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Structural properties of prime near-rings admitting derivations acting on specific ideals

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Abstract: Consider a near-ring Ω with a semigroup ideal $\mathscr{B}$ and a Jordan ideal $\mathscr{K}$. We analyze the structural properties of prime near-rings through a nonzero derivation $\Lambda : \Omega \to \Omega$ by imposing suitable conditions to specific subsets. The primary objective is to establish results that lead to significant structural constraints and ensuring the existence of a nonzero derivation, induces commutativity of Ω. Our results extend prior findings in the literature by providing new insights into the structural properties of near-rings. Additionally, we include an illustrative example to validate our theoretical claims.

Keywords: prime near-ring, Jordan ideal, semigroup ideal, derivations

MSC: 16N60, 16W25, 16Y30.

1. Introduction

A near-ring is an algebraic system equipped with two operations: addition " $+$ ", which forms a group, and an associative multiplication " $\cdot$ ", adhering to a single distributive law. Specifically, a structure $(\Omega, +, \cdot)$ is a right near-ring if it satisfies $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$ for all $n_1, n_2, n_3 \in \Omega$, or a left near-ring if $n_1 \cdot (n_2 + n_3) = n_1 \cdot n_2 + n_1 \cdot n_3$. A right near-ring is zero-symmetric if $n_1 \cdot 0 = 0$ for all $n_1 \in \Omega$, with $0 \cdot n_1 = 0$ following from the right distributive property. Throughout the paper, Ω represents a zero-symmetric right near-ring. The multiplicative center of Ω is defined as $\mathscr{Z}(\Omega) = \{n_1 \in \Omega \mid n_1 n_2 = n_2 n_1 \text{ for all } n_2 \in \Omega\}$. For arbitrary elements $n_1, n_2 \in \Omega$, the symbols $(n_1 \circ n_2) = n_1 n_2 + n_2 n_1$ and $[n_1, n_2] = n_1 n_2 - n_2 n_1$ represent the classical Jordan and Lie product, respectively. A near-ring Ω is prime if $n_1 \Omega n_2 = \{0\}$ implies $n_1 = 0$ or $n_2 = 0$. A non-void subset $\mathscr{B}$ is called a semigroup ideal of Ω that is both a left semigroup ideal ($\Omega\mathscr{B} \subseteq \mathscr{B}$) and a right semigroup ideal ($\mathscr{B}\Omega \subseteq \mathscr{B}$). An additive subgroup $\mathscr{K}$ of a near-ring Ω is defined as a Jordan ideal of Ω that is both a right Jordan ideal ($k_1 \circ n_1 \in \mathscr{K}$) and a left Jordan ideal ($n_1 \circ k_1 \in \mathscr{K}$) for all $k_1 \in \mathscr{K};\ n_1 \in \Omega$.

Derivations are mappings that preserve algebraic structures through specific product rules. In the framework of near-rings, Bell and Mason [1] characterized a derivation as an additive map $\Lambda : \Omega \to \Omega$ satisfying $\Lambda(n_1 n_2) = \Lambda(n_1) n_2 + n_1 \Lambda(n_2)$, while an analogous formulation was later investigated by Wang [2], by expressing the same condition in a reversed order as $\Lambda(n_1 n_2) = n_1 \Lambda(n_2) + \Lambda(n_1) n_2$ for all $n_1, n_2 \in \Omega$.

Numerous results have been explored for prime, semiprime rings and near-rings with specific constraints on such mappings [3–10]. Bergen et al. [11] established a classical result that if a 2- torsion-free prime ring $\mathscr{R}$ that possesses a nonzero derivation Λ with a nonzero Lie ideal $\mathscr{L}$ and satisfies $\Lambda(\mathscr{L}) \subseteq \mathscr{Z}(\mathscr{R})$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{R})$. For instance, it follows from [12] that a prime near-ring Ω endowed with a nonzero derivation $\Lambda : \Omega \to \Omega$ satisfying the identity $\Lambda([n_1, n_2]) = [\Lambda(n_1), n_2]$ for all $n_1, n_2 \in \Omega$, is commutative. Subsequently, Boua et al. [6] proved that a 2-torsion-free prime near-ring Ω containing a nonzero Lie ideal $\mathscr{L}$ and admits a generalized derivation $\mathscr{F}$ associated with a derivation d satisfying any of the subsequent conditions: (i) $\mathscr{F}([i_1, n_1]) = (i_1 \circ n_1)$, (ii) $\mathscr{F}(i_1 \circ n_1) = [i_1, n_1]$, (iii) $\mathscr{F}([i_1, n_1]) \pm (i_1 \circ n_1) \in \mathscr{Z}(\Omega)$ for all $i_1 \in \mathscr{L}, n_1 \in \Omega$, then $(\Omega, +)$ is abelian.

Drawing from these advancements, we want to pursue this line of investigation and our results pertaining to the conditions imposed on specific subsets providing a natural extension of prior findings derived from analogous assumptions on the entire near-ring.

This manuscript is organized as follows: §2 provides key preliminary results. §3 presents our main theorems on commutativity in near-rings and provides example in support of this discussion. §4 discusses potential directions for future research.

2. Preliminaries

The subsequent lemmas are essential for proving our main findings. Note that the proofs of Lemmas 1, 4 and 5 can be found in [13], [14] and [2] respectively, in the context of left near-ring. Similar results can be obtained for right near-ring.

Lemma 1. [13, Lemma 1.3 & Lemma 1.4] *Consider a prime near-ring Ω, endowed with a nonzero semigroup ideal $\mathscr{B}$.*

(i) *If $n_1, n_2 \in \Omega$ satisfy $n_1 \mathscr{B} n_2 = \{0\}$, it follows that either $n_1 = 0$ or $n_2 = 0$.*
(ii) *For any $n_1 \in \Omega$, if either $n_1 \mathscr{B} = \{0\}$ or $\mathscr{B} n_1 = \{0\}$, then necessarily $n_1 = 0$.*

Lemma 2. [12, Lemma 2] *Consider a near-ring Ω that possesses a derivation $\Lambda : \Omega \to \Omega$. Then, the subsequent identities are valid:*

(i) $n_1(\Lambda(n_2)n_3 + n_2\Lambda(n_3)) = n_1\Lambda(n_2)n_3 + n_1n_2\Lambda(n_3); \quad \forall\, n_1, n_2, n_3 \in \Omega.$
(ii) $n_1(n_2\Lambda(n_3) + \Lambda(n_2)n_3) = n_1n_2\Lambda(n_3) + n_1\Lambda(n_2)n_3; \quad \forall\, n_1, n_2, n_3 \in \Omega.$

Lemma 3. [15, Lemma 1 & Lemma 3] *Consider a prime near-ring Ω, endowed with a nonzero Jordan ideal $\mathscr{K}$.*

(i) *If $\mathscr{K} \subseteq \mathscr{Z}(\Omega)$ and Ω satisfies 2-torsion-free condition, then Ω is necessarily a commutative ring.*
(ii) *For any element $n_1 \in \Omega$, if $\mathscr{K} n_1 = \{0\}$, then necessarily $n_1 = 0$.*

Lemma 4. [7, Corollary 3] *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that satisfies $\Lambda(\mathscr{K}) = \{0\}$, then $\Lambda = 0$ or $\mathscr{K}$ is commutative under multiplication of Ω.*

Lemma 5. [14, Lemma 1.8] *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$. If Ω has a derivation Λ that is not identically trivial, then $\Lambda^2(\mathscr{B}) \neq \{0\}$.*

Lemma 6. [2, Lemma 2] *Consider a prime near-ring Ω that possesses a derivation $\Lambda : \Omega \to \Omega$. Then the derivation preserves centrality, i.e., for any central element $n_1 \in \mathscr{Z}(\Omega)$, the element $\Lambda(n_1)$ also lies in $\mathscr{Z}(\Omega)$.*

Lemma 7. [1, Theorem 2] *Consider a prime near-ring Ω with no nonzero elements of order 2. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda(\Omega) \subseteq \mathscr{Z}(\Omega)$, then Ω is necessarily a commutative ring.*

3. Commutativity criteria for prime near-rings under derivation acting on Jordan ideal and semigroup ideal

This section focuses on the study of derivations of a right near-ring Ω under certain algebraic identities on two distinguished subsets as Jordan ideal $\mathscr{K}$ and semigroup ideal $\mathscr{B}$ of Ω. By restricting the assumptions to these subsets instead of the whole near-ring, we establish structural behavior of near-ring Ω. These findings demonstrate the significance of restricting the identities to Jordan ideal and semigroup ideal and highlight the role of the underlying hypotheses. Indeed, we establish the following results.

Theorem 1. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda([k_1, p_1]) = [\Lambda(k_1), p_1]$ for all $p_1 \in \mathscr{B}$; $k_1 \in \mathscr{K}$, then Ω is necessarily a commutative ring.*

Proof. By the given condition,

$$\Lambda([k_1, p_1]) = [\Lambda(k_1), p_1], \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{1}$$

Since $\mathscr{B}$ is a semigroup ideal of Ω, so for each $p_1 \in \mathscr{B}$ and $k_1 \in \mathscr{K} \subseteq \Omega$, we have $p_1k_1 \in \mathscr{B}$. Therefore, substitute p_1 with p_1k_1 in (1) to obtain

$$\Lambda([k_1, p_1])k_1 + [k_1, p_1]\Lambda(k_1) = [\Lambda(k_1), p_1k_1].$$

Applying (1), we see that

$$[\Lambda(k_1), p_1]k_1 + [k_1, p_1]\Lambda(k_1) = [\Lambda(k_1), p_1k_1].$$

Expanding by using right-distributivity,

$$\Lambda(k_1)p_1k_1 - p_1\Lambda(k_1)k_1 + k_1p_1\Lambda(k_1) - p_1k_1\Lambda(k_1) = \Lambda(k_1)p_1k_1 - p_1k_1\Lambda(k_1).$$

This simplifies to,

$$k_1p_1\Lambda(k_1) = p_1\Lambda(k_1)k_1, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{2}$$

Replacing p_1 with n_1p_1 in (2) and invoking (2), it gives

$$k_1n_1p_1\Lambda(k_1) = n_1p_1\Lambda(k_1)k_1 = n_1k_1p_1\Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

This implies that

$$[k_1, n_1]p_1\Lambda(k_1) = 0, \quad \forall\, k_1 \in \mathscr{K};\ p_1 \in \mathscr{B};\ n_1 \in \Omega.$$

i.e.,

$$[k_1, n_1]\mathscr{B}\Lambda(k_1) = \{0\}, \quad \forall\, k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Lemma 1(i), yielding

$$k_1 \in \mathscr{Z}(\Omega) \quad \text{or} \quad \Lambda(k_1) = 0, \quad \forall\, k_1 \in \mathscr{K}. \tag{3}$$

If for some $k_0 \in \mathscr{K}$ the second alternative holds, $\Lambda(k_0) = 0$. Substitute k_1 with k_0 in (1), we get $\Lambda(k_0p_1) = \Lambda(p_1k_0)$ for all $p_1 \in \mathscr{B}$, this reduces to

$$k_0\Lambda(p_1) = \Lambda(p_1)k_0, \quad \forall\, p_1 \in \mathscr{B}. \tag{4}$$

Putting $q_1\Lambda(p_1)$ for p_1 in this expression, leading to

$$k_0\Lambda(q_1\Lambda(p_1)) = \Lambda(q_1\Lambda(p_1))k_0, \quad \forall\, p_1, q_1 \in \mathscr{B}.$$

Applying the derivation property to find

$$k_0(\Lambda(q_1)\Lambda(p_1) + q_1\Lambda^2(p_1)) = (\Lambda(q_1)\Lambda(p_1) + q_1\Lambda^2(p_1))k_0, \quad \forall\, p_1, q_1 \in \mathscr{B}.$$

Owing to Lemma 2 and using (4), we derive

$$k_0q_1\Lambda^2(p_1) = q_1\Lambda^2(p_1)k_0, \quad \forall\, p_1, q_1 \in \mathscr{B}. \tag{5}$$

Replace q_1 with n_2q_1 in (5) and using (5) again, giving

$$k_0n_2q_1\Lambda^2(p_1) = n_2q_1\Lambda^2(p_1)k_0 = n_2k_0q_1\Lambda^2(p_1), \quad \forall\, p_1, q_1 \in \mathscr{B};\ n_2 \in \Omega.$$

This yields

$$[k_0, n_2]q_1\Lambda^2(p_1) = 0, \quad \forall\, p_1, q_1 \in \mathscr{B};\ n_2 \in \Omega.$$

Appealing to standard Lemma 1(i) with Lemma 5, we confirm that $k_0 \in \mathscr{Z}(\Omega)$. Therefore, (3) implies $k_1 \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$. Hence, $\mathscr{K} \subseteq \mathscr{Z}(\Omega)$ and by virtue of Lemma 3, Ω is necessarily commutative. □

As a consequence of Theorem 1, we obtain the following corollaries by specializing the subsets involved. Specifically, Corollary 1 is obtained by taking $\mathscr{B} = \Omega$, while Corollary 2 follows by taking $\mathscr{B} = \Omega$ and $\mathscr{K} = \Omega$, since every near-ring is both a semigroup ideal and a Jordan ideal of itself.

Corollary 1. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda([k_1,n_1]) = [\Lambda(k_1),n_1]$ for all $k_1 \in \mathscr{K}$; $n_1 \in \Omega$, then Ω is necessarily a commutative ring.*

Corollary 2. [12, Theorem 1] *Consider a prime near-ring Ω with no nonzero elements of order 2. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda([n_1,n_2]) = [\Lambda(n_1),n_2]$ for all $n_1,n_2 \in \Omega$, then Ω is necessarily a commutative ring.*

Theorem 2. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda([p_1,k_1]) = 0$ for all $p_1 \in \mathscr{B}$; $k_1 \in \mathscr{K}$, then Ω is necessarily a commutative ring.*

Proof. Suppose that
$$\Lambda([p_1,k_1]) = 0, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{6}$$

Replacing p_1 with p_1k_1 in (6) and invoking the derivation property, we get
$$\Lambda([p_1,k_1])k_1 + [p_1,k_1]\Lambda(k_1) = 0, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

This implies
$$p_1k_1\Lambda(k_1) = k_1p_1\Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Substitute p_1 with n_1p_1 in last relation and apply it again, we deduce
$$[k_1,n_1]p_1\Lambda(k_1) = 0, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Thus,
$$[k_1,n_1]\mathscr{B}\Lambda(k_1) = \{0\}, \quad \forall\, k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Utilizing Lemma 1(i), giving
$$k_1 \in \mathscr{Z}(\Omega) \quad \text{or} \quad \Lambda(k_1) = 0, \quad \forall\, k_1 \in \mathscr{K}. \tag{7}$$

If the first alternative $k_1 \in \mathscr{Z}(\Omega)$ occurs, then Lemma 6 yields that $\Lambda(k_1) \in \mathscr{Z}(\Omega)$. On the other hand, if $\Lambda(k_1) = 0$ for all $k_1 \in \mathscr{K}$, then centrality is trivial. Therefore, we have
$$\Lambda(k_1) \in \mathscr{Z}(\Omega), \quad \forall\, k_1 \in \mathscr{K}. \tag{8}$$

Returning to Eq. (6) and expand, we obtain
$$\Lambda(p_1)k_1 = k_1\Lambda(p_1), \quad \forall\, p_1 \in \mathscr{B}.$$

Replace p_1 with $p_1\Lambda(q_1)$ in this expression, we derive
$$(\Lambda(p_1)\Lambda(q_1) + p_1\Lambda^2(q_1))k_1 = k_1(\Lambda(p_1)\Lambda(q_1) + p_1\Lambda^2(q_1)), \quad \forall\, p_1,q_1 \in \mathscr{B}.$$

By applying Lemma 2 with $\Lambda(p_1)k_1 = k_1\Lambda(p_1)$ for all $p_1 \in \mathscr{B}$, simplifies to
$$p_1\Lambda^2(q_1)k_1 = k_1p_1\Lambda^2(q_1), \quad \forall\, p_1,q_1 \in \mathscr{B}. \tag{9}$$

Next, substitute n_2p_1 for p_1 in (9) and apply (9) again, resulting in
$$[k_1,n_2]p_1\Lambda^2(q_1) = 0, \quad \forall\, p_1,q_1 \in \mathscr{B};\ n_2 \in \Omega.$$

Employing Lemma 1(i) with Lemma 5, we find $k_1 \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$. Hence, $\mathscr{K} \subseteq \mathscr{Z}(\Omega)$. Thus, one can conclude the desired result through Lemma 3(i). $\square$

The following corollaries are immediate consequences of Theorem 2 by taking $\mathscr{B} = \Omega$ and $\mathscr{K} = \Omega$.

Corollary 3. [15, Theorem 3] *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda([k_1, n_1]) = 0$ for all $k_1 \in \mathscr{K}$; $n_1 \in \Omega$, then Ω is necessarily a commutative ring.*

Corollary 4. *Consider a prime near-ring Ω with no nonzero elements of order 2. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda([n_1, n_2]) = 0$ for all $n_1, n_2 \in \Omega$, then Ω is necessarily a commutative ring.*

The commutativity criteria established above does not extend to the Jordan product $(p_1 \circ k_1)$. In fact, we obtain the following nonexistence theorems.

Theorem 3. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $\Lambda(p_1) \circ k_1 = (p_1 \circ k_1)$ for all $p_1 \in \mathscr{B}$; $k_1 \in \mathscr{K}$.*

Proof. Assume that

$$\Lambda(p_1) \circ k_1 = (p_1 \circ k_1), \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{10}$$

Substitute p_1 with $p_1 k_1$ in (10) to obtain

$$\Lambda(p_1 k_1) k_1 + k_1 \Lambda(p_1 k_1) = (p_1 \circ k_1) k_1, \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Applying the derivation property with (10), this gives

$$\Lambda(p_1) k_1 k_1 + p_1 \Lambda(k_1) k_1 + k_1 (p_1 \Lambda(k_1) + \Lambda(p_1) k_1) = (\Lambda(p_1) \circ k_1) k_1.$$

Using Lemma 2(ii), leading

$$\Lambda(p_1) k_1 k_1 + p_1 \Lambda(k_1) k_1 + k_1 p_1 \Lambda(k_1) + k_1 \Lambda(p_1) k_1 = \Lambda(p_1) k_1 k_1 + k_1 \Lambda(p_1) k_1,$$

simplifies to

$$p_1 \Lambda(k_1) k_1 + k_1 p_1 \Lambda(k_1) = 0, \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

This implies that

$$p_1 \Lambda(k_1) k_1 = -k_1 p_1 \Lambda(k_1), \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Replace p_1 with $n_1 p_1$ in last equation and utilize it again, yielding

$$\begin{aligned} n_1 p_1 \Lambda(k_1) k_1 &= -k_1 n_1 p_1 \Lambda(k_1) \\ &= (-k_1) n_1 p_1 \Lambda(k_1), \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega, \end{aligned}$$

while the left hand side can be expressed as

$$\begin{aligned} n_1 p_1 \Lambda(k_1) k_1 &= n_1(-k_1 p_1 \Lambda(k_1)) \\ &= n_1((-k_1) p_1 \Lambda(k_1)), \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega. \end{aligned}$$

Combining these expressions, we arrive at

$$(-k_1) n_1 p_1 \Lambda(k_1) = n_1((-k_1) p_1 \Lambda(k_1)), \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

This implies that

$$[n_1, -k_1] p_1 \Lambda(k_1) = 0, \quad \forall\ p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Taking $-k_1$ for k_1 in this relation as $\mathscr{K}$ is an additive subgroup, we have

$$[n_1, k_1]\mathscr{B}\Lambda(-k_1) = \{0\}, \quad \forall\, k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Owing to Lemma 1(i) with the additivity of Λ, we find

$$k_1 \in \mathscr{Z}(\Omega) \quad \text{or} \quad \Lambda(k_1) = 0, \quad \forall\, k_1 \in \mathscr{K}. \tag{11}$$

If for some $k_0 \in \mathscr{K}$ the second alternative occurs, $\Lambda(k_0) = 0$. Then $\Lambda(n_2 \circ k_0) = 0$ for all $n_2 \in \Omega$. On simplifying, we get $\Lambda(n_2) \circ k_0 = 0$ for all $n_2 \in \Omega$. Particularly, for $n_2 \in \mathscr{B}$, (10) yielding $n_2 \circ k_0 = 0$. This can rewrite as, $n_2 k_0 = -k_0 n_2$ for all $n_2 \in \mathscr{B}$. Next, replace n_2 with $n_3 n_2$ in this relation, we deduce that

$$\begin{aligned} n_3 n_2 k_0 &= n_3(-k_0 n_2) \\ &= n_3((-k_0) n_2) \\ &= (-k_0) n_3 n_2, \quad \forall\, n_2 \in \mathscr{B};\ n_3 \in \Omega. \end{aligned}$$

This implies that $[(-k_0), n_3]n_2 = 0$ for all $n_2 \in \mathscr{B};\ n_3 \in \Omega$. In view of Lemma 1(ii), we acquire that $-k_0 \in \mathscr{Z}(\Omega)$, so $k_0 \in \mathscr{Z}(\Omega)$. Therefore, (11) gives $k \in \mathscr{Z}(\Omega)$ for all $k \in \mathscr{K}$. Hence, $\mathscr{K} \subseteq \mathscr{Z}(\Omega)$ and Ω is commutative by standard Lemma 3(i). Returning to (10) and apply given condition of 2-torsion-freeness, this becomes

$$\Lambda(p_1)k_1 = p_1 k_1, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{12}$$

Substitute p_1 by $n_4 p_1$ in (12) and expand, resulting in

$$\Lambda(n_4)p_1 k_1 + n_4 \Lambda(p_1) k_1 = n_4 p_1 k_1.$$

Applying (12), simplifies to $\Lambda(n_4)p_1 k_1 = 0$ for all $p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_4 \in \Omega$, i.e., $\Lambda(n_4)\mathscr{B}k_1 = \{0\}$ for all $k_1 \in \mathscr{K};\ n_4 \in \Omega$. Appealing to Lemma 1(i) with the assumption $\mathscr{K} \neq \{0\}$, we get $\Lambda(n_4) = 0$ for all $n_4 \in \Omega$. Hence, $\Lambda = 0$. $\square$

As a consequence of Theorem 3, we have the following Corollaries:

Corollary 5. [15, Theorem 5] *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero Jordan ideal $\mathscr{K}$. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $\Lambda(n_1) \circ k_1 = (n_1 \circ k_1)$ for all $k_1 \in \mathscr{K};\ n_1 \in \Omega$.*

Corollary 6. [12, Theorem 4] *Consider a prime near-ring Ω with no nonzero elements of order 2. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $\Lambda(n_1) \circ n_2 = (n_1 \circ n_2)$ for all $n_1, n_2 \in \Omega$.*

Theorem 4. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $[\Lambda(p_1), k_1] = (p_1 \circ k_1)$ for all $p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}$.*

Proof. By the assertion

$$[\Lambda(p_1), k_1] = (p_1 \circ k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{13}$$

Substituting $p_1 k_1$ for p_1 in (13) and, expand to get

$$\Lambda(p_1)k_1 k_1 + p_1 \Lambda(k_1) k_1 - k_1 p_1 \Lambda(k_1) - k_1 \Lambda(p_1) k_1 = \Lambda(p_1) k_1 k_1 - k_1 \Lambda(p_1) k_1,$$

reduces to,

$$p_1 \Lambda(k_1) k_1 = k_1 p_1 \Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Taking n_1p_1 for p_1 in this relation and apply it again, yields

$$[k_1, n_1]\mathscr{B}\Lambda(k_1) = \{0\}, \quad \forall\, k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

In light of Lemma 1(i), we conclude

$$k_1 \in \mathscr{Z}(\Omega) \ \text{ or } \ \Lambda(k_1) = 0, \quad \forall\, k_1 \in \mathscr{K}.$$

Suppose the first alternative $k_0 \in \mathscr{Z}(\Omega)$ occurs for some $k_0 \in \mathscr{K}$. Next, replace k_1 with k_0 in (13), we get $p_1 \circ k_0 = 0$ for all $p_1 \in \mathscr{B}$, which implies that $2p_1k_0 = 0$ for all $p_1 \in \mathscr{B}$. By 2-torsion-freeness, we arrive at $p_1k_0 = 0$ for all $p_1 \in \mathscr{B}$, i.e., $\mathscr{B}k_0 = \{0\}$. By applying Lemma 1(ii), leading to $k_0 = 0$, contradicting the assumption. Therefore, the second alternative holds, i.e.,

$$\Lambda(k_1) = 0, \quad \forall\, k_1 \in \mathscr{K}. \tag{14}$$

Thus, Eq. (14) implies $\Lambda(n_2 \circ k_1) = 0$ for all $k_1 \in \mathscr{K};\ n_2 \in \Omega$. Utilizing the derivation property, we deduce

$$\begin{aligned} 0 &= \Lambda(n_2k_1) + \Lambda(k_1n_2) \\ &= \Lambda(n_2)k_1 + n_2\Lambda(k_1) + \Lambda(k_1)n_2 + k_1\Lambda(n_2), \quad \forall\, k_1 \in \mathscr{K};\ n_2 \in \Omega, \end{aligned}$$

leading to

$$\Lambda(n_2)k_1 + k_1\Lambda(n_2) = 0, \quad \forall\, k_1 \in \mathscr{K};\ n_2 \in \Omega.$$

This can rewrite as

$$\Lambda(n_2)k_1 = -k_1\Lambda(n_2), \quad \forall\, k_1 \in \mathscr{K};\ n_2 \in \Omega. \tag{15}$$

Substitute n_2 with $\Lambda(n_2)$ in (15) to obtain

$$\Lambda^2(n_2)k_1 = -k_1\Lambda^2(n_2), \quad \forall\, k_1 \in \mathscr{K};\ n_2 \in \Omega. \tag{16}$$

Returning to Eq. (13), we can rewrite as

$$\Lambda(p_1)k_1 - k_1\Lambda(p_1) = (p_1 \circ k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Applying (15) in this expression, as (15) holds for each element in Ω, giving

$$\begin{aligned} (p_1 \circ k_1) &= \Lambda(p_1)k_1 - k_1\Lambda(p_1) \\ &= (-k_1 - k_1)\Lambda(p_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \end{aligned}$$

Since $\mathscr{K}$ is an additive subgroup, thus a short of computation leads to

$$(k_1 + k_1)\Lambda(p_1) = (p_1 \circ (-k_1)), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Next, Applying Λ both sides and using derivation property, resulting in

$$\begin{aligned} \Lambda(p_1 \circ (-k_1)) &= \Lambda(k_1 + k_1)\Lambda(p_1) + (k_1 + k_1)\Lambda^2(p_1) \\ &= \Lambda(k_1)\Lambda(p_1) + \Lambda(k_1)\Lambda(p_1) + k_1\Lambda^2(p_1) + k_1\Lambda^2(p_1). \end{aligned}$$

Since $-k_1 \in \mathscr{K}$ as $\mathscr{K}$ is an additive subgroup and $p_1 \in \mathscr{B} \subseteq \Omega$, thus $p_1 \circ (-k_1) \in \mathscr{K}$. Applying (14) in the last relation, we arrive at

$$2k_1\Lambda^2(p_1) = 0, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

By 2-torsion-freeness, leading

$$k_1\Lambda^2(p_1) = 0, \quad \forall\, p_1 \in \mathscr{B};\, k_1 \in \mathscr{K}.$$

Invoking Lemma 3(ii), we deduce $\Lambda^2(p_1) = 0$ for all $p_1 \in \mathscr{B}$. Replace p_1 with p_1p_2 in this relation, giving

$$\begin{aligned}
0 &= \Lambda^2(p_1p_2)\\
&= \Lambda(\Lambda(p_1)p_2 + p_1\Lambda(p_2))\\
&= \Lambda(\Lambda(p_1)p_2) + \Lambda(p_1\Lambda(p_2))\\
&= \Lambda^2(p_1)p_2 + \Lambda(p_1)\Lambda(p_2) + \Lambda(p_1)\Lambda(p_2) + p_1\Lambda^2(p_2), \quad \forall\, p_1, p_2 \in \mathscr{B}.
\end{aligned}$$

This simplifies to $2\Lambda(p_1)\Lambda(p_2) = 0$ for all $p_1, p_2 \in \mathscr{B}$. By 2-torsion-freeness, we see that $\Lambda(p_1)\Lambda(p_2) = 0$ for all $p_1, p_2 \in \mathscr{B}$. Substitute p_2 with p_2p_3 in last expression and applying Lemma 2(i), we see that $\Lambda(p_1)\mathscr{B}\Lambda(p_3) = \{0\}$ for all $p_1, p_3 \in \mathscr{B}$. This gives either $\Lambda(p_1) = 0$ or $\Lambda(p_3) = 0$ for all $p_1, p_3 \in \mathscr{B}$ by Lemma 1(i). Therefore, $\Lambda(\mathscr{B}) = \{0\}$, i.e., $\Lambda(p_4) = 0$ for all $p_4 \in \mathscr{B}$. Putting p_4n_3 for p_4, we have $\Lambda(p_4)n_3 + p_4\Lambda(n_3) = 0$, which leads to $p_4\Lambda(n_3) = 0$ for all $p_4 \in \mathscr{B}, n_3 \in \Omega$. Using Lemma 1(ii), we conclude $\Lambda(n_3) = 0$ for all $n_3 \in \Omega$, i.e., $\Lambda = 0$. □

As a consequence of Theorem 4, we obtain the following corollaries: Specifically, Corollary 7 is obtained by taking $\mathscr{B} = \Omega$, while Corollary 8 follows by taking $\mathscr{B} = \Omega$ and $\mathscr{K} = \Omega$.

Corollary 7. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero Jordan ideal $\mathscr{K}$. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $[\Lambda(n_1), k_1] = (n_1 \circ k_1)$ for all $k_1 \in \mathscr{K};\ n_1 \in \Omega$.*

Corollary 8. *Consider a prime near-ring Ω with no nonzero elements of order 2. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $[\Lambda(n_1), n_2] = (n_1 \circ n_2)$ for all $n_1, n_2 \in \Omega$.*

Theorem 5. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that is not identically trivial and satisfies $[\Lambda(p_1), k_1] = 0$ for all $p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}$, then Ω is necessarily a commutative ring.*

Proof. By assumption

$$[\Lambda(p_1), k_1] = 0, \quad \forall\, p_1 \in \mathscr{B};\, k_1 \in \mathscr{K}. \tag{17}$$

Replacing p_1 with $p_1\Lambda(q_1)$ and expand the commutator, leading

$$\Lambda(p_1\Lambda(q_1))k_1 = k_1\Lambda(p_1\Lambda(q_1)), \quad \forall\, p_1, q_1 \in \mathscr{B};\, k_1 \in \mathscr{K}.$$

Applying the derivation property,

$$(\Lambda(p_1)\Lambda(q_1) + p_1\Lambda^2(q_1))k_1 = k_1(\Lambda(p_1)\Lambda(q_1) + p_1\Lambda^2(q_1)), \quad \forall\, p_1, q_1 \in \mathscr{B};\, k_1 \in \mathscr{K}.$$

Appealing to Lemma 2(i) with (17), simplifies to

$$p_1\Lambda^2(q_1)k_1 = k_1p_1\Lambda^2(q_1), \quad \forall\, p_1, q_1 \in \mathscr{B};\, k_1 \in \mathscr{K}.$$

Substitute p_1 with n_1p_1 in this relation to obtain

$$\begin{aligned}
n_1p_1\Lambda^2(q_1)k_1 &= k_1n_1p_1\Lambda^2(q_1)\\
&= n_1k_1p_1\Lambda^2(q_1), \quad \forall\, p_1, q_1 \in \mathscr{B},\, k_1 \in \mathscr{K};\, n_1 \in \Omega.
\end{aligned}$$

This yields

$$[k_1, n_1]p_1\Lambda^2(q_1) = 0, \quad \forall\, p_1, q_1 \in \mathscr{B},\ k_1 \in \mathscr{K};\ n_1 \in \Omega,$$

which means that

$$[k_1, n_1]\mathscr{B}\Lambda^2(q_1) = \{0\}, \quad \forall\, q_1 \in \mathscr{B},\ k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Application of Lemma 1(i) confirms that either $k_1 \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$ or $\Lambda^2(q_1) = 0$ for all $q_1 \in \mathscr{B}$. But the second alternative gives a contradiction to Lemma 5 as $\Lambda^2(q_1) \neq 0$ for all $q_1 \in \mathscr{B}$. Therefore, $k_1 \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$. This assures that $\mathscr{K} \subseteq \mathscr{Z}(\Omega)$. Thus, Lemma 3 gives the desired conclusion. □

Taking $\mathscr{B} = \Omega$ and $\mathscr{K} = \Omega$ in Theorem 5, we get the following corollary:

Corollary 9. [13, Theorem 2.1] Consider a prime near-ring Ω with no nonzero elements of order 2. If Ω has a derivation Λ that is not identically trivial and satisfies $\Lambda(\Omega) \subseteq \mathcal{Z}(\Omega)$, then Ω is necessarily a commutative ring.

Theorem 6. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that is not identically trivial and satisfies $[\Lambda(p_1), k_1] = [p_1, k_1]$ for all $p_1 \in \mathscr{B}$; $k_1 \in \mathscr{K}$, then $\Lambda(\mathscr{K}) \subseteq \mathscr{Z}(\Omega)$.*

Proof. Assume that

$$[\Lambda(p_1), k_1] = [p_1, k_1], \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{18}$$

Substitute p_1 with p_1k_1 in (18) and using (18), giving

$$\Lambda(p_1k_1)k_1 - k_1\Lambda(p_1k_1) = [\Lambda(p_1), k_1]k_1.$$

Employing the derivation property with Lemma 2,

$$\Lambda(p_1)k_1k_1 + p_1\Lambda(k_1)k_1 - k_1p_1\Lambda(k_1) - k_1\Lambda(p_1)k_1 = \Lambda(p_1)k_1k_1 - k_1\Lambda(p_1)k_1,$$

leads to,

$$p_1\Lambda(k_1)k_1 = k_1p_1\Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Putting n_1p_1 for p_1 in this expression to find

$$[k_1, n_1]\mathscr{B}\Lambda(k_1) = \{0\}, \quad \forall\, k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

By applying Lemma 1(i), we confirm that either $k_1 \in \mathscr{Z}(\Omega)$ or $\Lambda(k_1) = 0$ for all $k_1 \in \mathscr{K}$. If the first case occurs, $k_1 \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$, then Lemma 6 assures that $\Lambda(k_1) \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$, while for the second alternative, centrality is trivial. Hence, we have $\Lambda(k_1) \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$, i.e., $\Lambda(\mathscr{K}) \subseteq \mathscr{Z}(\Omega)$. □

Taking $\mathscr{B} = \Omega$ in Theorem 6, we get the following corollary:

Corollary 10. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that is not identically trivial and satisfies $[\Lambda(n_1), k_1] = [n_1, k_1]$ for all $k_1 \in \mathscr{K}$; $n_1 \in \Omega$, then $\Lambda(\mathscr{K}) \subseteq \mathscr{Z}(\Omega)$.*

As an application of Theorem 6, we can obtain the following result by assuming $\mathscr{B} = \Omega$ and $\mathscr{K} = \Omega$.

Proposition 1. *Consider a prime near-ring Ω with no nonzero elements of order 2. If Ω has a derivation Λ that is not identically trivial and satisfies $[\Lambda(n_1), n_2] = [n_1, n_2]$ for all $n_1, n_2 \in \Omega$, then Ω is necessarily a commutative ring.*

Proof. Using the similar techniques as in Theorem 6, we can find $\Lambda(\Omega) \subseteq \mathscr{Z}(\Omega)$. An application of Lemma 7, gives the required result. □

Theorem 7. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $\Lambda(p_1) \circ k_1 = [p_1, k_1]$ for all $p_1 \in \mathscr{B}$; $k_1 \in \mathscr{K}$.*

Proof. By the given assertion

$$\Lambda(p_1) \circ k_1 = [p_1, k_1], \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{19}$$

Substitute p_1 with $p_1 k_1$ in (19), we get

$$\Lambda(p_1 k_1) \circ k_1 = [p_1, k_1]k_1, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Expanding this expression and, applying (19) to find

$$\Lambda(p_1 k_1)k_1 + k_1 \Lambda(p_1 k_1) = (\Lambda(p_1) \circ k_1)k_1, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Utilizing the derivation property, we deduce

$$\Lambda(p_1)k_1 k_1 + p_1 \Lambda(k_1)k_1 + k_1(p_1 \Lambda(k_1) + \Lambda(p_1)k_1) = \Lambda(p_1)k_1 k_1 + k_1 \Lambda(p_1)k_1.$$

Appealing to Lemma 2(ii), simplifies to,

$$p_1 \Lambda(k_1)k_1 = -k_1 p_1 \Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{20}$$

Replace p_1 with $n_1 p_1$ in (20), giving

$$\begin{aligned} n_1 p_1 \Lambda(k_1)k_1 &= -k_1 n_1 p_1 \Lambda(k_1) \\ &= (-k_1) n_1 p_1 \Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega. \end{aligned}$$

Consequently, the left hand side can be written as

$$\begin{aligned} n_1 p_1 \Lambda(k_1)k_1 &= n_1(-k_1 p_1 \Lambda(k_1)) \\ &= n_1((-k_1) p_1 \Lambda(k_1)), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega. \end{aligned}$$

Comparing these expressions to get

$$n_1((-k_1) p_1 \Lambda(k_1)) = (-k_1) n_1 p_1 \Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

This implies that

$$[n_1, -k_1] p_1 \Lambda(k_1) = 0, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Putting $-k_1$ for k_1 in this relation as $\mathscr{K}$ is an additive subgroup, we get

$$[n_1, k_1] \mathscr{B} \Lambda(-k_1) = \{0\}, \quad \forall\, k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Appealing to Lemma 1(i) with the additivity of Λ, resulting in

$$k_1 \in \mathscr{Z}(\Omega) \quad \text{or} \quad \Lambda(k_1) = 0, \qquad \forall\, k_1 \in \mathscr{K}. \tag{21}$$

Assume that the second alternative occurs, $\Lambda(k_0) = 0$ for some $k_0 \in \mathscr{K}$. Then $\Lambda(n_2 \circ k_0) = 0$ for all $n_2 \in \Omega$. By additivity of Λ, we infer that $\Lambda(n_2k_0) + \Lambda(k_0n_2) = 0$ for all $n_2 \in \Omega$. Developing the last relation, we obtain

$$\Lambda(n_2)k_0 + n_2\Lambda(k_0) + \Lambda(k_0)n_2 + k_0\Lambda(n_2) = 0,$$

which gives $\Lambda(n_2)k_0 + k_0\Lambda(n_2) = 0$ for all $n_2 \in \Omega$. Hence, $\Lambda(n_2) \circ k_0 = 0$ for all $n_2 \in \Omega$. Particularly, for $n_2 \in \mathscr{B}$, (19) yielding $[n_2, k_0] = 0$, which can rewrite as $n_2k_0 = k_0n_2$. Replacing n_2 with n_3n_2 in this expression and using it again, we get $[k_0, n_3]n_2 = 0$ for all $n_2 \in \mathscr{B}$, $n_3 \in \Omega$. Applying Lemma 1(ii), we acquire that $k_0 \in \mathscr{Z}(\Omega)$. Thus, (21) yields that $k_1 \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$. Therefore, we have $\mathscr{K} \subseteq \mathscr{Z}(\Omega)$ and Ω is commutative in light of Lemma 3(i). Returning to (19) and apply given condition of 2-torsion-freeness, giving

$$\Lambda(p_1)k_1 = 0, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{22}$$

Substitute p_1 by n_4p_1 (22), resulting in

$$\Lambda(n_4)p_1k_1 + n_4\Lambda(p_1)k_1 = 0, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_4 \in \Omega.$$

Using (22), it reduces to $\Lambda(n_4)p_1k_1 = 0$ for all $p_1 \in \mathscr{B};\ k_1 \in \mathscr{K};\ n_4 \in \Omega$. Owing to Lemma 1(i) with the assumption $\mathscr{K} \neq \{0\}$, we get $\Lambda(n_4) = 0$ for all $n_4 \in \Omega$, i.e., $\Lambda = 0$. $\square$

Taking $\mathscr{B} = \Omega$ and $\mathscr{K} = \Omega$ in Theorem 7, we have the following corollary:

Corollary 11. *Consider a prime near-ring Ω with no nonzero elements of order 2. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $\Lambda(n_1) \circ n_2 = [n_1, n_2]$ for all $n_1, n_2 \in \Omega$.*

Theorem 8. *Consider a prime near-ring Ω with no nonzero elements of order 2, containing a nonzero semigroup ideal $\mathscr{B}$ and a nonzero Jordan ideal $\mathscr{K}$. If Ω has a derivation Λ that satisfies $\Lambda([p_1, k_1]) = p_1 \circ \Lambda(k_1)$ for all $p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}$, then $\Lambda = 0$ or elements of $\mathscr{K}$ commute under multiplication of Ω.*

Proof. By the given condition,

$$\Lambda([p_1, k_1]) = p_1 \circ \Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{23}$$

Substitute p_1 with p_1k_1 in (23) to obtain

$$[p_1, k_1]\Lambda(k_1) + \Lambda([p_1, k_1])k_1 = p_1k_1 \circ \Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}.$$

Applying (23) and expand, simplifies to

$$k_1p_1\Lambda(k_1) = p_1\Lambda(k_1)k_1, \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}. \tag{24}$$

Next, replace p_1 with n_1p_1 in (24) and using (24), leading

$$[k_1, n_1]\mathscr{B}\Lambda(k_1) = \{0\}, \quad \forall\, k_1 \in \mathscr{K};\ n_1 \in \Omega.$$

Lemma 1(i), yielding

$$k_1 \in \mathscr{Z}(\Omega) \quad \text{or} \quad \Lambda(k_1) = 0, \quad \forall\, k_1 \in \mathscr{K}. \tag{25}$$

If the second alternative holds, $\Lambda(k_0) = 0$ for some $k_0 \in \mathscr{K}$. Substitute k_1 with k_0 in (23), we observe that $\Lambda(k_0p_1) = \Lambda(p_1k_0)$ for all $p_1 \in \mathscr{B}$, this reduces to

$$k_0\Lambda(p_1) = \Lambda(p_1)k_0, \quad \forall\, p_1 \in \mathscr{B}. \tag{26}$$

Putting $q_1\Lambda(p_1)$ for p_1 in (26), resulting in

$$k_0(q_1\Lambda^2(p_1) + \Lambda(q_1)\Lambda(p_1)) = (q_1\Lambda^2(p_1) + \Lambda(q_1)\Lambda(p_1))k_0, \quad \forall\, p_1, q_1 \in \mathscr{B}.$$

Appealing to Lemma 2 with (26), we derive

$$k_0 q_1 \Lambda^2(p_1) = q_1 \Lambda^2(p_1) k_0, \quad \forall\, p_1, q_1 \in \mathscr{B}. \tag{27}$$

Substitute q_1 with $n_2 q_1$ in (27) and using (27) again, giving

$$[k_0, n_2] q_1 \Lambda^2(p_1) = 0, \quad \forall\, p_1, q_1 \in \mathscr{B};\ n_2 \in \Omega.$$

In light of standard Lemma 1(i) with Lemma 5, we confirm that $k_0 \in \mathscr{Z}(\Omega)$. Therefore, (25) implies $k_1 \in \mathscr{Z}(\Omega)$ for all $k_1 \in \mathscr{K}$, i.e., $\mathscr{K} \subseteq \mathscr{Z}(\Omega)$ and Lemma 3 ensures that Ω is necessarily commutative. Thus, (23) follows that

$$\begin{aligned} 0 &= p_1 \circ \Lambda(k_1) \\ &= p_1 \Lambda(k_1) + \Lambda(k_1) p_1 \\ &= p_1 \Lambda(k_1) + p_1 \Lambda(k_1), \quad \forall\, p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}, \end{aligned}$$

which implies that $2p_1\Lambda(k_1) = 0$ for all $p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}$. In view of 2-torsion-free condition, we get $p_1\Lambda(k_1) = 0$ for all $p_1 \in \mathscr{B};\ k_1 \in \mathscr{K}$. Applying Lemma 1, we get $\Lambda(k_1) = 0$ for all $k_1 \in \mathscr{K}$, i.e., $\Lambda(\mathscr{K}) = \{0\}$. Hence, Lemma 4 yields that $\Lambda = 0$ or $\mathscr{K}$ is commutative. $\square$

As a consequence of Theorem 8, we find the following corollary by taking $\mathscr{B} = \Omega$ and $\mathscr{K} = \Omega$.

Corollary 12. *Consider a prime near-ring Ω with no nonzero elements of order 2. Then Ω does not admit a derivation Λ that is not identically trivial and satisfies $\Lambda([n_1, n_2]) = n_1 \circ \Lambda(n_2)$ for all $n_1, n_2 \in \Omega$.*

Example 1. Consider a zero-symmetric, non-abelian, 2-torsion-free right near-ring $\mathscr{M}$ and let

$$\Omega = \left\{ \begin{pmatrix} 0 & n_1 & n_2 \\ 0 & 0 & n_3 \\ 0 & 0 & 0 \end{pmatrix} \middle|\ 0, n_1, n_2, n_3 \in \mathscr{M} \right\}.$$

Then Ω is a zero-symmetric, non-abelian, 2-torsion-free right near-ring with respect to matrix addition and matrix multiplication, but Ω is not prime since for $x = \begin{pmatrix} 0 & n_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 & n_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $x\Omega y = \{0\}$ although $x \neq 0,\ y \neq 0$. Define subsets

$$\mathscr{B} = \left\{ \begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ 0 & 0 & 0 \end{pmatrix} \middle|\ 0, p_1, p_2 \in \mathscr{M} \right\};\ \mathscr{K} = \left\{ \begin{pmatrix} 0 & 0 & k_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle|\ 0, k_1 \in \mathscr{M} \right\}$$

with the map $\Lambda : \Omega \to \Omega$ as

$$\Lambda \begin{pmatrix} 0 & n_1 & n_2 \\ 0 & 0 & n_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & n_1 & n_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, for all

$$n = \begin{pmatrix} 0 & n_1 & n_2 \\ 0 & 0 & n_3 \\ 0 & 0 & 0 \end{pmatrix} \in \Omega;\ k = \begin{pmatrix} 0 & 0 & k_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{K};\ p = \begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{B},$$

we see that

(i) $k \circ n = kn + nk = O_{3\times 3} \in \mathscr{K};\ n \circ k = nk + kn = O_{3\times 3} \in \mathscr{K}.$

(ii) $np = \begin{pmatrix} 0 & 0 & n_1p_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{B};\ \ pn = O_{3\times 3} \in \mathscr{B}.$

(ii) $\Lambda(nn') = \Lambda(n)n' + n\Lambda(n') = \begin{pmatrix} 0 & 0 & n_1n'_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all $n, n' \in \Omega$, where $n' = \begin{pmatrix} 0 & n'_1 & n'_2 \\ 0 & 0 & n'_3 \\ 0 & 0 & 0 \end{pmatrix}$.

It is verified that $\mathscr{K}$ is a Jordan ideal and $\mathscr{B}$ is a semigroup ideal of near-ring Ω. Our verification confirms that Λ behaves as a non-trivial derivation satisfies

(i) $\Lambda([k_1, p_1]) = [\Lambda(k_1), p_1]$; (ii) $\Lambda([p_1, k_1]) = 0$; (iii) $\Lambda(p_1) \circ k_1 = (p_1 \circ k_1)$;
(iv) $[\Lambda(p_1), k_1] = 0$; (v) $[\Lambda(p_1), k_1] = (p_1 \circ k_1)$; (vi) $[\Lambda(p_1), k_1] = [p_1, k_1]$;
(vii) $\Lambda(p_1) \circ k_1 = [p_1, k_1]$; (viii) $\Lambda([p_1, k_1]) = p_1 \circ \Lambda(k_1)$ for all $p_1 \in \mathscr{B}$; $k_1 \in \mathscr{K}$

but Ω is non-commutative, indicating the essentiality of primeness.

4. Conclusions

In this paper, we establish conditions ensuring structural result concentrating more localized conditions imposed on specific subsets in the theory of derivations. More precisely, we analyze the action of a non-trivial derivation preserving certain commutators between the elements of semigroup ideal $\mathscr{B}$ and Jordan ideal $\mathscr{K}$. Our results demonstrate that several commutativity conclusions remain valid when the underlying assumptions are imposed only on these nonzero subsets, rather than on the entire near-ring. This provides further insight into the role of derivations in the structure theory of prime near-rings under localized hypotheses.

Finally, we conclude our work by posing two crucial questions: (i) One pertinent direction is to explore whether analogous conclusions can be derived by substituting derivation with generalized derivation or generalized semiderivation or multiplicative generalized semiderivation. (ii) A significant interest is to examine the validity of our principal findings persists when the underlying structure extend to semiprime near-rings.

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