $\{A_n\}$ converges statistically bornologically to $\{0\}$, but it does not converge bornologically (since the conditions fail for infinitely many indices).

Example 2 (Empty limit). Let $X = \mathbb{R}$ with the usual metric, and let \mathcal{S} be the family of bounded subsets. Define

$$A_n = \mathbb{R} \setminus [-n, n], \quad A = \emptyset.$$

Fix $S \in \mathcal{S}$ (bounded) and $\varepsilon > 0$. There exists $M > 0$ such that $S \subseteq [-M, M]$. For $n > M + \varepsilon$, we have $A_n \cap S = \emptyset = A \cap S$, so $A_n \cap S \subseteq A^\varepsilon = \emptyset$ and $A \cap S \subseteq A_n^\varepsilon$ hold (the latter because $\emptyset \subseteq A_n^\varepsilon$ is always true). The indices $n \leq M + \varepsilon$ form a finite (hence density zero) set. Therefore, $\{A_n\}$ converges statistically bornologically to \emptyset .

Proposition 1 (Monotonicity under inclusion). *Let $\{A_n\}$ and $\{B_n\}$ be sequences of nonempty subsets such that $\mathcal{S}\text{-st-lim } A_n = A$ and $\mathcal{S}\text{-st-lim } B_n = B$. If $A_n \subseteq B_n$ for all n and B is closed, then $A \subseteq B$.*

Proof. Fix $x \in A$ and suppose, for contradiction, that $x \notin B$. Since B is nonempty and closed under distance, we have

$$\rho := d(x, B) > 0.$$

Let $S := B(x, \rho/2) \in \mathcal{S}$. Choose $\varepsilon := \rho/4$.

By \mathcal{S} -statistical bornological convergence of A_n to A , the set

$$E_1 := \{n \in \mathbb{N} : A \cap S \subseteq A_n^\varepsilon\},$$

has density one. Since $x \in A \cap S$, for each $n \in E_1$ we have $d(x, A_n) < \varepsilon$, hence there exists $a_n \in A_n$ with $d(x, a_n) < \varepsilon$. In particular, $a_n \in S$ because $\varepsilon = \rho/4 < \rho/2$, so $a_n \in A_n \cap S$.

By \mathcal{S} -statistical bornological convergence of B_n to B , the set

$$E_2 := \{n \in \mathbb{N} : B_n \cap S \subseteq B^\varepsilon\},$$

has density one. Let $n \in E_1 \cap E_2$ (still density one). Then $a_n \in A_n \cap S \subseteq B_n \cap S$, hence $a_n \in B^\varepsilon$. So $d(a_n, B) < \varepsilon$, and therefore

$$d(x, B) \leq d(x, a_n) + d(a_n, B) < \varepsilon + \varepsilon = \rho/2,$$

contradicting $d(x, B) = \rho$. Thus $x \in B$, and since $x \in A$ was arbitrary, we conclude $A \subseteq B$. \square

5. A key Lemma: Local distance control

The following lemma establishes the precise relationship between bornological inclusion conditions and distance control on the relevant subsets. This lemma is formulated in terms of the natural "excess" functionals used in bornological convergence theory.

Convention. Throughout this section, we adopt the convention $\sup \emptyset := 0$ in the definition of the excess functionals. In particular, $e_S(A, B) \geq 0$ for all A, B, S .

Lemma 1. *Let $A, B \subseteq X$ be nonempty sets, let $S \subseteq X$, and let $\varepsilon > 0$. If*

$$A \cap S \subseteq B^\varepsilon \quad \text{and} \quad B \cap S \subseteq A^\varepsilon,$$

then

$$\sup_{x \in A \cap S} d(x, B) \leq \varepsilon \quad \text{and} \quad \sup_{x \in B \cap S} d(x, A) \leq \varepsilon.$$

Proof. If $x \in A \cap S$, then $x \in B^\varepsilon$, so $d(x, B) < \varepsilon$. Hence $\sup_{x \in A \cap S} d(x, B) \leq \varepsilon$. Similarly, if $x \in B \cap S$, then $x \in A^\varepsilon$, so $d(x, A) < \varepsilon$, giving $\sup_{x \in B \cap S} d(x, A) \leq \varepsilon$. \square

Remark 2. Lemma 1 does not claim control over $\sup_{x \in S} |d(x, A) - d(x, B)|$; such control would require additional assumptions (e.g., that $S \subseteq A^\varepsilon \cup B^\varepsilon$). The form given here is exactly what is needed for the functional characterization below, as it matches the natural "excess" functionals used in bornological convergence.

6. A functional characterization using excess functionals

For the functional characterization, we use the natural excess functionals from bornological convergence theory. Define for any $S \in \mathcal{S}$,

$$e_S(A_n, A) := \max \left\{ \sup_{x \in A_n \cap S} d(x, A), \sup_{x \in A \cap S} d(x, A_n) \right\}.$$

This quantity measures how far A_n is from A on the test set S . Note that $e_S(A_n, A) < \varepsilon$ is equivalent to the two inclusion conditions in Definition 2.

We now assume boundedness to ensure these quantities are finite and the Cesàro averages are well-defined.

Assumption 1. In this section, we assume that all sets A_n ($n \in \mathbb{N}$) and the limit set A are nonempty and bounded. Moreover, we assume that each $S \in \mathcal{S}$ is such that the distance functions are bounded on S (this holds automatically if S is bounded and the A_n are bounded).

Theorem 1. Let Assumption 1 hold. Then $\{A_n\}$ statistically bornologically converges to A with respect to \mathcal{S} if and only if for every $S \in \mathcal{S}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e_S(A_k, A) = 0.$$

Proof. (\Rightarrow) Suppose $\{A_n\}$ statistically bornologically converges to A . Fix $S \in \mathcal{S}$ and $\varepsilon > 0$. Define

$$E_{S,\varepsilon} := \{n \in \mathbb{N} : A_n \cap S \subseteq A^\varepsilon \text{ and } A \cap S \subseteq A_n^\varepsilon\}.$$

By definition, $\delta(E_{S,\varepsilon}) = 1$. For $n \in E_{S,\varepsilon}$, Lemma 1 gives

$$e_S(A_n, A) \leq \varepsilon.$$

Since all sets are bounded, there exists $M > 0$ such that $e_S(A_n, A) \leq M$ for all n (this follows from boundedness of S , A_n , and A). Then

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n e_S(A_k, A) &= \frac{1}{n} \sum_{k \in E_{S,\varepsilon} \cap [1, n]} e_S(A_k, A) + \frac{1}{n} \sum_{k \notin E_{S,\varepsilon}} e_S(A_k, A) \\ &\leq \varepsilon \cdot \frac{|E_{S,\varepsilon} \cap [1, n]|}{n} + M \cdot \left(1 - \frac{|E_{S,\varepsilon} \cap [1, n]|}{n}\right). \end{aligned}$$

Taking $n \rightarrow \infty$, the right-hand side converges to ε . Since ε was arbitrary,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e_S(A_k, A) = 0.$$

(\Leftarrow) Conversely, assume the Cesàro limit is zero for every $S \in \mathcal{S}$. Fix $S \in \mathcal{S}$ and $\varepsilon > 0$. Define

$$F_{S,\varepsilon} := \{n \in \mathbb{N} : e_S(A_n, A) \geq \varepsilon\}.$$

Then

$$\frac{1}{n} \sum_{k=1}^n e_S(A_k, A) \geq \frac{1}{n} \sum_{k \in F_{S,\varepsilon} \cap [1, n]} \varepsilon = \varepsilon \cdot \frac{|F_{S,\varepsilon} \cap [1, n]|}{n}.$$

Since the left-hand side tends to zero, we must have $\lim_{n \rightarrow \infty} \frac{|F_{S,\varepsilon} \cap [1, n]|}{n} = 0$, i.e., $\delta(F_{S,\varepsilon}) = 0$. Hence $E_{S,\varepsilon} := \mathbb{N} \setminus F_{S,\varepsilon}$ has density one.

Now for $n \in E_{S,\varepsilon}$, we have $e_S(A_n, A) < \varepsilon$. By definition of e_S , this implies both $\sup_{x \in A_n \cap S} d(x, A) < \varepsilon$ and $\sup_{x \in A \cap S} d(x, A_n) < \varepsilon$. The first inequality gives $A_n \cap S \subseteq A^\varepsilon$; the second gives $A \cap S \subseteq A_n^\varepsilon$. Therefore, both

inclusion conditions hold for all $n \in E_{S,\varepsilon}$, and $\delta(E_{S,\varepsilon}) = 1$. Hence $\{A_n\}$ statistically bornologically converges to A . \square

Remark 3. The functional characterization uses the excess functional $e_S(A_n, A)$ rather than $\sup_{x \in S} |d(x, A_n) - d(x, A)|$. This is more natural in the bornological context and avoids the technical issues with controlling distances on the entire set S . The two formulations are equivalent under additional assumptions (e.g., if $S \subseteq A^\varepsilon \cup A_n^\varepsilon$ for large n), but the excess formulation follows directly from Lemma 1.

7. Stability under Bi-Lipschitz embeddings

In general, mere Lipschitz continuity does not allow one to control sets of the form $F(A_n) \cap T$ with $T \subseteq Y$ using only information about $A_n \cap S$ in X , because points in $F(A_n) \cap F[S]$ need not admit a common preimage belonging to both A_n and S . To avoid this obstruction, we assume that F is an embedding with a Lipschitz inverse on its range.

Definition 3. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $F : X \rightarrow Y$ is called a *bi-Lipschitz embedding* if F is injective and there exist constants $L \geq 0$ and $M \geq 0$ such that

$$d_Y(F(x), F(x')) \leq L d_X(x, x') \quad \text{for all } x, x' \in X,$$

and, writing $R := F(X)$, the inverse $F^{-1} : R \rightarrow X$ satisfies

$$d_X(F^{-1}(y), F^{-1}(y')) \leq M d_Y(y, y') \quad \text{for all } y, y' \in R.$$

Lemma 2. Let $F : X \rightarrow Y$ be L -Lipschitz. Then for any $B \subseteq X$ and $\varepsilon > 0$,

$$F(B^\varepsilon) \subseteq (F(B))^{L\varepsilon}.$$

Proof. If $y \in F(B^\varepsilon)$, then $y = F(x)$ for some x with $d_X(x, B) < \varepsilon$. Choose $b \in B$ with $d_X(x, b) < \varepsilon$. Then

$$d_Y(y, F(B)) \leq d_Y(F(x), F(b)) \leq L d_X(x, b) < L\varepsilon,$$

so $y \in (F(B))^{L\varepsilon}$. \square

Lemma 3. Let $F : X \rightarrow Y$ be a bi-Lipschitz embedding, and set $R := F(X)$. Assume that $F^{-1} : R \rightarrow X$ is M -Lipschitz. Then for any $C \subseteq X$ and $\varepsilon > 0$,

$$(F(C))^\varepsilon \cap R \subseteq F(C^{M\varepsilon}).$$

Proof. Let $y \in (F(C))^\varepsilon \cap R$. Then $y = F(x)$ for some $x \in X$ and there exists $c \in C$ such that $d_Y(F(x), F(c)) < \varepsilon$. Since F^{-1} is M -Lipschitz on R ,

$$d_X(x, c) = d_X(F^{-1}(F(x)), F^{-1}(F(c))) \leq M d_Y(F(x), F(c)) < M\varepsilon,$$

hence $x \in C^{M\varepsilon}$ and therefore $y = F(x) \in F(C^{M\varepsilon})$. \square

Theorem 2. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $\mathcal{S} \subseteq \mathcal{P}(X)$ be an ideal covering X . Assume that $F : X \rightarrow Y$ is a bi-Lipschitz embedding with constants L and M , and let $R := F(X)$. Define an ideal on Y by

$$F(\mathcal{S}) := \{ T \subseteq Y : F^{-1}(T \cap R) \in \mathcal{S} \}.$$

If $\{A_n\}$ statistically bornologically converges to A with respect to \mathcal{S} , then $\{F(A_n)\}$ statistically bornologically converges to $F(A)$ with respect to $F(\mathcal{S})$.

Proof. Fix $T \in F(\mathcal{S})$ and $\varepsilon > 0$. Let $R := F(X)$ and set $S := F^{-1}(T \cap R) \in \mathcal{S}$. Define

$$\eta := \frac{\varepsilon}{LM}.$$

By the \mathcal{S} -statistical bornological convergence of A_n to A , the set

$$E := \left\{ n \in \mathbb{N} : A_n \cap S \subseteq A^{M\eta} \text{ and } A \cap S \subseteq A_n^{M\eta} \right\},$$

has natural density one.

We claim that for each $n \in E$,

$$F(A_n) \cap T \subseteq (F(A))^\varepsilon \quad \text{and} \quad F(A) \cap T \subseteq (F(A_n))^\varepsilon.$$

First inclusion. Let $y \in F(A_n) \cap T$. Since $y \in F(A_n) \subseteq R$, the inverse $F^{-1}(y)$ is defined. Moreover, $y \in T \cap R$ implies $F^{-1}(y) \in S$. Hence $F^{-1}(y) \in A_n \cap S \subseteq A^{M\eta}$, so $y \in F(A^{M\eta})$. By Lemma 2,

$$y \in F(A^{M\eta}) \subseteq (F(A))^{L(M\eta)} = (F(A))^{LM\eta} = (F(A))^\varepsilon.$$

Second inclusion. Let $y \in F(A) \cap T$. Then $y \in R$ and $F^{-1}(y) \in A$. Also $y \in T \cap R$ gives $F^{-1}(y) \in S$, hence $F^{-1}(y) \in A \cap S \subseteq A_n^{M\eta}$. Therefore $y \in F(A_n^{M\eta}) \subseteq (F(A_n))^{LM\eta} = (F(A_n))^\varepsilon$ by Lemma 2.

Thus both inclusions hold for all $n \in E$, and since $\delta(E) = 1$, we conclude that $\{F(A_n)\}$ statistically bornologically converges to $F(A)$ with respect to $F(\mathcal{S})$. \square

Remark 4. Theorem 2 shows that statistically bornological convergence is preserved under bi-Lipschitz embeddings. The use of the ideal $F(\mathcal{S}) = \{T \subseteq Y : F^{-1}(T \cap F(X)) \in \mathcal{S}\}$ eliminates the preimage-selection obstruction present for general (non-injective) Lipschitz maps.

8. Subsequence properties

For the subsequence theorem, we need the standard diagonal density lemma from statistical convergence theory.

Lemma 4 (Diagonal Density Lemma [7]). *Let $\{E_m\}_{m \in \mathbb{N}}$ be a countable family of subsets of \mathbb{N} each having natural density one. Then there exists a set $E \subseteq \mathbb{N}$ with natural density one such that $E \setminus E_m$ is finite for every m (i.e., E is eventually contained in each E_m).*

Using this lemma, we can prove the subsequence theorem.

Theorem 3. *Let $\{A_n\}$ statistically bornologically converge to A with respect to \mathcal{S} . Assume that \mathcal{S} has a countable base (i.e., there exists a countable family $\{S_m\}_{m \in \mathbb{N}} \subseteq \mathcal{S}$ such that every $S \in \mathcal{S}$ is contained in some S_m). Then there exists a subsequence $\{A_{n_j}\}$ such that $\{n_j\}$ has natural density one and $\{A_{n_j}\}$ converges bornologically to A (in the classical sense) with respect to \mathcal{S} .*

Proof. For each $m, k \in \mathbb{N}$, define

$$E_{m,k} := \left\{ n \in \mathbb{N} : A_n \cap S_m \subseteq A^{1/k} \text{ and } A \cap S_m \subseteq A_n^{1/k} \right\}.$$

By statistical bornological convergence, $\delta(E_{m,k}) = 1$ for each m, k .

By the Diagonal Density Lemma, there exists a set $E \subseteq \mathbb{N}$ with $\delta(E) = 1$ such that $E \setminus E_{m,k}$ is finite for every m, k . Let $\{n_j\}$ be the increasing enumeration of E .

Now fix any $S \in \mathcal{S}$ and $\varepsilon > 0$. Choose m such that $S \subseteq S_m$, and choose k with $1/k < \varepsilon$. Since $E \setminus E_{m,k}$ is finite, there exists J such that for all $j \geq J$, $n_j \in E_{m,k}$. Then for all $j \geq J$,

$$A_{n_j} \cap S \subseteq A_{n_j} \cap S_m \subseteq A^{1/k} \subseteq A^\varepsilon,$$

and similarly $A \cap S \subseteq A_{n_j}^\varepsilon$. Thus $\{A_{n_j}\}$ converges bornologically to A . \square

Remark 5. The countable base assumption on \mathcal{S} is satisfied in many standard situations. For example, if \mathcal{S} is the ideal of bounded subsets of a separable metric space (X, d) , then one can take as a countable base the family

of all closed balls $\overline{B}(x_m, r_j)$ with (x_m) a fixed countable dense subset of X and (r_j) the positive rationals. In contrast, if \mathcal{S} is the ideal of all finite subsets of X , then \mathcal{S} admits a countable base *only when X is countable*; in general (e.g. X uncountable) no countable base exists.

9. Relation to Wijsman statistical convergence

Wijsman statistical convergence of a sequence of (nonempty) sets $\{A_n\}$ to A means that for every $x \in X$ and every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| \geq \varepsilon\}) = 0.$$

See [8].

In general, statistically bornological convergence does *not* imply Wijsman statistical convergence: bornological conditions control the sets only through the local inclusions on test sets $S \in \mathcal{S}$, whereas Wijsman convergence requires pointwise control of the distance functions on all of X .

However, under a natural *uniform boundedness* hypothesis, one can recover Wijsman statistical convergence.

Assumption 2 (Uniform boundedness in a test set). There exists $S_0 \in \mathcal{S}$ such that

$$A \subseteq S_0 \quad \text{and} \quad A_n \subseteq S_0 \quad \text{for all } n \in \mathbb{N}.$$

Proposition 2. Assume \mathcal{S} -st- $\lim A_n = A$ and Assumption 2 holds. Then $\{A_n\}$ converges to A in the Wijsman statistical sense.

Proof. Fix $x \in X$. For each n , choose $a_n \in A_n$ with $d(x, a_n) \leq d(x, A_n) + \frac{1}{n}$ and choose $a \in A$ with $d(x, a) \leq d(x, A) + \frac{1}{n}$. By Assumption 2, we have $a_n, a \in S_0$.

Let $\varepsilon > 0$ and consider the density-one set

$$E := \{n \in \mathbb{N} : A_n \cap S_0 \subseteq A^\varepsilon \text{ and } A \cap S_0 \subseteq A_n^\varepsilon\}.$$

For $n \in E$, from $a_n \in A_n \cap S_0 \subseteq A^\varepsilon$ we get $d(a_n, A) < \varepsilon$, hence there exists $b_n \in A$ with $d(a_n, b_n) < \varepsilon$. Then

$$d(x, A) \leq d(x, b_n) \leq d(x, a_n) + d(a_n, b_n) < d(x, A_n) + \frac{1}{n} + \varepsilon.$$

Similarly, from $a \in A \cap S_0 \subseteq A_n^\varepsilon$ we obtain a point $c_n \in A_n$ with $d(a, c_n) < \varepsilon$, and hence

$$d(x, A_n) \leq d(x, c_n) \leq d(x, a) + d(a, c_n) < d(x, A) + \frac{1}{n} + \varepsilon.$$

Combining the two inequalities yields, for all $n \in E$,

$$|d(x, A_n) - d(x, A)| < \varepsilon + \frac{1}{n}.$$

Therefore,

$$\{n : |d(x, A_n) - d(x, A)| \geq 2\varepsilon\} \subseteq \mathbb{N} \setminus E \cup \{1, \dots, N\},$$

for some N , so this set has natural density zero. Since x and ε were arbitrary, we conclude Wijsman statistical convergence. \square

Remark 6. Assumption 2 is automatic, for example, if all sets A_n and A are contained in a fixed bounded ball and \mathcal{S} contains all bounded sets (in particular, all balls).

10. Concluding remarks

We have introduced statistically bornological convergence, addressing the foundational issues raised in previous versions. The key improvements include:

- Clear handling of the empty set with explicit conventions.

- A rigorous functional characterization using excess functionals $e_S(A_n, A)$ rather than suprema over the entire test set, avoiding technical issues with distance control.
- A corrected Lipschitz stability proof for bi-Lipschitz embeddings with a properly defined pushforward ideal.
- A subsequence theorem using the standard diagonal density lemma.
- A precise relationship with Wijsman statistical convergence via the functional characterization.

Remark 7 (Ideal on \mathbb{N}). More generally, one may replace natural density with an arbitrary ideal \mathcal{I} on \mathbb{N} , leading to \mathcal{I} -bornological convergence. Many of the results extend naturally to this setting, provided the ideal satisfies appropriate properties (e.g., containing the Fréchet filter). This connects with the rich theory of ideal convergence [12].

Remark 8 (Open Questions). Several questions remain open:

- Can the boundedness assumption in Theorem 1 be relaxed or removed? This would require a different approach to handle infinite distance values.
- What is the topological nature of statistically bornological convergence? Does it arise from a pseudometric on an appropriate hyperspace?
- How does this notion behave for nets (rather than sequences) and in non-metrizable spaces? Statistical convergence for nets is more delicate and requires filters or ideals.
- The stability under Lipschitz maps for general (non-injective) maps remains open and requires further investigation.

Conflicts of Interest: This work does not have any conflicts of interest.

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