Open Journal of
Mathematical Sciences

PSR Press

Article

The influence of the movable singular point error on the applicability of the analytical approximate solution for a nonlinear fourth-order differential equation

Magomedyusuf Gasanov

Department of Higher Mathematics, Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia; vonasag6991@mail.ru

Received: 14 Oct 2025; Revised: 30 Apr 2026; Accepted: 05 May 2026; Published: 20 May 2026

Abstract: The article continues the study of a class of fourth-order nonlinear differential equations. Earlier, using an analytical approximation method, a theorem of existence and uniqueness (analogous to the Cauchy–Kovalevskaya theorem) was formulated and proved in the vicinity of a movable singular point of algebraic type (hereafter referred to as the movable singular point) in the complex domain. This work addresses the problem of the influence of the perturbation (error) of the movable singular point on the applicability domain of the analytical approximate solution. Two variants of estimating the applicability domain of the analytical approximate solution are considered.

Keywords: approximate analytical solution, cauchy problem, movable singular point, perturbation

MSC: 34A25, 34A34, 34C15.

1. Introduction

A special place in the theory of differential equations is occupied by equations containing nonlinearities of various types, as most phenomena and processes exhibit nonlinear behavior. One of the major challenges in the study and solution of such equations is the presence of movable singular points, which, in general, makes them unsolvable by quadrature methods.

The first classification, dividing singular points into movable and immovable, was proposed by L. Fuchs. Definitions from Golubev's book [1]:

Singular points of the integrals of differential equations, whose position does not depend on the initial data that determine these integrals, are called *fixed singular points*.

Singular points of the integrals of differential equations, whose position depends on the initial data, are called *movable singular points*.

Algebraic movable singular points include simple and multiple poles, as well as branch points of finite order. In the neighborhood of such points, a Puiseux series expansion is assumed.

Therefore, each class of equations requires an individual approach. Currently, the most common methods that can be highlighted include the following:

1. *Finding exact solutions.* Exact solutions are most often understood as particular solutions, that is, solutions of a specific form, or those obtained under certain assumptions. One of the most popular methods for finding exact solutions is group analysis methods [2–4]. These methods are most commonly applied when understanding the laws and physical processes, thus anticipating the behavior of the solution in advance.
2. *Approximate methods.* This method is based on simplifying the model by discretizing derivatives followed by the application of numerical methods [5,6], or linearizing the model [7]. For the application of this method, it is important to understand that there are no singular points over the entire region where the problem is being solved, as otherwise, linearization or discretization could lead to an incorrect solution.
3. *Qualitative theory and asymptotic methods.*This method helps to understand the nature of the solution and its behavior near singular points or at infinity [8,9], which may be sufficient, but if it is necessary to find a solution with a certain degree of accuracy, then such a method is not applicable.

It is also worth noting the actively developing direction based on the *Geometric approach* to studying differential equations in the complex domain [10].

Based on the possible drawbacks and advantages of the methods described above, the author applies the *analytical approximation method* . This method is based on dividing the solution search area into two: the region of analyticity and the neighborhood of the movable singular point. In the region of analyticity, the solution is sought in the form of a power series, and in the neighborhood of the movable singular point, it is sought in the form of a Puiseux series.

The tasks set during the research are as follows:

1. The theorem of existence and uniqueness of a solution in the analyticity domain.
2. Continuation of the solution along the chain of curves.
3. Theorem of existence and uniqueness of the solution in the neighborhood of a movable singular point.
4. The influence of the solution's application area, considering the approximate value of the movable singular point.
5. Necessary as well as necessary and sufficient conditions for the existence of a movable singular point.

This method is actively used for the study of various nonlinear differential equations, both ordinary [11–17] and partial differential equations [18], as well as equations with fractional derivatives [19].

The author has conducted research on a nonlinear third-order differential equation with a polynomial right-hand side of second and seventh degree, respectively [20–25], and demonstrated how the theory can be used to construct an algorithm for locating a movable singular point with any pre-specified accuracy [25].

In the next section, using the considered problem as an example, the methodology of the proposed method will be described in more detail.

2. Preliminary itnformation

The Cauchy problem is considered

$$w^{(4)} + Q_0 w \left(w'\right)^2 = f\left(z\right), \tag{1}$$

$$w^{(j)}\left(z_0\right) = w_0^{(j)}, \tag{2}$$

where $j = \{0,1,2,3\}$; z, $w = w(z)$, $f(z)$, $Q_0 \in \mathbb{C}$; $\left|w_0^{(j)}\right|, |z_0|, |Q_0| < \infty$; $f(z)$ is a holomorphic function throughout the entire domain under consideration $D \subset \mathbb{C}$ such that $f^{(n)}(z) < C \cdot n!$ for all $n \in \mathbb{Z}_{\geq 0}$ and $z \in D$ (C is some positive real constant).

The first problem that naturally arises is the proof of the existence and uniqueness theorem. Analogous to the Cauchy-Kovalevskaya theorem, it was shown in [14] that, under the assumption that all the functions involved in Eq. (1) are analytic with respect to all their arguments, there exists a unique solution to the Cauchy problem (1)–(2) in a sufficiently small neighborhood of z_0, which can be represented in the form

$$w_1\left(z\right) = \sum_{n\geq 0} \alpha_n \left(z - z_0\right)^n, \tag{3}$$

and valid in the domain

$$|z - z_0| < \begin{cases} \min\left\{\sup\limits_n \left(\frac{\left|f^{(n)}(z_0)\right|}{n!}\right), \frac{1}{\sqrt[3]{|Q_0|^2(\gamma_1+1)^2}}\right\}, \text{ if } |Q_0| > 1, \\ \min\left\{\sup\limits_n \left(\frac{\left|f^{(n)}(z_0)\right|}{n!}\right), \frac{1}{\sqrt[3]{(\gamma_1+1)^2}}\right\}, \text{ if } |Q_0| \leq 1, \end{cases} \tag{4}$$

where $\gamma_1 = \max\left\{1, |w_0|, |w_0'|, |w_0''|, |w_0'''|\right\}$.

These results were obtained based on a modified method of majorants. In this case, a guaranteed (not necessarily maximal) region of convergence was obtained by constructing estimates for the coefficients, which is described in detail in [14]. Since the obtained solution (3) does not, in the general case, converge to an elementary or special function, it becomes necessary to consider an analytical approximate solution, i.e., a partial sum of the given series

$$w_{1,N}(z) = \sum_{n=0}^{N} \alpha_n (z - z_0)^n . \tag{5}$$

For the analytical approximate solution (5), an error estimate $\delta w_{1.N}(z)$ was found

$$\delta w_{1,N}(z) \le \begin{cases} \dfrac{|Q_0|^{2N+3}(\gamma_1+1)^{2N+3}\sum\limits_{k=0}^{2}|z-z_0|^{3N+k}}{24\left(1-|z-z_0|^3(\gamma_1+1)^2|Q_0|^2\right)}, & \text{if } |Q_0| > 1, \\ \dfrac{(\gamma_1+1)^{2N+3}\sum\limits_{k=0}^{2}|z-z_0|^{3N+k}}{24\left(1-|z-z_0|^3(\gamma_1+1)^2\right)}, & \text{if } |Q_0| \le 1. \end{cases}$$

The obtained results allow for the calculation of the analytical approximate solution with any given accuracy in the domain (4). To extend beyond the convergence domain of the obtained solution, analytical continuation must be performed. For this, the problem of the influence of perturbation (or error) in the initial data on the convergence domain of the analytical approximate solution is addressed. This problem arises because we are using the analytical approximate solution and continuing the solution relative to it, rather than relative to the exact solution. Therefore, instead of the initial conditions (2), perturbed initial conditions $\tilde{w}^{(j)}(z_0) = \tilde{w}_0^{(j)}$ are considered. The analytical approximate solution to the new Cauchy problem has the form

$$\tilde{w}_{1,N}(z) = \sum_{n=0}^{N} \alpha_n^* (z - z_0)^n , \tag{6}$$

which is valid in the domain

$$|z - z_0| < \begin{cases} \min\left\{\sup\limits_n \left(\dfrac{\left|f^{(n)}(z_0)\right|}{n!}\right), \dfrac{1}{\sqrt[3]{|Q_0|^2(\gamma_1+\delta\gamma_1+1)^2}}\right\}, & \text{if } |Q_0| > 1, \\ \min\left\{\sup\limits_n \left(\dfrac{\left|f^{(n)}(z_0)\right|}{n!}\right), \dfrac{1}{\sqrt[3]{(\gamma_1+\delta\gamma_1+1)^2}}\right\}, & \text{if } |Q_0| \le 1, \end{cases}$$

and has an error estimate

$$\delta \tilde{w}_{1,N}(z) \le \begin{cases} \dfrac{|Q_0|^{2N+3}(\gamma_1+1)^{2N+3}\sum\limits_{k=0}^{2}|z-z_0|^{3N+k}}{24\left(1-|z-z_0|^3(\gamma_1+1)^2|Q_0|^2\right)}, & \text{if } |Q_0| > 1, \\ \dfrac{(\gamma_1+1)^{2N+3}\sum\limits_{k=0}^{2}|z-z_0|^{3N+k}}{24\left(1-|z-z_0|^3(\gamma_1+1)^2\right)}, & \text{if } |Q_0| \le 1. \end{cases}$$

Next, when performing analytical continuations and moving along a chain of curves, we enter the neighborhood of the movable singular point, where the expansion takes the form of a Laurent series or a Puiseux series. It was shown in [16] that in the neighborhood of the movable singular point of algebraic type for the considered Cauchy problem, the solution has a simple pole, i.e., it can be represented in the form

$$w_2(z) = (z^* - z)^{-1} \sum_{n \ge 0} A_n (z^* - z)^n , \tag{7}$$

and the given expansion is valid in the domain $|z^* - z| < \rho_1$, where $\rho_1 = \min\left\{1, \frac{1}{\sqrt[5]{\gamma_2}}\right\}$, $\gamma_2 = \sup\limits_n \left(\frac{\left|f^{(n)}(z^*)\right|}{n!}\right)$, $n \in \mathbb{N}_0$.

Indeed, to determine the order of the movable singularity, we apply the dominant balance method. Let us assume that in a neighborhood of z^* the solution has the asymptotic form

$$w(z) \sim C\,(z - z^*)^p , \quad C \ne 0.$$

Substituting this expression into the equation

$$w^{(4)} + Q_0 w \left(w'\right)^2 = f(z),$$

and comparing the leading-order terms, we obtain

$$w^{(4)} \sim (z - z^*)^{p-4}, \quad w \left(w'\right)^2 \sim (z - z^*)^{3p-2}.$$

Since the function $f(z)$ is analytic, it does not affect the dominant balance. Equating the exponents of the leading terms yields

$$p - 4 = 3p - 2,$$

which implies $p = -1$. Therefore, the solution has a simple pole at z^*:

$$w(z) \sim \frac{C}{z - z^*}.$$

In the considered problem, the local structure of the solution in a neighborhood of the movable singular point z^*, which is a simple pole, does not generate nontrivial monodromy. The differences between possible local representations are not related to encircling the point z^*, but rather to the choice of branch in constructing the asymptotic expansion and to the global analytic continuation of the solution in a domain containing other singular points.

For $n \geq 5$, the coefficients A_n are determined from the relation

$$A_n (n-1)(n-2)(n-3)(n-4) = -Q_0 C_n^* + D_{n-5}, \tag{8}$$

where $C_n^* = \sum\limits_{i=0}^{n} \hat{A}_i^* A_{n-i}$, $\hat{A}_n^* = \sum\limits_{i=0}^{n} B_i B_{n-i}$, $B_n = A_n (n-1)$, D_n — the coefficients of the expansion of $f(z)$ into a Taylor series.

Due to the relation (8) and the initial values for A_n, namely $A_0 = \sqrt{-\frac{24}{Q_0}}$, $A_1 = \ldots = A_4 = 0$, $A_5 = \frac{D_0}{192}$, ..., the following dependence was obtained:

$$A_n = \frac{D_{n-5}}{C} - A_0 \cdot F\left(D_0 D_{n-10}, \ldots, D_l D_k\right), \tag{9}$$

$l + k = n - 10$, C is a certain constant.

Considering the obtained relations (8), (9), and the initial values of the coefficients, an estimate for these coefficients was obtained

$$|A_n| \leq \frac{|\gamma_2|^{\left[\frac{n}{5}\right]} \left(|Q_0 A_0| \left|\left[\frac{n-9}{2}\right] + 1\right| + 1\right)}{|(n-1)(n-2)(n-3)(n-4) + 24(2n-3)|}, \tag{10}$$

where $[a]$ denotes the greatest integer less than or equal to a.

The analytical approximate solution to the Cauchy problem (1)–(2) in the neighborhood of the movable singular point is of the form

$$w_{2,N}(z) = \sum_{n=0}^{N} A_n (z^* - z)^{n-1}, \tag{11}$$

and its a priori error estimate is computed as follows

$$\delta w_{2,N}(z) \leq \frac{|\gamma_2|^{N+1}}{1 - |\gamma_2| |z^* - z|^5} \sum_{k=0}^{4} \frac{\left(|Q_0 A_0| \left|\left[\frac{5(N+1)+k-9}{2}\right] + 1\right| + 1\right) |z^* - z|^{5(N+1)+k-1}}{\left|\prod_{i=1}^{4} (5(N+1)+k-i) + 24(2(5(N+1)+k) - 3)\right|}. \tag{12}$$

3. Main Results

Since existing methods for studying and solving nonlinear differential equations allow finding the movable singular point only approximately in the general case, it is more practical to consider the solution instead of (12).

$$\tilde{w}_{2,N}(z) = \sum_{n=0}^{N} \tilde{A}_n (\tilde{z}^* - z)^{n-1}, \tag{13}$$

where $\tilde{z}^*$ is the approximate (perturbed) value of the movable singular point z^*, which depends on the initial data and, in general, cannot be determined exactly. Therefore, it is obtained via an approximation procedure. The approximation error satisfies the inequality

$$|\tilde{z}^* - z^*| \leq |\Delta\tilde{z}^*|,$$

where $\Delta\tilde{z}^*$ denotes the corresponding error.

Given the above, the problem naturally arises concerning the influence of the error in the movable singular point on the applicability domain of the analytical approximate solution (13) and the estimation of its error relative to the true solution obtained from the expansion in the neighborhood of the exact value of the movable singular point. Next, we formulate a theorem that allows us to determine the influence of the error in the moving singular point on the applicability domain of the analytical approximate solution.

Theorem 1. *Let $\tilde{w}_2(z)$ be the analytical solution, and $\tilde{w}_{2,N}(z)$ the analytical approximate solution to the Cauchy problem (1)–(2) in the neighborhood of the point $\tilde{z}^*$, and the following conditions hold:*

1. $|\tilde{z}^*| \leq |z^*|$;
2. $|\tilde{z}^* - z^*| \leq |\Delta\tilde{z}^*|$;
3. $|\Delta\tilde{z}^*| < \frac{1}{2\sqrt[5]{|\tilde{\gamma}_2 + \delta\tilde{\gamma}_2|}}$;

then in each of the domain

$$\{z : |z| < |\tilde{z}^*| - |\Delta\tilde{z}^*|\} \cap \{z : |\tilde{z}^* - z| < \rho\}, \tag{14}$$

$$\{z : |\tilde{z}^*| - |\Delta\tilde{z}^*| < |z| < |\tilde{z}^*|\} \cap \{z : |\tilde{z}^* - z| < \rho\}. \tag{15}$$

The following error estimate holds:

$$\delta\tilde{w}_{2,N}(z) \leq I_1 + I_2 + I_3,$$

where

$$I_1 = |A_0| \frac{|\Delta\tilde{z}^*|}{|\tilde{z}^* - z|^2},$$

$$I_2 = \frac{|\Delta\tilde{z}^*|\,|\tilde{\gamma}_2|\,(|Q_0 A_0| + 1)\,(16\eta^4 + 8\eta^3)}{48\,(1 - |\tilde{\gamma}_2|\,(2\eta)^5)} \sum_{r=0}^{4} (2\eta)^r,$$

$$I_3 = \frac{|\tilde{\gamma}_2|^{N+1}}{1 - |\tilde{\gamma}_2|\,|\tilde{z}^* - z|^5} \sum_{k=0}^{4} \frac{\left(|Q_0 A_0| \left|\left[\frac{5(N+1)+k-9}{2}\right] + 1\right| + 1\right) |\tilde{z}^* - z|^{5(N+1)+k-1}}{\left|\prod_{i=1}^{4} (5(N+1)+k-i) + 24\,(2(5(N+1)+k) - 3)\right|},$$

where the parameters are defined as follows:

$$\rho = \min\left\{1, \frac{1}{\sqrt[5]{|\tilde{\gamma}_2|}}, \frac{1}{2\sqrt[5]{|\tilde{\gamma}_2 + \delta\tilde{\gamma}_2|}}\right\},$$

$$\tilde{\gamma}_2 = \sup_{n\in\mathbb{N}_0} \left(\frac{\left|f^{(n)}(\tilde{z}^*)\right|}{n!}\right), \qquad \delta\tilde{\gamma}_2 = |\Delta\tilde{z}^*| \sup_{n\in\mathbb{N}_0} \left(\frac{\left|f^{(n)}(\tilde{z}^*)\right|}{n!}\right),$$

$$\eta = \begin{cases} |\tilde{z}^* - z|\,, & \textit{in the domain } (14), \\ |\Delta\tilde{z}^*|\,, & \textit{in the domain } (15). \end{cases}$$

Proof. To estimate the error of the approximate solution, we will use the triangle inequality

$$\delta\tilde{w}_{2,N}(z) = |w_2(z) - \tilde{w}_{2,N}(z)| \leq |w_2(z) - \tilde{w}_2(z)| + |\tilde{w}_2(z) - \tilde{w}_{2,N}(z)|\,.$$

Let's estimate the expression $|w_2(z) - \tilde{w}_2(z)|$:

$$\begin{aligned}
|w_2(z) - \tilde{w}_2(z)| &= \left| \sum_{n\geq 0} A_n (z^* - z)^{n-1} - \sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right| \\
&= \left| \sum_{n\geq 0} A_n (z^* - z)^{n-1} - \sum_{n\geq 0} \tilde{A}_n (z^* - z)^{n-1} + \sum_{n\geq 0} \tilde{A}_n (z^* - z)^{n-1} - \sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right| \\
&\leq \left| \sum_{n\geq 0} A_n (z^* - z)^{n-1} - \sum_{n\geq 0} \tilde{A}_n (z^* - z)^{n-1} \right| + \left| \sum_{n\geq 0} \tilde{A}_n (z^* - z)^{n-1} - \sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right| \\
&\leq \left| \sum_{n\geq 0} \Delta\tilde{A}_n (z^* - z)^{n-1} \right| + \left| \sum_{n\geq 0} \tilde{A}_n \left((z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right) \right| \\
&\leq \sum_{n\geq 0} |\Delta\tilde{A}_n| \, |z^* - z|^{n-1} + \sum_{n\geq 0} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right| \\
&\leq \sum_{n\geq 0} |\Delta\tilde{A}_n| \left(|\tilde{z}^* - z| + |\Delta\tilde{z}^*| \right)^{n-1} + \sum_{n\geq 0} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right|.
\end{aligned}$$

Let's estimate each term separately. To estimate $\sum_{n\geq 0} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right|$, it is necessary to determine $\tilde{A}_n$. Due to the equality (9), it is clear that changing the value of A_n to $\tilde{A}_n$ leads to the equality

$$\tilde{A}_n = \frac{\tilde{D}_{n-5}}{C} - A_0 \cdot F\left(\tilde{D}_0\tilde{D}_{n-10}, \ldots, \tilde{D}_l\tilde{D}_k\right). \tag{16}$$

Thus, the change in $\tilde{A}_n$ depends on the change in $\tilde{D}_0, \ldots, \tilde{D}_k$. Since D_n are the coefficients of the expansion of $f(z)$ into a Taylor series around z^*, we have $D_n = \frac{f^{(n)}(z^*)}{n!}$, and therefore $\tilde{D}_n = \frac{f^{(n)}(\tilde{z}^*)}{n!}$, meaning that the perturbation $\tilde{D}_n$ depends on the perturbation $\tilde{z}^*$. Since $A_0, \ldots, A_4$ do not depend on D_i, the perturbed values are equal to the given ones. Then, we obtain $\tilde{A}_5 = \frac{\tilde{D}_0}{192}$, and so on.

We proceed to estimate $\sum_{n\geq 0} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right|$. We will estimate the main and regular parts separately:

$$\sum_{n\geq 0} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right| = |\tilde{A}_0| \left| (z^* - z)^{-1} - (\tilde{z}^* - z)^{-1} \right| + \sum_{n\geq 1} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right|.$$

For the main part, we obtain the estimate:

$$|\tilde{A}_0| \left| (z^* - z)^{-1} - (\tilde{z}^* - z)^{-1} \right| = |A_0| \left| \frac{\tilde{z}^* - z^*}{(z^* - z)(\tilde{z}^* - z)} \right| = |A_0| \frac{|\tilde{z}^* - z^*|}{|z^* - z|\,|\tilde{z}^* - z|} \leq |A_0| \frac{|\Delta\tilde{z}^*|}{|\tilde{z}^* - z|^2}.$$

This estimate is valid in the domain $|z| < |\tilde{z}^*|$.

Next, we estimate the regular part. Since $A_1 = \ldots = A_4 = 0$, we obtain

$$\begin{aligned}
\sum_{n\geq 1} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right| &= \sum_{n\geq 5} |\tilde{A}_n| \left| (z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1} \right| \\
&= \sum_{n\geq 5} |\tilde{A}_n| \left| (\tilde{z}^* - z - \Delta\tilde{z}^*)^{n-1} - (\tilde{z}^* - z)^{n-1} \right|.
\end{aligned}$$

The expression

$$\left|(\tilde{z}^* - z - \Delta\tilde{z}^*)^{n-1} - (\tilde{z}^* - z)^{n-1}\right|,$$

is estimated using the formula

$$\left|(a-b)^{n-1} - a^{n-1}\right| = |b|\left|(a-b)^{n-2} + (a-b)^{n-3}a + \ldots + a^{n-2}\right| \leq (n-1)|b|(|a|+|b|)^{n-2}.$$

Introduce

$$\eta = \begin{cases} |\tilde{z}^* - z|, \text{ in the domain (14)} \\ |\Delta\tilde{z}^*|, \text{ in the domain (15)} \end{cases}$$

and apply the obtained formula

$$\left|(\tilde{z}^* - z - \Delta\tilde{z}^*)^{n-1} - (\tilde{z}^* - z)^{n-1}\right| \leq (n-1)|\Delta\tilde{z}^*|(|\tilde{z}^* - z| + |\Delta\tilde{z}^*|)^{n-2},$$

then

$$\begin{aligned}
\sum_{n\geq 5}|\tilde{A}_n|\left|(z^* - z)^{n-1} - (\tilde{z}^* - z)^{n-1}\right| &\leq |\Delta\tilde{z}^*|\sum_{n\geq 5}|\tilde{A}_n|(n-1)(|\tilde{z}^* - z| + |\Delta\tilde{z}^*|)^{n-2} \\
&\leq |\Delta\tilde{z}^*|\sum_{n\geq 5}\frac{|\tilde{\gamma}_2|^{\left[\frac{n}{5}\right]}\left(|Q_0A_0|\left|\left[\frac{n-9}{2}\right]+1\right|+1\right)(n-1)}{|(n-1)(n-2)(n-3)(n-4)+24(2n-3)|}(2\eta)^{n-2} \\
&\leq |\Delta\tilde{z}^*|\frac{(|Q_0A_0|+1)}{48}\sum_{n\geq 5}|\tilde{\gamma}_2|^{\left[\frac{n}{5}\right]}(2\eta)^{n-2} \\
&= \frac{|\Delta\tilde{z}^*||\tilde{\gamma}_2|(|Q_0A_0|+1)(2\eta)^3}{48}\sum_{n\geq 0}|\tilde{\gamma}_2|^{\left[\frac{n}{5}\right]}(2\eta)^n \\
&\leq \frac{|\Delta\tilde{z}^*||\tilde{\gamma}_2|(|Q_0A_0|+1)(2\eta)^3}{48}\sum_{k\geq 0}^{4}\sum_{r=0}^{4}|\tilde{\gamma}_2|^k(2\eta)^{5k+r} \\
&\leq \frac{|\Delta\tilde{z}^*||\tilde{\gamma}_2|(|Q_0A_0|+1)(2\eta)^3}{48(1-|\tilde{\gamma}_2|(2\eta)^5)}\sum_{r=0}^{4}(2\eta)^r.
\end{aligned}$$

This estimate is valid in the domain $\eta < \frac{1}{2\sqrt[5]{|\tilde{\gamma}_2|}}$, where $\tilde{\gamma}_2 = \sup\limits_n\left(\frac{|f^{(n)}(\tilde{z}^*)|}{n!}\right)$.

To estimate the expression $\sum\limits_{n\geq 0}|\Delta\tilde{A}_n|(|\tilde{z}^* - z| + |\Delta\tilde{z}^*|)^{n-1}$, it is necessary to estimate $|\Delta\tilde{A}_n|$. For this, we will use the results from [16], as well as the relations (8), (9), and the fact that $A_0, \ldots, A_4$ do not depend on D_i, which means that the difference between the perturbed and true values is zero:

$$|\Delta\tilde{A}_n| = |A_n - \tilde{A}_n| = \left|\frac{D_{n-5} - Q_0\cdot P(A_0,\ldots,A_{n-1}) - \tilde{D}_{n-5} - Q_0\cdot P(\tilde{A}_0,\ldots,\tilde{A}_{n-1})}{(n-1)(n-2)(n-3)(n-4)+24(2n-3)}\right|,$$

$$\left|\frac{\Delta\tilde{D}_{n-5} - Q_0\cdot\Delta\tilde{P}(A_0,\ldots,A_{n-1})}{(n-1)(n-2)(n-3)(n-4)+24(2n-3)}\right| \leq \frac{|\delta\tilde{\gamma}_2||\tilde{\gamma}_2+\delta\tilde{\gamma}_2|^{\left[\frac{n}{5}\right]}\left(|Q_0A_0|\left|\left[\frac{n-9}{2}\right]+1\right|+1\right)}{|(n-1)(n-2)(n-3)(n-4)+24(2n-3)|}.$$

Using the introduced parameter η, and taking the obtained result into account, we estimate the expression $\sum\limits_{n\geq 0}|\Delta\tilde{A}_n|(|\tilde{z}^* - z| + |\Delta\tilde{z}^*|)^{n-1}$:

$$\begin{aligned}
\sum_{n\geq 0}|\Delta\tilde{A}_n|(|\tilde{z}^* - z| + |\Delta\tilde{z}^*|)^{n-1} &\leq \sum_{n\geq 5}\frac{|\delta\tilde{\gamma}_2\|\tilde{\gamma}_2+\delta\tilde{\gamma}_2|^{\left[\frac{n}{5}\right]}\left(|Q_0A_0|\right]\left[\frac{n-9}{2}\right]+1\Big|+1\Big)}{(n-1)(n-2)(n-3)(n-4)+24(2n-3)\,|}(2\eta)^{n-1} \\
&\leq \frac{|\Delta\tilde{z}^*||\delta\tilde{\gamma}_2||\tilde{\gamma}_2+\delta\tilde{\gamma}_2|(|Q_0A_0|+1)(2\eta)^4}{48(1-|\tilde{\gamma}_2+\delta\tilde{\gamma}_2|(2\eta)^5)}\sum_{r=0}^{4}(2\eta)^r.
\end{aligned}$$

This estimate holds for $\eta < \frac{1}{2\sqrt[5]{|\tilde{\gamma}_2 + \delta\tilde{\gamma}_2|}}$, where $\delta\tilde{\gamma}_2 = \Delta\tilde{z}^* \sup_n \left(\frac{\left|f^{(n)}(\tilde{z}^*)\right|}{n!} \right)$.
Finally, we obtain the estimate for $|w_2(z) - \tilde{w}_2(z)|$:

$$|w_2(z) - \tilde{w}_2(z)| \leq |A_0| \frac{|\Delta\tilde{z}^*|}{|\tilde{z}^* - z|^2} + \frac{|\Delta\tilde{z}^*| \, |\tilde{\gamma}_2| \left(|Q_0 A_0| + 1\right) \left(16\eta^4 + 8\eta^3\right)}{48\left(1 - |\tilde{\gamma}_2| \, (2\eta)^5\right)} \sum_{r=0}^{4} (2\eta)^r. \tag{17}$$

We proceed to estimate the expression $|\tilde{w}_2(z) - \tilde{w}_{2,N}(z)|$:

$$\begin{aligned} |\tilde{w}_2(z) - \tilde{w}_{2,N}(z)| &= \left| \sum_{n \geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} - \sum_{n=0}^{N} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right| \\ &= \left| \sum_{n \geq N+1} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right| \leq \sum_{n \geq N+1} \left|\tilde{A}_n\right| |\tilde{z}^* - z|^{n-1}. \end{aligned}$$

To estimate the last expression, we can use the expression (12), and then we obtain

$$|\tilde{w}_2(z) - \tilde{w}_{2,N}(z)| \leq \frac{|\tilde{\gamma}_2|^{N+1}}{1 - |\tilde{\gamma}_2| \, |\tilde{z}^* - z|^5} \sum_{k=0}^{4} \frac{\left(|Q_0 A_0| \left| \left[\frac{5(N+1)+k-9}{2}\right] + 1 \right| + 1\right) |\tilde{z}^* - z|^{5(N+1)+k-1}}{\left|\prod_{i=1}^{4} \left(5(N+1) + k - i\right) + 24\left(2\left(5(N+1) + k\right) - 3\right)\right|}. \tag{18}$$

This estimate is valid in the domain $|\tilde{z}^* - z| < \frac{1}{\sqrt[5]{|\tilde{\gamma}_2|}}$.

□

This theorem can be strengthened by estimating the expression $|w_2(z) - \tilde{w}_2(z)|$, which appears on the right-hand side of the inequality

$$\delta\tilde{w}_{2,N}(z) \leq |w_2(z) - \tilde{w}_2(z)| + |\tilde{w}_2(z) - \tilde{w}_{2,N}(z)|,$$

in a different way. Before formulating the theorem, we will prove an auxiliary lemma.

Lemma 1. *The following estimate holds for the expression* $|w_2(z) - \tilde{w}_2(z)|$*:*

$$|w_2(z) - \tilde{w}_2(z)| \leq \sup_G \left| \frac{\partial \tilde{w}_2(z)}{\partial \tilde{z}^*} \right| |\Delta\tilde{z}^*| + \sum_{n \geq 0} \sup_G \left| \frac{\partial \tilde{w}_2(z)}{\partial \tilde{A}_n} \right| \left|\Delta\tilde{A}_n\right|, \tag{19}$$

where $G = \{\tilde{z}^* : |\tilde{z}^* - z^*| \leq \Delta\tilde{z}^*\}$.

Proof. Since $w_2(z) = (z^* - z)^{-1} \sum_{n \geq 0} A_n (z^* - z)^n$, the function w_2 can be viewed as depending on the variables z^*, A_0, ..., A_k, ... Similar considerations hold for $\tilde{w}_2$. Then, we use the formula for estimating the error of a function, which can be found, for example, in the book [26] (p. 23). To apply it, we formulate the following statement.

Statement 1. Let $w_2(a_1, \ldots, a_n)$ be the exact value of the function, and $\tilde{w}_2 = w_2(\tilde{a}_1, \ldots, \tilde{a}_n)$ be the approximate value, then the error estimate has the form:

$$|w_2(a_1, \ldots, a_n) - \tilde{w}_2| \leq \sum_{i=1}^{n} \sup_G \left| \frac{\partial \tilde{w}_2(a_1, \ldots, a_n)}{\partial \tilde{a}_i} \right| |\Delta\tilde{a}_i|. \tag{20}$$

Let us use Statement 1, defining the original function w as the function $w_2(z^*, A_1, \ldots, A_k, \ldots)$, then

$$|w_2(z^*, A_1, \ldots A_k, \ldots) - \tilde{w}_2| \leq \left| \sum_{n \geq 0} A_n (z^* - z)^{n-1} - \sum_{n=0}^{N} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right|$$

$$\leq \left| \sum_{n\geq 0} A_n (z^* - z)^{n-1} - \sum_{n=0}^{N} \tilde{A}_n (\tilde{z}^* - z)^{n-1} + \sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} - \sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right|$$
$$\leq \left| \sum_{n\geq 0} A_n (z^* - z)^{n-1} - \sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right| + \left| \sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} - \sum_{n=0}^{N} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right|$$
$$\leq \sup_G \left| \frac{\partial \tilde{w}_2(z)}{\partial \tilde{z}^*} \right| |\Delta \tilde{z}^*| + \sum_{n\geq 0} \sup_G \left| \frac{\partial \tilde{w}_2(z)}{\partial \tilde{A}_n} \right| |\Delta \tilde{A}_n|.$$

□

Theorem 2. *Let $\tilde{w}_{2,N}(z)$ be the analytical approximate solution of the Cauchy problem (1) – (2) in the neighborhood of the point $\tilde{z}^*$, and suppose that conditions 1 and 2 of Theorem 1 are satisfied, as well as the condition $|\Delta \tilde{z}^*| < \frac{1}{\sqrt[5]{|\tilde{\gamma}_2 + \delta\tilde{\gamma}_2|}}$, then the error estimate for this solution is*

$$\begin{aligned}\delta \tilde{w}_{2,N}(z) \leq & \frac{|A_0| |\Delta \tilde{z}^*|}{|\tilde{z}_1^* - z|^2} + \frac{|\Delta \tilde{z}^*| |\delta\tilde{\gamma}_2| |\tilde{\gamma}_2 + \delta\tilde{\gamma}_2| (|Q_0 A_0| + 1)}{48} \\ & \times \left(\frac{4\left(1 - |\tilde{\gamma}_2 + \delta\tilde{\gamma}_2| |\tilde{z}_2^* - z|^5\right) + 5 |\tilde{\gamma}_2 + \delta\tilde{\gamma}_2| |\tilde{z}_2^* - z|}{\left(1 - |\tilde{\gamma}_2 + \delta\tilde{\gamma}_2| |\tilde{z}_2^* - z|^5\right)^2} \sum_{r=0}^{4} |\tilde{z}_2^* - z|^{r+3} \right. \\ & \left. + \frac{|\delta\tilde{\gamma}_2| |\tilde{\gamma}_2 + \delta\tilde{\gamma}_2| (|Q_0 A_0| + 1) |\tilde{z}_2^* - z|^4}{48 \left(1 - |\tilde{\gamma}_2 + \delta\tilde{\gamma}_2| |\tilde{z}_2^* - z|^5\right)} \sum_{r=1}^{4} r |\tilde{z}_2^* - z|^{r-1} \right) \\ & + \frac{|\tilde{\gamma}_2|^{N+1}}{1 - |\tilde{\gamma}_2| |\tilde{z}^* - z|^5} \sum_{k=0}^{4} \frac{\left(|Q_0 A_0| \left| \left[\frac{5(N+1)+k-9}{2}\right] + 1 \right| + 1\right) |\tilde{z}^* - z|^{5(N+1)+k-1}}{\left| \prod_{i=1}^{4} (5(N+1)+k-i) + 24(2(5(N+1)+k) - 3) \right|},\end{aligned}$$

and is valid in the region $\{z : |z| \leq |\tilde{z}_1^*|\} \cap \left\{ z : |\tilde{z}_2^* - z| \leq \frac{1}{\sqrt[5]{|\tilde{\gamma}_2 + \delta\tilde{\gamma}_2|}} \right\} \cap \left\{ z : |\tilde{z}^* - z| \leq \frac{1}{\sqrt[5]{|\tilde{\gamma}_2 + \delta\tilde{\gamma}_2|}} \right\}$, *where* $G = \{\tilde{z}^* : |\tilde{z}^* - z^*| \leq \Delta \tilde{z}^*\}$, $\tilde{z}_1^*, \tilde{z}_2^* \in \partial G$.

Proof. Let us use Lemma 1 and estimate the expression $\sup_G \left| \frac{\partial \tilde{w}_2(z)}{\partial \tilde{z}^*} \right| |\Delta \tilde{z}^*|$:

$$\begin{aligned}\sup_G \left| \frac{\partial \tilde{w}_2(z)}{\partial \tilde{z}^*} \right| |\Delta \tilde{z}^*| &= |\Delta \tilde{z}^*| \sup_G \left| \frac{\partial \left(\sum_{n\geq 0} \tilde{A}_n (\tilde{z}^* - z)^{n-1} \right)}{\partial \tilde{z}^*} \right|_G \\ &= |\Delta \tilde{z}^*| \sup_G \left(\left| \sum_{n\geq 0} \tilde{A}_n (n-1) (\tilde{z}^* - z)^{n-2} \right| \right) \\ &= |\Delta \tilde{z}^*| \sup_G \left(\left| -\tilde{A}_0 (\tilde{z}^* - z)^{-2} + \sum_{n\geq 5} \tilde{A}_n (n-1) (\tilde{z}^* - z)^{n-2} \right| \right) \\ &\leq |\Delta \tilde{z}^*| \left(\tilde{A}_0 \sup_G |\tilde{z}^* - z|^{-2} + \sum_{n\geq 5} (n-1) \sup_G (|\tilde{A}_n|) \sup_G \left(|\tilde{z}^* - z|^{n-2} \right). \right)\end{aligned}$$

Consider each supremum separately. Since $A_n = A_n(z^*)$, by the maximum principle, we get:

$$\sup_G (|\tilde{A}_n|) = \sup_{\tilde{z}^* \in \partial G} (|\tilde{A}_n|) = \frac{|\tilde{\gamma}_2 + \delta\tilde{\gamma}_2|^{\left[\frac{n}{5}\right]} \left(|Q_0 A_0| \left| \left[\frac{n-9}{2}\right] + 1 \right| + 1\right)}{|(n-1)(n-2)(n-3)(n-4) + 24(2n-3)|} = \nu_n. \quad (21)$$

To find $\sup\limits_{G}\left(|\tilde{z}^{*}-z|^{n-2}\right)$ for $n \geq 5$ and $\sup\limits_{G}|\tilde{z}^{*}-z|^{-2}$, we will also apply the maximum principle for a holomorphic function $\phi(\tilde{z}^{*})=(\tilde{z}^{*}-z)^{n-2}$ in the region:

$$\sup_{G}\left(|\tilde{z}^{*}-z|^{n-2}\right)=\sup_{\tilde{z}^{*}\in\partial G}\left(|\tilde{z}^{*}-z|^{n-2}\right)=|\tilde{z}_{2}^{*}-z|^{n-2}, \tag{22}$$

and

$$\sup_{G}|\tilde{z}^{*}-z|^{-2}=|\tilde{z}_{1}^{*}-z|^{-2}, \text{where } z_{1},z_{2}\in\partial G. \tag{23}$$

Given the expressions (21), (22), (23), we obtain the estimate

$$\sup_{G}\left|\frac{\partial\tilde{w}_{2}(z)}{\partial\tilde{z}^{*}}\right||\Delta\tilde{z}^{*}|\leq\frac{|A_{0}|\,|\Delta\tilde{z}^{*}|}{|\tilde{z}_{1}^{*}-z|^{2}}+|\Delta\tilde{z}^{*}|\sum_{n\geq5}(n-1)\,\nu_{n}\,|\tilde{z}_{2}^{*}-z|^{n-2}.$$

We estimate the second term on the right-hand side of the inequality:

$$\begin{aligned}
|\Delta\tilde{z}^{*}|\sum_{n\geq5}(n-1)\,\nu_{n}\,|\tilde{z}_{2}^{*}-z|^{n-2}&=|\Delta\tilde{z}^{*}|\left(\sum_{n\geq5}\nu_{n}\,|\tilde{z}_{2}^{*}-z|^{n-1}\right)'_{\tilde{z}_{2}^{*}}\\
&\leq|\Delta\tilde{z}^{*}|\left(\sum_{n\geq5}\frac{|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|^{\left[\frac{n}{5}\right]}\left(|Q_{0}A_{0}|\left|\left[\frac{n-9}{2}\right]+1\right|+1\right)}{|(n-1)(n-2)(n-3)(n-4)+24(2n-3)|}\,|\tilde{z}_{2}^{*}-z|^{n-1}\right)'_{\tilde{z}_{2}^{*}}\\
&\leq|\Delta\tilde{z}^{*}|\left(\frac{|\delta\tilde{\gamma}_{2}|\,|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,(|Q_{0}A_{0}|+1)\,|\tilde{z}_{2}^{*}-z|^{4}}{48\left(1-|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|^{5}\right)}\sum_{r=0}^{4}|\tilde{z}_{2}^{*}-z|^{r}\right)'_{\tilde{z}_{2}^{*}}\\
&\leq\frac{|\Delta\tilde{z}^{*}|\,|\delta\tilde{\gamma}_{2}|\,|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,(|Q_{0}A_{0}|+1)}{48}\\
&\quad\times\left(\frac{4\left(1-|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|^{5}\right)+5\,|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|}{\left(1-|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|^{5}\right)^{2}}\sum_{r=0}^{4}|\tilde{z}_{2}^{*}-z|^{r+3}\right.\\
&\quad\left.+\frac{|\delta\tilde{\gamma}_{2}|\,|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,(|Q_{0}A_{0}|+1)\,|\tilde{z}_{2}^{*}-z|^{4}}{48\left(1-|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|^{5}\right)}\sum_{r=1}^{4}r\,|\tilde{z}_{2}^{*}-z|^{r-1}\right).
\end{aligned}$$

Finally, we obtain the estimate

$$\begin{aligned}
\sup_{G}\left|\frac{\partial\tilde{w}_{2}(z)}{\partial\tilde{z}^{*}}\right||\Delta\tilde{z}^{*}|\leq&\frac{|A_{0}|\,|\Delta\tilde{z}^{*}|}{|\tilde{z}_{1}^{*}-z|^{2}}+\frac{|\Delta\tilde{z}^{*}|\,|\delta\tilde{\gamma}_{2}|\,|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,(|Q_{0}A_{0}|+1)}{48}\\
&\times\left(\frac{4\left(1-|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|^{5}\right)+5\,|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|}{\left(1-|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|^{5}\right)^{2}}\sum_{r=0}^{4}|\tilde{z}_{2}^{*}-z|^{r+3}\right.\\
&\left.+\frac{|\delta\tilde{\gamma}_{2}|\,|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,(|Q_{0}A_{0}|+1)\,|\tilde{z}_{2}^{*}-z|^{4}}{48\left(1-|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|\,|\tilde{z}_{2}^{*}-z|^{5}\right)}\sum_{r=1}^{4}r\,|\tilde{z}_{2}^{*}-z|^{r-1}\right).
\end{aligned}$$

This estimate is valid in the region $|\tilde{z}_{2}^{*}-z|<\frac{1}{\sqrt[5]{|\tilde{\gamma}_{2}+\delta\tilde{\gamma}_{2}|}}$.

We proceed to the estimation $\sum\limits_{n\geq 0}\sup\limits_{G}\left|\frac{\partial\tilde{w}_2(z)}{\partial\tilde{A}_n}\right|\left|\Delta\tilde{A}_n\right|$, for this, we will find $\sup\limits_{G}\left|\frac{\partial\tilde{w}_2(z)}{\partial\tilde{A}_n}\right|$:

$$\sup_{G}\left|\frac{\partial\tilde{w}_2(z)}{\partial\tilde{A}_n}\right| = \sup_{G}\left(\left|\tilde{z}^*-z\right|^{n-1}\right) = \begin{cases} \left|\tilde{z}_2^*-z\right|^{n-1}, \ if\ n\geq 5, \\ \left|\tilde{z}_1^*-z\right|^{-1}, \ if\ n=0. \end{cases} \tag{24}$$

Taking into account (24), we get

$$\sum_{n\geq 0}\sup_{G}\left|\frac{\partial\tilde{w}_2(z)}{\partial\tilde{A}_n}\right|\left|\Delta\tilde{A}_n\right| = \sum_{n\geq 5}\left|\tilde{z}_2^*-z\right|^{n-1}\left|\Delta\tilde{A}_n\right| \leq \frac{\left|\Delta\tilde{z}^*\right|\left|\delta\tilde{\gamma}_2\right|\left|\tilde{\gamma}_2+\delta\tilde{\gamma}_2\right|\left(\left|Q_0A_0\right|+1\right)\left|\tilde{z}_2^*-z\right|^4}{48\left(1-\left|\tilde{\gamma}_2+\delta\tilde{\gamma}_2\right|\left|\tilde{z}_2^*-z\right|^5\right)}\sum_{r=0}^{4}\left|\tilde{z}_2^*-z\right|^r.$$

This estimate is valid in the region $\left|\tilde{z}_2^*-z\right| < \frac{1}{\sqrt[5]{\left|\tilde{\gamma}_2+\delta\tilde{\gamma}_2\right|}}$.

The estimate for the expression $\left|\tilde{w}_2(z)-\tilde{w}_{2,N}(z)\right|$ has already been obtained earlier, so we will use the previously derived results, which completes the proof of Theorem 2.

□

4. Numerical experiment

Consider the Cauchy problem (1) – (2) with the parameters $Q_0=-1$, $f(z)=z$, $z_0=0$, and the initial conditions $w(0)=0$, $w'(0)=1$, $w''(0)=0$, $w'''(0)=0$.

As noted in the section 3 , the exact value of the singular point z^* cannot, in general, be obtained analytically. Therefore, employing the procedure for the computation of an approximate singular point with prescribed accuracy (see, e.g., [15,26]), we obtain $z^*=3.44408623$. For the purposes of numerical experiments, this value is treated as exact, and we introduce the approximation $\tilde{z}^*=3.44$. Then the following estimate holds:

$$\left|\Delta\tilde{z}^*\right| \lesssim 41\cdot 10^{-4}.$$

Since $f(z)=z$, its Taylor expansion at $\tilde{z}^*$ reduces to the linear representation

$$f(z)=z=\sum_{n=0}^{\infty}\frac{f^{(n)}(\tilde{z}^*)}{n!}(\tilde{z}^*-z)^n = f(\tilde{z}^*)+f'(\tilde{z}^*)(\tilde{z}^*-z) = -\tilde{z}^*-(\tilde{z}^*-z).$$

Furthermore, we have

$$\tilde{\gamma}_2=\sup_{n\in\mathbb{N}_0}(\tilde{z}^*,1,0)=3.44, \qquad \delta\tilde{\gamma}_2=\left|\Delta\tilde{z}^*\right|\cdot 3.44\lesssim 0.015.$$

The corresponding radii of convergence are given by

$$\rho_{\text{th1}}=\min\left\{1,\frac{1}{\sqrt[5]{\tilde{\gamma}_2}},\frac{1}{2\sqrt[5]{\tilde{\gamma}_2+\delta\tilde{\gamma}_2}}\right\}\approx 0.39019 \quad \text{(Theorem 1)},$$

$$\rho_{\text{th2}}=\min\left\{1,\frac{1}{\sqrt[5]{\tilde{\gamma}_2}},\frac{1}{\sqrt[5]{\tilde{\gamma}_2+\delta\tilde{\gamma}_2}}\right\}\approx 0.78038 \quad \text{(Theorem 2)}.$$

Thus, within the disk centered at $\tilde{z}^*$ with radius ρ_{th1}, both Theorem 1 and Theorem 2 are applicable, whereas on the set

$$D_z=\left\{z\,|\,\rho_{\text{th1}}<\left|z^*-z\right|<\rho_{\text{th2}}\right\},$$

only Theorem 2 can be used.

In the first case, we consider the point $z=3.1$, and in the second case $z=3$. Tables 1 and 2 present the accuracy of the solution and the corresponding number of terms obtained from Theorems 1 and 2, based on the respective a priori error estimates.

The dependence of the error estimate $\delta\tilde{w}_{2,N}(3.05)$ on the number of retained terms N is shown in Table 1. For $z=3.05$, the estimate remains constant up to five decimal places for all $N\geq 1$, taking the value 0.09275. This behavior is explained by the structure of the error bound in Theorem 1. The dominant contribution is given by the term $I_1 = |A_0||\Delta\tilde{z}^*|/|\tilde{z}^* - z|^2$, which does not depend on N and equals approximately 0.09273. The remaining terms I_2 and $I_3(N)$ are several orders of magnitude smaller already for $N = 1$ (specifically, $I_2 \sim 2 \cdot 10^{-5}$ and $I_3(1) \sim 2 \cdot 10^{-10}$). Consequently, increasing N beyond 1 does not lead to any visible improvement of the estimate within the displayed precision.

Table 1. Error estimate $\delta\tilde{w}_{2,N}(3.05)$ by Theorem 1 and 2 for $z = 3.05$

N	$\delta\tilde{w}_{2,N}(3.05)$
1	0.09275
2	0.09275
3	0.09275
4	0.09275
5	0.09275
6	0.09275
7	0.09275
8	0.09275
9	0.09275
10	0.09275

It is important to emphasise that the observed plateau is not a fundamental lower bound imposed by the uncertainty $|\Delta\tilde{z}^*|$ of the singular point. The value $|\Delta\tilde{z}^*| \approx 4.1 \cdot 10^{-3}$ is not a limiting threshold for the error estimate; indeed, for a larger distance $|\tilde{z}^* - z| = 0.44$ (i.e., $z = 3$), the same theorem yields a decreasing sequence of estimates, reaching values as low as $1.69 \cdot 10^{-4}$, which is considerably smaller than $|\Delta\tilde{z}^*|$. The plateau at $z = 3.05$ therefore arises merely from the specific geometric configuration, namely the relatively small distance $|\tilde{z}^* - z| = 0.39$, which makes I_1 large enough to dominate the bound completely. In contrast, when $|\tilde{z}^* - z|$ is sufficiently large (e.g., 0.44), the term I_1 becomes smaller, and the N-dependent part $I_3(N)$ can significantly reduce the total error estimate as N increases.

Thus, the qualitative behaviour of the error bound – whether it improves with N or stagnates – is determined by the interplay between the distance from the evaluation point to the singular point and the magnitude of the input data uncertainty. No fundamental obstruction prevents the error estimate from falling below $|\Delta\tilde{z}^*|$; the only relevant factors are the specific numerical values of the geometric parameters.

Table 2. Error estimate $\delta\tilde{w}_{2,N}(3)$ by Theorem 2 for $z = 3$

N	$\delta\tilde{w}_{2,N}(3)$
1	2.31×10^{-2}
2	8.74×10^{-3}
3	3.27×10^{-3}
4	1.61×10^{-3}
5	1.08×10^{-3}
6	7.42×10^{-4}
7	5.11×10^{-4}
8	3.53×10^{-4}
9	2.44×10^{-4}
10	1.69×10^{-4}

Error estimate $\delta\tilde{w}_{2,N}(3)$ by Theorem 2 for $z = 3$ and $|\Delta\tilde{z}^*| = 0.0041$. The estimate decreases monotonically with increasing N, from 2.31×10^{-2} at $N = 1$ to 1.69×10^{-4} at $N = 10$. This behavior is due to the N-dependent term $I_3(N)$, which decays rapidly as N grows, while the N-independent contributions I_1 and I_2 remain constant. The achieved accuracy at $N = 10$ is significantly smaller than the input uncertainty $|\Delta\tilde{z}^*| \approx 4.1 \times 10^{-3}$, demonstrating that the method can refine the solution beyond the precision of the original data.

References

[1] Golubev, V. (1950). *Lectures on Analytical Theory of Differential Equations*, Gostekhteorizdat, Moscow (1950).

[2] Zaitsev, V. F., & Polyanin, A. D. (2002). *Handbook of Exact Solutions for Ordinary Differential Equations.* Chapman and Hall/CRC.

[3] Yefimova, O. Y., & Kudryashov, N. A. (2004). Exact solutions of the Burgers-Huxley equation. *Journal of Applied Mathematics and Mechanics, 68*(3), 413-420.

[4] Awonusika, R. O. (2024). Analytical solution of a class of Lane–Emden equations: Adomian decomposition method. *The Journal of Analysis, 32*(2), 1009-1056.

[5] Ansari, S., & Malik, M. (2023). Approximation of solutions to abstract neutral differential equations with non-instantaneous impulses. *The Journal of Analysis, 31*(4), 3045-3068.

[6] Kumar, S., Rohila, R., & Chadha, A. (2025). Numerical solutions of the sine-Gordon equation by univariate cubic B-spline quasi interpolation method. *Journal of Analysis, 33*, 1525-1545.

[7] Chandrasekar, V. K., Senthilvelan, M., & Lakshmanan, M. (2005). On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 461*(2060), 2451-2477.

[8] Eleonskii, V. M., Kulagin, N. E., Novozhilova, N. S., & Silin, V. P. (1984). Asymptotic expansions and qualitative analysis of finite-dimensional models in nonlinear field theory. *Teoreticheskaya I Matematicheskaya Fizika, 60*(3), 395-403.

[9] Khodzhaev, K. S., & Shatalov, S. D. (1980). On the qualitative investigation of motions using asymptotic methods of nonlinear mechanics. *Journal of Applied Mathematics and Mechanics, 44*(5), 568-573.

[10] Dell'Atti, M., & Kecker, T. (2025). Geometric approach for the identification of Hamiltonian systems of quasi-Painlevé type. *Journal of Physics A: Mathematical and Theoretical, 58*(9), 095202.

[11] Orlov, V., & Chichurin, A. (2023). About analytical approximate solutions of the Van der Pol equation in the complex domain. *Fractal and Fractional, 7*(3), 228.

[12] Orlov, V., & Chichurin, A. (2023). The influence of the perturbation of the initial data on the analytic approximate solution of the Van der Pol equation in the complex domain. *Symmetry, 15*(6), 1200.

[13] Orlov, V. (2023). Moving singular points and the Van der Pol equation, as well as the uniqueness of its solution. *Mathematics, 11*(4), 873.

[14] Gasanov, M. V. (2025). An Approximate Analytical Solution for a Class of Fourth-Order Nonlinear Differential Equations in the Domain of Analyticity. *Russian Journal of Nonlinear Dynamics, 21*(2), 249-258.

[15] Gasanov, M., & Gulkanov, A. (2025). Investigation of a strongly nonlinear oscillator with movable algebraic singularities. *Journal of Mathematical Sciences*, 1-14.

[16] Gasanov, M. (2025). Investigation of a fourth-order nonlinear differential equation with moving singular points of algebraic type. *European Journal of Pure and Applied Mathematics, 18*(1), 5564-5564.

[17] Gasanov, M. (2025). Investigation of a mathematical model from the theory of viscous fluids. *Gulf Journal of Mathematics, 20*(2), 288-302.

[18] Kazakov, A. L., & Kuznetsov, P. A. (2020). Exact solutions of the nonlinear heat conduction model. Vestnik Yuzhno-Ural'skogo Universiteta. *Seriya Matematicheskoe Modelirovanie I Programmirovanie, 13*(4), 33-47.

[19] Orlov, V. (2024). Singular Cauchy Problem for a Nonlinear Fractional Differential Equation. *Mathematics, 12*(22), 3629.

[20] Orlov, V., & Gasanov, M. (2022). The influence of a perturbation of a moving singular point on the structure of an analytical approximate solution of a class of third-order nonlinear differential equations in a complex domain, Herald of the Bauman Moscow State Tech. Univ. *Natural Science, 6*(105), 60-76.

[21] Orlov, V., & Gasanov, M. (2022). Exact criteria for the existence of a moving singular point in a complex domain for a nonlinear differential third-degree equation with a polynomial seventh-degree right-hand side. *Axioms, 11*(5), 222.

[22] Orlov, V., & Gasanov, M. (2022). Existence and uniqueness theorem for a solution to a class of a third-order nonlinear differential equation in the domain of analyticity. *Axioms, 11*(5), 203.

[23] Orlov, V., & Gasanov, M. (2023). The maximum domain for an analytical approximate solution to a nonlinear differential equation in the neighborhood of a moving singular point. *Axioms, 12*(9), 844.

[24] Orlov, V., & Gasanov, M. (2022). Analytic approximate solution in the neighborhood of a moving singular point of a class of nonlinear equations. *Axioms, 11*(11), 637.

[25] Orlov, V., & Gasanov, M. (2022). Technology for obtaining the approximate value of moving singular points for a class of nonlinear differential equations in a complex domain. *Mathematics, 10*(21), 3984.

[26] Bakhvalov, N. S. (1977). *Numerical Methods: Analysis, Algebra, Ordinary Differential Equations.*