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The Z_6 transformation: A novel method for solving differential equations

Hong Lai Zhu

School of Physics and Electronic Information, Huaibei Normal University, Anhui 235000, China; honglaizhu@qq.com

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Abstract: This paper introduces the Z_6 transformation for the first time as a constructive framework for generating and solving differential equations. Its core idea is to construct a new solvable equation from two known solvable equations through a predefined transformation between the dependent and independent variables. This framework unifies two classic techniques-coordinate transformation and substitution of the dependent variable-into a single, systematic procedure. Furthermore, by iterating this construction process and combining it with the principle of common solutions proposed in this paper, an infinite number of solvable equations can be generated. Using this framework, we have obtained, for the first time, general solutions for a variety of differential equations, including general solutions for the Laplace equation in cylindrical and spherical coordinates, as well as general solutions for several classes of linear and nonlinear second-order partial differential equations with varying coefficients, among others. We also solved two boundary value problems exactly, demonstrating the practicality of this framework. The Z_6 transformation, together with the Z_A method we previously proposed, provides two complementary approaches for systematically expanding the range of solvable differential equations, offering powerful tools for both theoretical analysis and practical applications.

Keywords: Z_6 transformation, constructive framework, explicit analytical solutions, generation of solvable equations, principle of common solutions

MSC: 34A25, 34A30, 34A34, 35A09, 35A25.

1. Introduction

Since their inception, differential equations have remained a key area of mathematical research, as they possess not only significant theoretical value but also immense practical value [1–6].

In the early stages of research in this field, mathematicians generally sought exact analytical solutions. However, as they encountered a bottleneck, qualitative theory and numerical analysis gradually became the mainstream [7–9]. Nevertheless, the almost perfect properties of analytical solutions-especially general solutions-shine as brightly as stars. Although the path was once fraught with difficulties, we have published a series of papers since 2022, bringing renewed attention to research on general solutions [10–16].

In a previous paper, we proposed the Z_A method for solving differential equations [15]. This method demonstrates that any solvable equation can serve as a source equation; not only does it have an infinite number of corresponding sub-equations, but these equations are all theoretically solvable. Numerous examples demonstrate that the Z_A method can solve not only ordinary differential equations but also partial differential equations; not only linear equations but also nonlinear equations; and general or analytical solutions for a wide range of equations have been obtained for the first time.

In the research methods for ordinary differential equations, transformations of the dependent variable are frequently used. In the study of partial differential equations, transformations of the independent variables are frequently employed. Through comprehensive and in-depth research, we have discovered that these two methods can be integrated into a new approach-the Z_6 transformation. Its effectiveness is comparable to that of the Z_A method; not only can it effectively solve many types of partial differential equations and ordinary differential equations, but it can also generate an infinite number of solvable equations.

In §2, we will introduce the Z_6 transformation for partial differential equations and its two corollaries. In §3, we will use them to solve some typical partial differential equations and demonstrate their effectiveness. §4 introduces the Z_6 transformation for ordinary differential equations and its applications. In §5, we present the principle of common solutions and its preliminary applications, and in §6, we summarize this paper.

2. Z_6 transformations of partial differential equations

First, we introduce the Z_6 transformation of partial differential equations.

Z_6 transformations. In the domain D, $(D \subseteq \mathbb{R}^n)$, if the solution $v = v(y_1, \ldots y_n)$ of a PDE $G(y_1, \ldots y_n, v, v_{y_1}, \ldots v_{y_n}, v_{y_1 y_2}, \ldots) = 0$ is known, set x_i are independent and y_i are independent too, (i=1,2,…n), $x_i = x_i(y_1, y_2, \ldots y_n), y_i = y_i(x_1, x_2 \ldots x_n)$, which are known, and $v = v(x_1, \ldots x_n) = h(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots)$, then the solution of $G(y_1, \ldots y_n, v, v_{y_1}, \ldots v_{y_n}, v_{y_1 y_2}, \ldots) = F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$ is the solution of $h(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = v(x_1, \ldots x_n)$.

If the Z_6 transformation is used to solve partial differential equations, the equation we need to solve is

$$F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0. \tag{1}$$

Since $x_i = x_i(y_1, y_2, \ldots y_n)$ and $y_i = y_i(x_1, x_2 \ldots x_n)$ are known, using the given transformation

$$v(x_1, \ldots x_n) = h(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots). \tag{2}$$

Eq. (1) can be transformed into a known solvable equation $G(y_1, \ldots y_n, v, v_{y_1}, \ldots v_{y_n}, v_{y_1 y_2}, \ldots) = 0$, namely

$$G(y_1, \ldots y_n, v, v_{y_1}, \ldots v_{y_n}, v_{y_1 y_2}, \ldots) = F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0. \tag{3}$$

Using (2), if $u(x_1, \ldots x_n)$ can be solved, that solution is also the solution to Eq. (1).

To construct a new solvable equation using the Z_6 transformation, two known solvable source equations are required

$$G(y_1, \ldots y_n, v, v_{y_1}, \ldots v_{y_n}, v_{y_1 y_2}, \ldots) = 0, \tag{4}$$

$$h(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = v(x_1, \ldots x_n), \tag{5}$$

where Eq. (4) is explicitly solvable and is referred to as the first source equation, while Eq. (5) is a non-homogeneous equation and is referred to as the second source equation. Through appropriate transformations, a new solvable equation is derived, which is Eq. (1)

$$F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0.$$

Moreover, the solution to Eq. (5) is the solution to the Eq. (1).

Any explicitly solvable equation can serve as the first source equation, and any solvable non-homogeneous equation can serve as the second source equation. If Eq. (1) is explicitly solvable or a non-homogeneous equation, it can be used as a new source equation to derive new solvable equations.

There are two special cases of the Z_6 transformation, namely Corollaries 1 and 2.

Corollary 1. *In the domain D, $(D \subseteq \mathbb{R}^n)$, $(n \geq 2)$, if x_i are independent and y_i are independent too, $(i = 1, 2, \ldots n)$ by $x_i = x_i(y_1, y_2, \ldots y_n), y_i = y_i(x_1, x_2 \ldots x_n)$, which are known, a mth-order PDE $G(y_1, \ldots y_n, u, u_{y_1}, \ldots u_{y_n}, u_{y_1 y_2}, \ldots) = 0$ is converted into another mth-order PDE $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$, and the solution $u = u(y_1, \ldots y_n)$ of $G(y_1, \ldots y_n, u, u_{y_1}, \ldots u_{y_n}, u_{y_1 y_2}, \ldots) = 0$ is known, then the solution of $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$ is $u = u(y_1, \ldots y_n) = u(x_1, \ldots x_n)$.*

Using $x_i = x_i(y_1, \ldots y_n)$ and $y_i = y_i(x_1, \ldots x_n)$, $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots) = 0$ could be converted into $G(y_1, \ldots y_n, u, u_{y_1}, \ldots u_{y_n}, u_{y_1 y_2}, \ldots) = 0$. Because the solution $u = u(y_1, \ldots y_n)$ of $G = 0$ is known, by $y_i = y_i(x_1, \ldots x_n), u = g(y_1, \ldots y_n)$ may be varied into $u = u(x_1, \ldots x_n)$, and it is the solution of $F = 0$.

Corollary 2. *In the domain D, $(D \subseteq \mathbb{R}^n)$, if the solution $v = v(x_1, \ldots x_n)$ of a PDE $G(x_1, \ldots x_n, v, v_{x_1}, \ldots v_{x_n}, v_{x_1 x_2}, \ldots) = 0$ is known, set $v = h(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots)$, then the* *solution of* $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = G(x_1, \ldots x_n, v, v_{x_1}, \ldots v_{x_n}, v_{x_1x_2}, \ldots) = 0$ *is the solution of* $h(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = v(x_1, \ldots x_n)$.

3. Applications of the Z_6 transformation in partial differential equations

Many mathematical physics equations under non-Cartesian coordinate system have been investigated deeply [17–22]. If a solution to a partial differential equation has already been obtained in one coordinate system, Corollary 1 can be used to directly obtain its solution in any coordinate system. Such as the Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ in spherical coordinates and cylindrical coordinates are written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0,$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

In cylindrical coordinates,

$$x = rcos\theta, y = rsin\theta, z = z. \tag{6}$$

In spherical coordinates,

$$x = rsin\theta cos\varphi, y = rsin\theta sin\varphi, z = rcos\theta. \tag{7}$$

The basic general solution of Laplace equation in the Cartesian coordinate system is [10]

$$u = f_1\left(x\sqrt{-k_1^2 - k_2^2} + k_1 y + k_2 z + k_3\right) + f_2\left(-x\sqrt{-k_4^2 - k_5^2} + k_4 y + k_5 z + k_6\right) + k_7 x + k_8 y + k_9 z + k_{10}. \tag{8}$$

We need to interpret (8) in the complex plane; to obtain real solutions, choose appropriate complex constants so that the overall result is real. Based on Corollary 1, we can directly obtain the general solutions of the Laplace equation in cylindrical and spherical coordinates, namely Theorems 1 and 2.

Theorem 1. *In* $\mathbb{R}^3$,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} = 0, (r > 0). \tag{9}$$

The basic general solution of Eq. (9) is

$$\begin{aligned} u(r,\theta,z) = {} & f_1\left(rcos\theta\sqrt{-k_1^2 - k_2^2} + k_1 rsin\theta + k_2 z + k_3\right) \\ & + f_2\left(-rcos\theta\sqrt{-k_4^2 - k_5^2} + k_4 rsin\theta + k_5 z + k_6\right) + k_7 rcos\theta + k_8 rsin\theta + k_9 z + k_{10}, \end{aligned} \tag{10}$$

where f_1 and f_2 are arbitrary second differentiable functions, $k_1 - k_{10}$ are arbitrary constants.

Theorem 2. *In* $\mathbb{R}^3$,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0, \qquad (r > 0,\ \sin\theta \neq 0). \tag{11}$$

The basic general solution of Eq. (11) is

$$\begin{aligned} u(r,\theta,\varphi) = {} & f_1\left(rsin\theta cos\varphi\sqrt{-k_1^2 - k_2^2} + k_1 rsin\theta sin\varphi + k_2 rcos\theta + k_3\right) \\ & + f_2\left(-rsin\theta cos\varphi\sqrt{-k_4^2 - k_5^2} + k_4 rsin\theta sin\varphi + k_5 rcos\theta + k_6\right) \\ & + k_7 rsin\theta cos\varphi + k_8 rsin\theta sin\varphi + k_9 rcos\theta + k_{10}, \end{aligned} \tag{12}$$

where f_1 and f_2 are arbitrary second differentiable functions, $k_1 - k_{10}$ are arbitrary constants.

Next, we use Corollary 2 to prove Theorems 3 and 4.

Theorem 3. *In $\mathbb{R}^3$, the general solution of*

$$u_{xx} + u_{xy} + u_{xz} + b\left(x,y,z\right)\left(u_x + u_y + u_z\right) = c\left(x,y,z\right), \tag{13}$$

is

$$u = \phi\left(p,q\right) + \frac{\int v\left(p,q,r\right)dr}{c_{31} + c_{32} + c_{33}}, \tag{14}$$

and

$$v(x,y,z) = \mathrm{e}^{-\int b(x,y,z)\,dx}\left[\varphi(y,z) + \int c(x,y,z)\,\mathrm{e}^{\int b(x,y,z)\,dx}\,dx\right]. \tag{15}$$

where $b(x,y,z)$ and $c(x,y,z)$ are arbitrary known functions, $\phi(p,q)$ and $\varphi(y,z)$ are arbitrary functions. p,q, and r are mutually independent variables satisfying

$$p = -\left(c_{12} + c_{13}\right)x + c_{12}y + c_{13}z,$$

$$q = -\left(c_{22} + c_{23}\right)x + c_{22}y + c_{23}z,$$

$$r = c_{31}x + c_{32}y + c_{33}z,$$

$$\frac{\partial\left(p,q,r\right)}{\partial\left(x,y,z\right)} \neq 0,$$

$$c_{31} + c_{32} + c_{33} \neq 0.$$

Proof. The general solution of

$$v_x + b\left(x,y,z\right)v = c\left(x,y,z\right), \tag{16}$$

is

$$v(x,y,z) = \mathrm{e}^{-\int b(x,y,z)\,dx}\left[\varphi(y,z) + \int c(x,y,z)\mathrm{e}^{\int b(x,y,z)\,dx}\,dx\right].$$

where $\phi(y,z)$ is an arbitrary function [23]. By Corollary 2, set

$$u_x + u_y + u_z = v(x,y,z). \tag{17}$$

The general solution of (17) is [11]

$$u = \phi(p,q) + \frac{\displaystyle\int v(p,q,r)\,dr}{c_{31} + c_{32} + c_{33}},$$

where

$$\begin{aligned} p &= -(c_{12} + c_{13})x + c_{12}y + c_{13}z, \\ q &= -(c_{22} + c_{23})x + c_{22}y + c_{23}z, \\ r &= c_{31}x + c_{32}y + c_{33}z, \\ &\frac{\partial(p,q,r)}{\partial(x,y,z)} \neq 0, \\ &c_{31} + c_{32} + c_{33} \neq 0. \end{aligned}$$

We get

$$x = \frac{1}{\Delta}\Big[(c_{22}c_{33} - c_{23}c_{32})p - (c_{12}c_{33} - c_{13}c_{32})q + (c_{12}c_{23} - c_{13}c_{22})r\Big],$$

$$y = \frac{1}{\Delta}\Big[\big((c_{22} + c_{23})c_{33} + c_{23}c_{31}\big)p - \big((c_{12} + c_{13})c_{33} + c_{13}c_{31}\big)q + (c_{12}c_{23} - c_{13}c_{22})r\Big],$$

$$z=\frac{1}{\Delta}\Big[-\big((c_{22}+c_{23})c_{32}+c_{22}c_{31}\big)p+\big((c_{12}+c_{13})c_{32}+c_{12}c_{31}\big)q+(c_{12}c_{23}-c_{13}c_{22})r\Big],$$

where

$$\Delta=-(c_{12}+c_{13})(c_{22}c_{33}-c_{23}c_{32})-c_{12}\big[-(c_{22}+c_{23})c_{33}-c_{23}c_{31}\big]+c_{13}\big[-(c_{22}+c_{23})c_{32}-c_{22}c_{31}\big]\neq 0.$$

Then

$$v_x=u_{xx}+u_{xy}+u_{xz}.$$

$$v_x+b(x,y,z)v=u_{xx}+u_{xy}+u_{xz}+b(x,y,z)(u_x+u_y+u_z)=c(x,y,z). \tag{18}$$

By Corollary 2, the general solution to (13) is

$$u=\phi(p,q)+\frac{\int v(p,q,r)\,dr}{c_{31}+c_{32}+c_{33}},$$

where

$$v(x,y,z)=\mathrm{e}^{-\int b(x,y,z)\,dx}\left[\varphi(y,z)+\int c(x,y,z)\mathrm{e}^{\int b(x,y,z)\,dx}\,dx\right].$$

So the theorem is proved. □

Theorem 4. *In $\mathbb{R}^3$, the general solution of*

$$u_{xx}+u_{xy}+u_{xz}+A(x,y,z)(u_x+u_y+u_z)^n=0 \tag{19}$$

is

$$u=\phi(p,q)+\frac{\int v(p,q,r)\,dr}{c_{31}+c_{32}+c_{33}}, \tag{20}$$

and

$$v(x,y,z)=\left[(1-n)\left(\varphi(y,z)-\int A(x,y,z)\,dx\right)\right]^{\frac{1}{1-n}}, \tag{21}$$

where $A(x,y,z)$ is an arbitrary known function, while $\phi(p,q)$ and $\varphi(y,z)$ are arbitrary functions. The variables p, q, and r are mutually independent and satisfy

$$\begin{aligned} p&=-(c_{12}+c_{13})x+c_{12}y+c_{13}z,\\ q&=-(c_{22}+c_{23})x+c_{22}y+c_{23}z,\\ r&=c_{31}x+c_{32}y+c_{33}z,\\ \frac{\partial(p,q,r)}{\partial(x,y,z)}&\neq 0,\\ c_{31}+c_{32}+c_{33}&\neq 0. \end{aligned}$$

Proof. By Corollary 2, set

$$v(x,y,z)=u_x+u_y+u_z. \tag{22}$$

The general solution of (22) is [11]

$$u=\phi(p,q)+\frac{\int v(p,q,r)\,dr}{c_{31}+c_{32}+c_{33}},$$

where

$$p=-(c_{12}+c_{13})x+c_{12}y+c_{13}z,$$

$$\begin{aligned} q &= -(c_{22}+c_{23})x + c_{22}y + c_{23}z, \\ r &= c_{31}x + c_{32}y + c_{33}z, \\ \frac{\partial(p,q,r)}{\partial(x,y,z)} &\neq 0, \\ c_{31}+c_{32}+c_{33} &\neq 0. \end{aligned}$$

We get

$$\begin{aligned} x &= \frac{1}{\Delta}\Big[(c_{22}c_{33}-c_{23}c_{32})p - (c_{12}c_{33}-c_{13}c_{32})q + (c_{12}c_{23}-c_{13}c_{22})r\Big], \\ y &= \frac{1}{\Delta}\Big[\big((c_{22}+c_{23})c_{33}+c_{23}c_{31}\big)p - \big((c_{12}+c_{13})c_{33}+c_{13}c_{31}\big)q + (c_{12}c_{23}-c_{13}c_{22})r\Big], \\ z &= \frac{1}{\Delta}\Big[-\big((c_{22}+c_{23})c_{32}+c_{22}c_{31}\big)p + \big((c_{12}+c_{13})c_{32}+c_{12}c_{31}\big)q + (c_{12}c_{23}-c_{13}c_{22})r\Big], \end{aligned}$$

where

$$\Delta = -(c_{12}+c_{13})(c_{22}c_{33}-c_{23}c_{32}) - c_{12}\big[-(c_{22}+c_{23})c_{33}-c_{23}c_{31}\big] + c_{13}\big[-(c_{22}+c_{23})c_{32}-c_{22}c_{31}\big] \neq 0.$$

Then

$$v_x = u_{xx} + u_{xy} + u_{xz}.$$

Therefore,

$$v_x + A(x,y,z)v^n = u_{xx} + u_{xy} + u_{xz} + A(x,y,z)(u_x+u_y+u_z)^n = 0. \tag{23}$$

For

$$v_x + A(x,y,z)v^n = 0, \tag{24}$$

the general solution of (24) is [15]

$$v = \left[(1-n)\left(\varphi(y,z) - \int A(x,y,z)\,dx\right)\right]^{\frac{1}{1-n}}.$$

By Corollary 2, the general solution to (20) is

$$u = \phi(p,q) + \frac{\int v(p,q,r)\,dr}{c_{31}+c_{32}+c_{33}},$$

where

$$v(x,y,z) = \left[(1-n)\left(\varphi(y,z) - \int A(x,y,z)\,dx\right)\right]^{\frac{1}{1-n}}.$$

So the theorem is proved. □

Below, we use the Z_6 transformation to prove Theorems 5 and 6.

Theorem 5. *In $\mathbb{R}^2$, the general solution of*

$$au_{xx} + (a+b)u_{xy} + bu_{yy} + A(x,y)(u_x+u_y) = B(x,y), \tag{25}$$

is

$$u = \phi(r) + \frac{\int v(r,s)\,ds}{k+l}, \tag{26}$$

and

$$v(p,q) = \mathrm{e}^{-\int A(x,y)\,dp}\left[\varphi(q) + \int B(x,y)\mathrm{e}^{\int A(x,y)\,dp}\,dp\right], \tag{27}$$

where a and b are random known constants, $A(x,y)$ and $B(x,y)$ are arbitrary known functions, and $\phi(r)$ and $\varphi(q)$ are arbitrary functions. The independent variables p,q and r,s, and the parameters c and d, satisfy

$$\begin{aligned} x &= ap + cq, & y &= bp + dq,\\ ad - bc &\neq 0, & &\\ r &= x - y, & s &= kx + ly,\\ k + l &\neq 0. & & \end{aligned}$$

Proof. By Z_6 transformation, set

$$v\,(x,y) = u_x + u_y. \tag{28}$$

The general solution of Eq. (28) is [11]

$$u = \phi\,(r) + \frac{\int v\,(r,s)\,ds}{k+l},$$

where

$$r = x - y, s = kx + ly,$$

and

$$x = \frac{s+lr}{k+l}, y = \frac{s-kr}{k+l},$$
$$k + l \neq 0.$$

According to Eq. (28), we get

$$v_x = u_{xx} + u_{xy},$$
$$v_y = u_{xy} + u_{yy}.$$

Set

$$x = ap + cq, y = bp + dq,$$
$$ad - bc \neq 0.$$

Then

$$p = \frac{xd - yc}{ad - bc} = \frac{d\,(s+lr) - c\,(s-kr)}{(k+l)\,(ad-bc)},$$
$$q = \frac{ya - xb}{ad - bc} = \frac{a\,(s-kr) - b\,(s+lr)}{(k+l)\,(ad-bc)}.$$

For

$$v_p + A\,(ap+cq, bp+dq)\,v = B\,(ap+cq, bp+dq)\,, \tag{29}$$

the general solution of Eq. (29) is [23]:

$$v(p,q) = \mathrm{e}^{-\int A(ap+cq,bp+dq)\,dp}\left[\varphi(q) + \int B(ap+cq,bp+dq)\mathrm{e}^{\int A(ap+cq,bp+dq)\,dp}\,dp\right]. \tag{30}$$

Namely

$$v(p,q) = \mathrm{e}^{-\int A(x,y)\,dp}\left[\varphi(q) + \int B(x,y)\,\mathrm{e}^{\int A(x,y)\,dp}\,dp\right].$$

Because

$$v_p = av_x + bv_y = au_{xx} + au_{xy} + bu_{xy} + bu_{yy},$$

we have

$$\begin{aligned} v_p + A(ap+cq, bp+dq)v &= au_{xx} + au_{xy} + bu_{yy} + bu_{xy} + A(ap+cq, bp+dq)(u_x + u_y) \\ &= au_{xx} + (a+b)u_{xy} + bu_{yy} + A(x,y)(u_x + u_y) \\ &= B(ap+cq, bp+dq) = B(x,y). \end{aligned}$$

According to the Z_6 transformation, the general solution of (25) is

$$u = \phi(r) + \frac{\int v(r,s)\, ds}{k+l},$$

where

$$v(p,q) = \mathrm{e}^{-\int A(x,y)\, dp} \left[\varphi(q) + \int B(x,y)\, \mathrm{e}^{\int A(x,y)\, dp}\, dp \right].$$

So the theorem is proved. □

Theorem 6. *In $\mathbb{R}^2$, the general solution of*

$$au_{xx} + (a+b)u_{xy} + bu_{yy} + A(x,y)(u_x + u_y)^n = 0, \tag{31}$$

is

$$u = \phi(r) + \frac{\int v(r,s)\, ds}{k+l},$$

and

$$v(p,q) = \left[(1-n) \left(\varphi(q) - \int A(p,q)\, dp \right) \right]^{\frac{1}{1-n}},$$

where a and b are random known constants, $A(x,y)$ is an arbitrary known function, and $\phi(r)$ and $\varphi(q)$ are arbitrary functions. The independent variables p, q and r, s, and the parameters c and d, satisfy

$$\begin{aligned} x &= ap + cq, & y &= bp + dq, \\ r &= x - y, & s &= kx + ly, \\ ad - bc &\neq 0, & k + l &\neq 0. \end{aligned}$$

Proof. By Z_6 transformation, set

$$v\,(x,y) = u_x + u_y. \tag{32}$$

The general solution of Eq. (32) is [11]

$$u = \phi\,(r) + \frac{\int v\,(r,s)\, ds}{k+l},$$

where

$$r = x - y, s = kx + ly,$$

$$k + l \neq 0,$$

and

$$x = \frac{s + lr}{k+l}, y = \frac{s - kr}{k+l}.$$

According to Eq.(32), we get

$$v_x = u_{xx} + u_{xy},$$

$$v_y = u_{xy} + u_{yy}.$$

Set

$$x = ap + cq, y = bp + dq,$$
$$ad - bc \neq 0.$$

Then

$$p = \frac{xd - yc}{ad - bc} = \frac{d(s + lr) - c(s - kr)}{(k + l)(ad - bc)},$$
$$q = \frac{ya - xb}{ad - bc} = \frac{a(s - kr) - b(s + lr)}{(k + l)(ad - bc)}.$$

For

$$v_p + A(p, q) v^n = 0, \tag{33}$$

the general solution of Eq. (33) is [15]:

$$v = \left((1 - n)\left(\varphi(q) - \int A(p, q)\, dp\right)\right)^{\frac{1}{1-n}}. \tag{34}$$

Because of

$$v_p = av_x + bv_y = au_{xx} + au_{xy} + bu_{xy} + bu_{yy},$$

$$\begin{aligned} v_p + A(p, q) v^n &= au_{xx} + au_{xy} + bu_{yy} + bu_{xy} + A(p, q)\left(u_x + u_y\right)^n \\ &= au_{xx} + (a + b) u_{xy} + bu_{yy} + A(x, y)\left(u_x + u_y\right)^n = 0. \end{aligned}$$

According to Z_6 transformation, the general solution of Eq. (32) is

$$u = \phi(r) + \frac{\int v(r, s)\, ds}{k + l},$$

where

$$v(p, q) = \left((1 - n)\left(\varphi(q) - \int A(p, q)\, dp\right)\right)^{\frac{1}{1-n}}.$$

So the theorem is proved. □

Using the general solution often provides an effective way to obtain the exact solution to definite solution problems [10–16]. Below, we analyze two typical cases.

Example 1. Find the exact solution of $u_{xx} + 3u_{xy} + 2u_{yy} + (x - y)\left(u_x + u_y\right)^2 = 0$ under the conditions $u(x, 0) = x, u(x, x) = x$.

Solution. According to Theorem 6, we have

$$a = 1, b = 2, A(x, y) = x - y, n = 2.$$

Set

$$c = 1, d = 1, k = 2, l = 1.$$

Then

$$x = p + q, y = 2p + q,$$
$$r = x - y, s = 2x + y.$$

And

$$x = \frac{r + s}{3}, y = \frac{s - 2r}{3},$$
$$p = y - x, q = 2x - y,$$

$$p=-r,q=\frac{s+4r}{3},$$

$$A\left(p,q\right)=x-y=\left(p+q\right)-\left(2p+q\right)=-p, \tag{35}$$

$$\int A\left(p,q\right)\,dp=\int\left(-p\right)\,dp=-\frac{p^2}{2}=-\frac{r^2}{2}=-\frac{\left(x-y\right)^2}{2}, \tag{36}$$

$$v=\left(-\left(\varphi\left(q\right)-\int A\left(p,q\right)\,dp\right)\right)^{-1}=\left(-\left(\varphi\left(2x-y\right)+\frac{\left(x-y\right)^2}{2}\right)\right)^{-1}=\left(-\varphi\left(\frac{4r+s}{3}\right)-\frac{r^2}{2}\right)^{-1}. \tag{37}$$

So the general solution of $u_{xx}+3u_{xy}+2u_{yy}+\left(x-y\right)\left(u_x+u_y\right)^2=0$ is

$$u=\phi\left(r\right)+\frac{\int v\left(r,s\right)ds}{k+l}=\phi\left(x-y\right)+\frac{1}{3}\int\left(-\varphi\left(\frac{4r+s}{3}\right)-\frac{r^2}{2}\right)^{-1}ds. \tag{38}$$

When $u(x,x)=x,r=0,s=3x$, that is

$$u\left(0,s\right)=\frac{s}{3}.$$

According to (38), we get

$$u\left(0,s\right)=\phi\left(0\right)+\frac{1}{3}\int\left(-\varphi\left(\frac{s}{3}\right)\right)^{-1}ds. \tag{39}$$

Because of

$$\frac{d\left[u\left(0,s\right)\right]}{ds}=\frac{1}{3}\left(-\varphi\left(\frac{s}{3}\right)\right)^{-1}=\frac{1}{3}.$$

So

$$-\varphi\left(\frac{s}{3}\right)=1.$$

Namely

$$\varphi\equiv-1. \tag{40}$$

Then

$$u\left(r,s\right)=\phi\left(r\right)+\frac{1}{3}\int\left(-\varphi\left(\frac{4r+s}{3}\right)-\frac{r^2}{2}\right)^{-1}ds=\phi\left(r\right)+\frac{1}{3}\int\left(1-\frac{r^2}{2}\right)^{-1}ds.$$

That is

$$u\left(r,s\right)=\phi\left(r\right)+\frac{2s}{3\left(2-r^2\right)}. \tag{41}$$

When $u(x,0)=x,r=x,s=2x$, that is, when $s=2r,u=r$, so

$$\phi\left(r\right)+\frac{4r}{3\left(2-r^2\right)}=r\Longrightarrow\phi\left(r\right)=\frac{2r-3r^3}{3\left(2-r^2\right)}.$$

So

$$u\left(r,s\right)=\frac{2r-3r^3+2s}{3\left(2-r^2\right)}. \tag{42}$$

Substituting $r=x-y,s=2x+y$ into Eq. (42), the exact solution to this definite solution problem is

$$u\left(x,y\right)=\frac{2x-\left(x-y\right)^3}{2-\left(x-y\right)^2},(\left(x-y\right)^2\neq2\right). \tag{43}$$

Example 2. Find the exact solution of

$$u_{xx}+u_{xy}+(x-y)^2(u_x+u_y)=(x-y)^2,$$

under the conditions

$$u(x,0)=\varphi(x),\qquad u(y,y)=\psi(y),$$

where $\varphi(x)$ and $\psi(y)$ are arbitrary known functions satisfying $\psi(0) = \varphi(0)$.

Solution. By Theorem 5, we have $a = 1$ and $b = 0$. Set $c = 1$, $d = 1$, $k = 0$, and $l = 1$. Then

$$\begin{aligned} x &= p + q, & y &= q, \\ p &= x - y, & q &= y, \\ r &= x - y, & s &= x. \end{aligned}$$

Moreover,

$$A(x,y) = B(x,y) = (x-y)^2 = p^2. \tag{44}$$

Hence,

$$\begin{aligned} \int A\,dp &= \int p^2\,dp = \frac{p^3}{3}, & \mathrm{e}^{-\int A\,dp} &= \mathrm{e}^{-p^3/3}, \\ \int B\mathrm{e}^{\int A\,dp}\,dp &= \int p^2\mathrm{e}^{p^3/3}\,dp = \mathrm{e}^{p^3/3}. \end{aligned}$$

Therefore,

$$\begin{aligned} v(p,q) &= \mathrm{e}^{-\int A(x,y)\,dp}\left[\varphi(q) + \int B(x,y)\mathrm{e}^{\int A(x,y)\,dp}\,dp\right] \\ &= \mathrm{e}^{-p^3/3}\left[\varphi(q) + \mathrm{e}^{p^3/3}\right] \\ &= \mathrm{e}^{-p^3/3}\varphi(q) + 1 \\ &= \mathrm{e}^{-(x-y)^3/3}\varphi(y) + 1 \\ &= \mathrm{e}^{-r^3/3}\varphi(s-r) + 1. \end{aligned}$$

Thereupon,

$$\begin{aligned} \frac{\int v(r,s)\,ds}{k+l} &= \int \left[\mathrm{e}^{-r^3/3}\varphi(s-r) + 1\right] ds \\ &= \mathrm{e}^{-r^3/3}\int \varphi(s-r)\,ds + s \\ &= \mathrm{e}^{-r^3/3}g(s-r) + s \\ &= \mathrm{e}^{-(x-y)^3/3}g(y) + x. \end{aligned}$$

Thus, the general solution of

$$u_{xx} + u_{xy} + (x-y)^2(u_x + u_y) = (x-y)^2,$$

is

$$u = f(x-y) + x + \mathrm{e}^{-(x-y)^3/3}g(y), \tag{45}$$

where f and g are any known functions. Then

$$\begin{aligned} u(x,0) &= f(x) + x + \mathrm{e}^{-x^3/3}g(0) = \varphi(x), \\ u(y,y) &= f(0) + y + g(y) = \psi(y). \end{aligned}$$

Set $C_1 = g(0)$. Then

$$f(x) = \varphi(x) - x - C_1\mathrm{e}^{-x^3/3},$$

whereupon

$$f(x-y) = \varphi(x-y) - x + y - C_1\mathrm{e}^{-(x-y)^3/3}. \tag{46}$$

Also,

$$f(0) = \varphi(0) - C_1 = \psi(y) - y - g(y).$$

Hence,

$$\begin{aligned} g(y) &= \psi(y) - y - \varphi(0) + C_1, \\ g(0) &= \psi(0) - \varphi(0) + C_1 = C_1. \end{aligned}$$

Therefore,

$$\psi(0) = \varphi(0). \tag{47}$$

Consequently,

$$\begin{aligned} u(x,y) &= f(x-y) + x + \mathrm{e}^{-(x-y)^3/3} g(y) \\ &= \left[\varphi(x-y) - (x-y) - C_1 \mathrm{e}^{-(x-y)^3/3}\right] + x + \mathrm{e}^{-(x-y)^3/3}\left[\psi(y) - y - \varphi(0) + C_1\right]. \end{aligned}$$

Thus, the exact solution to this definite solution problem is

$$u(x,y) = \varphi(x-y) + y + \mathrm{e}^{-(x-y)^3/3}\left[\psi(y) - y - \varphi(0)\right]. \tag{48}$$

4. The Z_6 transformation in ordinary differential equations and its applications

The Z_6 transformation in ordinary differential equations is defined as follows.

Z_6 **transformation.** In the domain D, $(D \subseteq \mathbb{R}^1)$, if the solution $w = g(v)$ of an ODE $G\left(v, w, w', w'', \dots w^{(n)}\right) = 0$ is known, $v = v(x)$, $x = x(v)$ are known too, and set $w = h\left(x, y, y', y'', \dots y^{(m)}\right)$, then the solution of $F\left(x, y, y', y'', \dots y^{(m+n)}\right) = G\left(v, w, w', w'', \dots w^{(n)}\right) = 0$ is the solution of $h\left(x, y, y', y'', \dots y^{(m)}\right) = g(v)$.

If the Z_6 transformation is used to solve ordinary differential equations, the equation we need to solve is

$$F\left(x, y, y', y'', \dots y^{(m+n)}\right) = 0, \tag{49}$$

using $w = h\left(x, y, y', y'', \dots y^{(m)}\right)$ and $x = x(v)$, we can transform Eq. (49) into

$$G\left(v, w, w', w'', \dots w^{(n)}\right) = 0. \tag{50}$$

Since the solution $w = g(v)$ of Eq. (50) is known, the solution $y = y(x)$ obtained using $w = h\left(x, y, y', y'', \dots y^{(m)}\right) = g(v)$ is the solution to Eq. (49).

To construct a new solvable equation using the Z_6 transformation, two known solvable source equations are required

$$\begin{gathered} G\left(v, w, w', w'', \dots w^{(n)}\right) = 0, \\ h\left(x, y, y', y'', \dots y^{(m)}\right) = g(v), \end{gathered} \tag{51}$$

where Eq. (50) is explicitly solvable and is referred to as the first source equation, while Eq. (51) is a non-homogeneous equation and is referred to as the second source equation. Through an appropriate transformation, a new solvable Eq. (49) is derived, and the solution to Eq. (51) is also the solution to Eq. (49).

Below, we use Z_6 transformation to analyze a specific example.

Example 3. Given $\frac{dw}{dv} - w^n = 0, w = y' + \psi(x)y, v = \varphi(x), \psi(x)$ and $\varphi(x)$ are arbitrary known functions, and $\varphi'(x) \neq 0$, use Z_6 transformation to find the solution to the new equation they form.

Solution. When $n = 1$

$$\frac{dw}{dv} - w = 0, w = y' + \psi(x)y, v = \varphi(x).$$

Whereupon

$$\frac{dw}{dv} = \frac{dw}{dx} \cdot \frac{dx}{dv} = \frac{y'' + \psi(x)y' + \psi'(x)y}{\varphi'(x)}, (\varphi'(x) \neq 0).$$

And

$$\frac{dw}{dv} - w = \frac{y'' + \psi(x)y' + \psi'(x)y}{\varphi'(x)} - (y' + \psi(x)y) = 0$$
$$\Longrightarrow y'' + \psi(x)y' + \psi'(x)y - \varphi'(x)y' - \varphi'(x)\psi(x)y = 0.$$

Namely

$$y'' + [\psi(x) - \varphi'(x)]y' + [\psi'(x) - \varphi'(x)\psi(x)]y = 0. \tag{52}$$

The general solution of

$$\frac{dw}{dv} - w = 0,$$

is

$$w = Ce^{v},$$

where C is a random known constant. Moreover,

$$w = y' + \psi(x)y = Ce^{v} = Ce^{\varphi(x)}.$$

The general solution of

$$y' + \psi(x)y = Ce^{\varphi(x)},$$

is

$$y(x) = e^{-\int \psi(x)\,dx}\left[C\int e^{\varphi(x)}e^{\int \psi(x)\,dx}\,dx + D\right]. \tag{53}$$

By Z_6 transformation, we know that (53) is also the general solution of Eq. (52), where C and D are both arbitrary constants.

When $n \neq 1$

$$\frac{dw}{dv} = w^n \Longrightarrow w = [(1-n)(v+C)]^{\frac{1}{1-n}},$$
$$\frac{dw}{dv} = \frac{dw}{dx} \cdot \frac{dx}{dv} = \frac{y'' + \psi(x)y' + \psi'(x)y}{\varphi'(x)} = w^n.$$

Namely

$$y'' + \psi(x)y' + \psi'(x)y = \varphi'(x)[y' + \psi(x)y]^n. \tag{54}$$

Equation (54) is the new equation derived. Since

$$y' + \psi(x)y = [(1-n)(\varphi(x)+C)]^{\frac{1}{1-n}}, \tag{55}$$

the general solution of (54) is

$$y = e^{-\int \psi(x)\,dx}\left[\int [(1-n)(\varphi(x)+C)]^{\frac{1}{1-n}} e^{\int \psi(x)\,dx}\,dx + D\right]. \tag{56}$$

By the Z_6 transformation, we know that (56) is also the general solution of (54), where C and D are arbitrary constants. For

$$[(1-n)(\varphi(x)+C)]^{\frac{1}{1-n}},$$

in the field of real numbers, we require the base to be positive and take the real-valued principal branch; in the field of complex numbers, we take the principal argument branch. Readers may choose the appropriate domain of interpretation based on the specific physical context.

5. The principle of common solutions

In the Z_6 transformation of partial differential equations, we find that the new solvable Eq. (1) and the second source Eq. (5) have the same solutions. In the Z_6 transformation of ordinary differential equations, we find that the new solvable Eq. (49) and the second source Eq. (51) have the same solutions. Is it possible to construct new equations with the same solution using known equations with the same solution? Below, we address this question by introducing the principle of common solutions.

Principle of common solutions. In $\mathbb{R}^n$, if the equations $F_i = 0$ have a common solution $u = f(x_1, x_2, \ldots x_n)$, $(i = 1, 2, \ldots m)$, and the equation $G(F_1, F_2, \ldots F_m) = 0$ satisfies $G(0, 0, \ldots 0) = 0$, then $u = f(x_1, x_2, \ldots x_n)$ is also a solution to $G(F_1, F_2, \ldots F_m) = 0$.

In $\mathbb{R}^n$, both homogeneous and nonhomogeneous equations can be transformed into the form $F_i = 0$. Since the new equation $G(F_1, F_2, \ldots F_m) = 0$ satisfies $G(0, 0, \ldots 0) = 0$, if $F_i = 0$, the common solution is $u = f(x_1, x_2, \ldots x_n)$. Substituting this into $G(F_1, F_2, \ldots F_m) = 0$ necessarily yields $0 = 0$, therefore $u = f(x_1, x_2, \ldots x_n)$ is also a solution to $G(F_1, F_2, \ldots F_m) = 0$.

The principle of common solutions applies not only to differential equations, but also to algebraic equations, functional equations, and other fields.

From the proof of Theorem 3, we can see that the common solution of

$$u_x + u_y + u_z = v(x, y, z),$$

and

$$u_{xx} + u_{xy} + u_{xz} + b(x, y, z)(u_x + u_y + u_z) = c(x, y, z),$$

is

$$u = \phi(p, q) + \frac{\int v(p, q, r)\, dr}{c_{31} + c_{32} + c_{33}},$$

where

$$v(x, y, z) = \mathrm{e}^{-\int b(x,y,z)\, dx} \left[\varphi(y, z) + \int c(x, y, z) \mathrm{e}^{\int b(x,y,z)\, dx}\, dx \right].$$

By the principle of common solutions, set

$$F_1 = u_x + u_y + u_z - v(x, y, z) = 0. \tag{57}$$

and

$$F_2 = u_{xx} + u_{xy} + u_{xz} + b(x, y, z)(u_x + u_y + u_z) - c(x, y, z) = 0. \tag{58}$$

Thus, $F_2 + F_1^2 = 0$. Therefore, a solution of

$$u_{xx} + u_{xy} + u_{xz} + b(x, y, z)(u_x + u_y + u_z) - c(x, y, z) + \left[u_x + u_y + u_z - v(x, y, z) \right]^2 = 0, \tag{59}$$

is given by (14).

From Example 3, we can see that the common solution of

$$y' + \psi(x) y = C\mathrm{e}^{\varphi(x)},$$

and

$$y'' + [\psi(x) - \varphi'(x)] y' + [\psi'(x) - \varphi'(x)\psi(x)] y = 0,$$

is

$$y(x) = \mathrm{e}^{-\int \psi(x)\, dx} \left[C \int \mathrm{e}^{\varphi(x)} \mathrm{e}^{\int \psi(x)\, dx}\, dx + D \right].$$

By the principle of common solutions, set

$$F_1 = y' + \psi(x) y - C\mathrm{e}^{\varphi(x)} = 0,$$

and

$$F_2 = y'' + [\psi(x) - \varphi'(x)]y' + [\psi'(x) - \varphi'(x)\psi(x)]y = 0.$$

Thus, $F_2 - F_1 = 0$. Therefore, a solution of

$$\begin{aligned} y'' &+ [\psi(x) - \varphi'(x) - 1]y' \\ &+ [\psi'(x) - \varphi'(x)\psi(x) - \psi(x)]y + Ce^{\varphi(x)} = 0, \end{aligned}$$

is given by (53).

Since there are infinitely many specific forms of the equation

$$G(F_1, F_2) = 0,$$

that satisfy $G(0, 0) = 0$, it is possible to construct an infinite number of solvable ordinary and partial differential equations using Z_6 transformations and the principle of common solutions.

6. Discussion and conclusion

The Z_6 transformation proposed in this paper serves as a unified constructive framework for differential equations. Unlike simple variable substitution, this framework establishes a systematic set of procedures: by taking two known solvable equations as source equations and combining them with a transformation of the independent variables, a new solvable equation can be constructed. This approach successfully unifies two classic techniques-coordinate transformation and dependent variable substitution-into a single, self-consistent methodology.

If a solution to a partial differential equation has already been obtained in one coordinate system, Corollary 1 can be used to directly obtain its solution in any coordinate system. A thorough literature review has revealed an interesting historical precedent. The idea of directly obtaining solutions to partial differential equations in different coordinate systems through coordinate transformations-rather than solving them anew in each new coordinate system-has been documented only sporadically throughout history. Forsyth (1898) studied coordinate transformations that yield new solutions to the Laplace equation [24], while Jeffery (1917) conducted a similar investigation of Whittaker's solutions [25]. However, these works were limited to the Laplace equation and involved only specific types of transformations. The key point is that they did not develop a general systematic methodology applicable to other equations or broader classes of transformations. To the best of our knowledge, there are no reports in the recent literature of this idea being developed into a unified framework for solvable differential equations. Corollary 1 of the Z_6 transformation effectively resolves this problem; thus, we have not only revived a long-neglected mathematical idea but also elevated it to a rigorous and universally applicable methodology.

Regarding the general solutions to the Laplace equation in cylindrical and spherical coordinates obtained in this paper, one question warrants attention: Can these general solutions be used to express all harmonic functions in the respective coordinate systems? This is a highly challenging new problem, and it will also be one of the key areas of our future research.

The most notable feature of the Z_6 transformation lies in its generative capability. In our previous work, we proposed the Z_A method, which demonstrates that any solvable equation can serve as a source equation to generate an infinite number of sub-equations, and that the corresponding equations for all of these sub-equations are theoretically solvable [15]. The Z_6 transformation has similar capabilities but operates in a complementary manner. The Z_A method generates new equations by applying mathematical operations, such as differential operators, to a single source equation, whereas the Z_6 transformation is achieved by combining two known solvable equations-an explicitly solvable first source equation and a solvable non-homogeneous second source equation-with a variable transformation. The resulting new equation inherits the solvability of both. Based on the Z_6 transformation, we have also discovered the principle of common solutions for equations, which can be used to directly discover an infinite number of new solvable ordinary and partial differential equations.

The significance of this work lies in two aspects. Theoretically, the Z_6 transformation and the Z_A method together provide constructive proofs, demonstrating that the class of differential equations with explicit solutions is far more extensive than traditionally believed-in fact, an infinite number of solvable equations can be systematically discovered. In practice, this framework provides applied mathematicians and physicists with a flexible and powerful tool for generating closed-form solutions to newly encountered equations, which is particularly valuable for model validation, asymptotic analysis, and benchmarking in numerical simulations. We believe that the generative paradigm introduced in this paper-which focuses not only on solving given equations but also on systematically constructing new solvable equations-opens up a promising new avenue for revitalizing research in the theory of exact solutions.

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