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Article

Titchmarsh and Boas-type theorems within the framework of the linear canonical Sturm–Liouville transform

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Abstract: We introduce Lipschitz classes $\mathrm{Lip}^M(\eta, 2)$ and $\mathrm{Lip}^M(\eta, \infty)$ of functions associated with the canonical Sturm–Liouville operator $\mathscr{L}^M$; and we prove a new versions of Titchmarsh and Boas-type theorems for the canonical Sturm–Liouville transform $\mathscr{F}^M$. An application to the canonical Sturm–Liouville multipliers is given. Titchmarsh and Boas-type theorems for the canonical Fourier–Bessel transform and the canonical Fourier–Jacobi transform are special cases of this work.

Keywords: Canonical Sturm–Liouville transform, Lipschitz classes, Titchmarsh-type theorem, Boas-type theorem

MSC: 44A05, 44A20.

1. Introduction

Titchmarsh's results [1, Theorems 84 and 85], giving a relationship between the smoothness of a function and the integrability of its Fourier transform. More precisely, the second Titchmarsh theorem characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Lipschitz condition by means of an asymptotic estimate growth of the norm of its Fourier transform as follows. Let $0 < \eta < 1$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents

(i) $\|f(x+y) - f(x)\|_{L^2(\mathbb{R})} = O(y^\eta)$ as $y \to 0$.

(ii) $\displaystyle\int_{|\lambda|\geq s} |\widehat{f}(\lambda)|^2 d\lambda = O(s^{-2\eta})$ as $s \to \infty$, where $\widehat{f}$ stands for the Fourier transform of f.

There are many analogues of Titchmarsh's theorem are also proved by Daher et al. for the Fourier transform [2], by El Hamma et al. for the Fourier–Bessel transform [3], for the Fourier–Jacobi transform [4–7]. Recently, Soltani and Aloui [8] established the analogue of Titchmarsh's theorem for the Sturm–Liouville transform

Related to Titchmarsh's theorems, Boas [9] found necessary and sufficient conditions on the Fourier coefficients of a function to belong to a Lipschitz class, which is one of the classical topics of harmonic analysis and approximation theory. Móricz [10] studied the continuity and regularity properties of a function f with an absolutely convergent Fourier series. Then, the same author [11] extended these results as follows. If $f \in L^1(\mathbb{R})$, and $0 < \beta \leq k$, $k \in \mathbb{N}$, one has

$$\int_{|\lambda|<s} |\lambda|^k |\widehat{f}(\lambda)| \mathrm{d}\lambda = O(s^{k-\beta}) \quad \text{for all} \quad s > 0,$$

then $\widehat{f} \in L^1(\mathbb{R})$ and f satisfies the smooth Lipschitz condition of order k.

The extension of classical Boas-type theorems to the framework of harmonic analysis constitutes a significant contribution that will interest researchers working on integral transforms and differential operators, such as the Bessel transform [12], the generalized Bessel transform [13,14], the q-Bessel Fourier transform [15, 16], the deformed Hankel transform [17,18], the Hartley–Bessel transforms [19], the second Hankel–Clifford transform [20], the Dunkl transform [21,22], the Laguerre transform [23], the free metaplectic transform [24], the linear canonical transform [25], and also the quaternion transform [26,27].

Here, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2, \mathbb{R})$ such that $b > 0$. We define the canonical Sturm–Liouville operator $\mathscr{L}^M$ on $(0, \infty)$ by

$$\mathscr{L}^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is a positive function satisfying certain conditions.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator $\mathscr{L}^M$ is reduced to the Sturm–Liouville operator $\mathscr{L}$:

$$\mathscr{L} := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}.$$

The classical Sturm–Liouville operator $\mathscr{L}$ plays an important role in analysis [28,29]. In particular, the two references [30,31] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with the operator $\mathscr{L}$.

Using the Sturm–Liouville harmonic analysis [30,31], for all $\lambda \in \mathbb{C}$, the system

$$\mathscr{L}^M u = -\left(\frac{\lambda^2}{b^2} + \rho^2\right)u, \quad u(0) = e^{i\frac{d}{2b}\lambda^2}, \; u'(0) = 0,$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)}\varphi_{\frac{\lambda}{b}}(x), \quad x \geq 0,$$

where $\varphi_\lambda(x)$ is the Sturm–Liouville kernel [8,32].

In this paper, we introduce the canonical Sturm–Liouville transform $\mathscr{F}^M$:

$$\mathscr{F}^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^M(x) f(x) A(x)\mathrm{d}x, \quad \lambda \geq 0.$$

The canonical Sturm–Liouville transform $\mathscr{F}^M$ can be regarded as a generalization of the Sturm–Liouville transform $\mathscr{F}$ (see [8,32,33]):

$$\mathscr{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) A(x)\mathrm{d}x, \quad \lambda \geq 0.$$

The main objective of this work is to give two versions of Titchmarsh and Boas-type theorems for the canonical Sturm–Liouville transform $\mathscr{F}^M$. These versions are taken for $f \in L^2(\mathbb{R}_+, A(x)\mathrm{d}x)$ (resp. $f \in L^1(\mathbb{R}_+, A(x)\mathrm{d}x)$) and for $\mathscr{F}(f)$ meets certain conditions of integrability. These results are proven by means of the canonical Sturm–Liouville translation defined later in the §2. We conclude this work by providing an application of Theorems 7 to the study of the generalized Bessel potential.

Titchmarsh and Boas-type results for the canonical Fourier–Bessel and Fourier–Jacobi transforms are special cases of this work. However, we left the same problem as an open topic for other Fourier-type transformations.

This paper is organized as follows. In §2, we recall some results about the Sturm–Liouville transform $\mathscr{F}$ and the Sturm–Liouville translation T_y. Next, we introduce the canonical Sturm–Liouville operator $\mathscr{L}^M$, and we investigate the properties of the canonical Sturm–Liouville transform $\mathscr{F}^M$ and the canonical Sturm–Liouville translation T_y^M associated with this operator. In §3, we define the M-canonical Sturm–Liouville Lipschitz classes, and we show a versions of Titchmarsh and Boas-type theorems for the canonical Sturm–Liouville transform $\mathscr{F}^M$. We give an application to the canonical Sturm–Liouville Bessel potential. In the last section, we discuss the special cases of Titchmarsh and Boas-type theorems for the canonical Fourier–Bessel and Fourier–Jacobi transforms.

2. The canonical Sturm–Liouville operator

In this section we recall some results about the Sturm–Liouville and the canonical Sturm–Liouville operators.

2.1. The Sturm–Liouville transform

We consider the second-order differential operator $\mathscr{L}$ defined on $(0,\infty)$ by

$$\mathscr{L} := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that A satisfies the following conditions:

(i) A is increasing and $\lim_{x\to\infty} A(x) = \infty$,

(ii) $\frac{A'}{A}$ is decreasing and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0$,

(iii) there exists a constant $\delta > 0$, such that

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x}D(x), \quad \text{if } \rho > 0,$$

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + e^{-\delta x}D(x), \quad \text{if } \rho = 0,$$

where D is an infinitely differentiable function on $(0,\infty)$, bounded and with bounded derivatives on all intervals $[x_0,\infty)$, for $x_0 > 0$.

This operator was studied in [30,31], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\mathscr{L}u = -(\lambda^2+\rho^2)u, \quad u(0) = 1,\ u'(0) = 0,$$

admits a unique solution, denoted by φ_λ, with the following properties:

- for $x \geq 0$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;
- for $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$.

(II) For nonzero $\lambda \in \mathbb{C}$, the equation $\mathscr{L}u = -(\lambda^2+\rho^2)u$ has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{e^{i\lambda x}}{\sqrt{A(x)}}V(x,\lambda),$$

with

$$\lim_{x\to\infty} V(x,\lambda) = 1.$$

Consequently there exists a function (spectral function) $\lambda \mapsto c(\lambda)$, such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x \geq 0,$$

for nonzero $\lambda \in \mathbb{C}$.

Moreover, the function $\lambda \mapsto |c(\lambda)|^{-2}$ is continuous on $[0,\infty)$ and there exist positive constants k_1, k_2, k, such that (see [34, page 99])

$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},$$

for all λ such that $\mathrm{Im}\lambda \leq 0$ and $|\lambda| \geq k$.

Lemma 1 ([8]). *The Sturm–Liouville function $\varphi_\lambda(x)$; $\lambda, x \geq 0$, possesses the following properties.*

(i) $|\varphi_\lambda(x)| \leq 1$,
(ii) $1-\varphi_\lambda(x) \leq \frac{1}{2}(\lambda^2+\rho^2)x^2$.
We denote by
• *μ the measure defined on $[0,\infty)$ by*

$$d\mu(x) := A(x)dx,$$

and by $L^p(\mu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0,\infty)$, such that

$$\|f\|_{L^p(\mu)} := \left[\int_0^\infty |f(x)|^p d\mu(x)\right]^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\mu)} := ess \sup_{x\in[0,\infty)} |f(x)| < \infty;$$

• *ν the measure defined on $[0,\infty)$ by*

$$d\nu(\lambda) := \frac{d\lambda}{2\pi|c(\lambda)|^2},$$

and by $L^p(\nu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0,\infty)$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Sturm–Liouville transform is the Fourier transform associated with the operator $\mathscr{L}$ and is defined for $f \in L^1(\mu)$ by

$$\mathscr{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x)f(x)\mathrm{d}\mu(x), \quad \lambda \geq 0. \tag{1}$$

Theorem 1 ([28–31])**.** (i) *Plancherel theorem. The Sturm–Liouville transform $\mathscr{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular,*

$$\|f\|_{L^2(\mu)} = \|\mathscr{F}(f)\|_{L^2(\nu)}.$$

(ii) *Inversion theorem. Let $f \in L^1(\mu)$, such that $\mathscr{F}(f) \in L^1(\nu)$. Then*

$$f(x) = \int_0^\infty \varphi_\lambda(x)\mathscr{F}(f)(\lambda)d\nu(\lambda), \quad a.e. \quad x \geq 0.$$

The Sturm–Liouville function φ_λ satisfies the product formula [30,31]

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)w(x,y,z)\mathrm{d}\mu(z) \quad \text{for} \quad x,y \geq 0; \tag{2}$$

where $w(x,y,.)$ is a measurable positive function on $[0,\infty)$, with support in $[|x-y|, x+y]$, satisfying

$$\int_0^\infty w(x,y,z)\mathrm{d}\mu(z) = 1,$$

$$w(x,y,z) = w(y,x,z) \quad \text{for} \quad z \geq 0, \tag{3}$$

$$w(x,y,z) = w(x,z,y) \quad \text{for} \quad z > 0. \tag{4}$$

We now define the generalized translation operator induced by (2). For $f \in L^1(\mu)$, the linear operator

$$T_y f(x) := \int_0^\infty f(z)w(x,y,z)\mathrm{d}\mu(z), \quad x,y \geq 0, \tag{5}$$

will be called Sturm–Liouville translation.

As a first remark, we note that the relation (3) means that

$$T_y f(x) = T_x f(y), \quad x,y \geq 0.$$

Theorem 2 ([8,32]). (i) *For all* $y \geq 0$ *and* $f \in L^p(\mu)$, $p \in [1, \infty]$, *we have*

$$\|T_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

(ii) *For* $f \in L^2(\mu)$ *and* $y \geq 0$, *we have*

$$\mathscr{F}(T_y f)(\lambda) = \varphi_\lambda(y)\mathscr{F}(f)(\lambda), \quad \lambda \geq 0.$$

2.2. The canonical Sturm–Liouville translation operator

Throughout this paper, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitary matrix in $SL(2, \mathbb{R})$ such that $b > 0$. We define the canonical Sturm–Liouville operator $\mathscr{L}^M$ on $(0, \infty)$ by

$$\mathscr{L}^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x\frac{A'(x)}{A(x)} + i\frac{a}{b}\right),$$

where A is the positive function given in Section 2.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator $\mathscr{L}^M$ is reduced to the Sturm–Liouville operator $\mathscr{L}$:

$$\mathscr{L} := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x}.$$

For all $\lambda \in \mathbb{C}$, the equation

$$\mathscr{L}^M u = -\left(\frac{\lambda^2}{b^2} + \rho^2\right)u, \quad u(0) = e^{i\frac{d}{2b}\lambda^2},\ u'(0) = 0,$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)}\varphi_{\frac{\lambda}{b}}(x), \quad x \geq 0.$$

For $f \in L^1(\mu)$, we define the canonical Sturm–Liouville transform $\mathscr{F}^M(f)$ by

$$\mathscr{F}^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^M(x) f(x)\mathrm{d}\mu(x), \quad \lambda \geq 0.$$

This transform can be written as

$$\mathscr{F}^M(f)(\lambda) = e^{i\frac{d}{2b}\lambda^2}\mathscr{F}(e^{i\frac{a}{2b}x^2}f)(\frac{\lambda}{b}), \quad f \in L^1(\mu), \tag{6}$$

where $\mathscr{F}$ is the Sturm–Liouville transform given by (1).

We denote by ν_b, $b > 0$ the measure defined on $[0, \infty)$ by

$$\mathrm{d}\nu_b(\lambda) := \frac{\mathrm{d}\lambda}{2\pi b|c(\frac{\lambda}{b})|^2},$$

and by $L^p(\nu_b)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0, \infty)$, such that $\|f\|_{L^p(\nu_b)} < \infty$.

Theorem 3. (i) *Let* $f \in L^1(\mu)$, *such that* $\mathscr{F}^M(f) \in L^1(\nu_b)$. *Then*

$$f(x) = \int_0^\infty \varphi_\lambda^N(x)\mathscr{F}^M(f)(\lambda)d\nu_b(\lambda), \quad a.e \quad x \geq 0,$$

where N is the matrix given by $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

(ii) *For* $f \in L^2(\mu)$ *we have*

$$\|\mathscr{F}^M(f)\|_{L^2(\nu_b)} = \|f\|_{L^2(\mu)}.$$

Proof. (i) follows from Theorem 1 (ii) and relation (6).
(ii) follows from Theorem 1 (i) and relation (6). □

For $f \in L^1(\mu)$, we define the canonical Sturm–Liouville translation operators by

$$T_y^N f(x) := e^{-i\frac{a}{2b}(x^2+y^2)} \int_0^\infty f(z) e^{i\frac{a}{2b}z^2} w(x,y,z) \mathrm{d}\mu(z), \quad x,y \geq 0.$$

It is easy to prove the following results.

Theorem 4. *The operators* T_y^N*,* $y \geq 0$*, satisfy:*
(i) $T_y^N f(x) = T_x^N f(y)$, $x, y \geq 0$.
(ii) $T_y^N f(x) = e^{-i\frac{a}{2b}(x^2+y^2)} T_y \left(f(z) e^{i\frac{a}{2b}z^2} \right)(x)$*, where* T_y *is the Sturm–Liouville translation given by (5).*
(iii) $T_y^M \varphi_\lambda^M(x) = e^{-i\frac{d}{2b}\lambda^2} \varphi_\lambda^M(x) \varphi_\lambda^M(y)$.

Theorem 5. (i) *For all* $y \geq 0$ *and* $f \in L^p(\mu)$*,* $p \in [1, \infty]$*, we have*

$$\|T_y^N f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

(ii) *For* $f \in L^2(\mu)$ *and* $y \geq 0$*, we have*

$$\mathscr{F}^M(T_y^N f)(\lambda) = e^{i\frac{d}{2b}\lambda^2} \varphi_\lambda^N(y) \mathscr{F}^M(f)(\lambda), \quad \lambda \geq 0,$$

where $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

Proof. (i) follows from Theorem 2(i) and Theorem 4(ii).
(ii) Let $f \in L^1 \cap L^2(\mu)$. Then

$$\begin{aligned} \mathscr{F}^M(T_y^N f)(\lambda) &= \int_0^\infty T_y^N f(x) \varphi_\lambda^M(x) \mathrm{d}\mu(x) \\ &= \int_0^\infty \left[e^{-i\frac{a}{2b}(x^2+y^2)} \int_0^\infty f(z) e^{i\frac{a}{2b}z^2} w(x,y,z) \mathrm{d}\mu(z) \right] \varphi_\lambda^M(x) \mathrm{d}\mu(x). \end{aligned}$$

By using Fubini's theorem, (3) and (4) we obtain

$$\mathscr{F}^M(T_y^N f)(\lambda) = e^{-i\frac{a}{2b}y^2} \int_0^\infty f(z) e^{i\frac{a}{2b}z^2} \left[\int_0^\infty \varphi_\lambda^M(x) e^{-i\frac{a}{2b}x^2} w(z,y,x) \mathrm{d}\mu(x) \right] \mathrm{d}\mu(z).$$

And by Theorem 4(iii) we deduce that

$$\mathscr{F}^M(T_y^N f)(\lambda) = e^{i\frac{d}{2b}\lambda^2} \varphi_\lambda^N(y) \mathscr{F}^M(f)(\lambda), \quad \lambda \geq 0. \tag{7}$$

Since $L^1 \cap L^2(\mu)$ is a dense in $L^2(\mu)$, the formula (7) remains valid for $f \in L^2(\mu)$. □

Let $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$. The finite difference Δ_y^N is defined as follows

$$\Delta_y^N f(x) := T_y^N f(x) - e^{-i\frac{a}{2b}y^2} f(x), \quad y \geq 0. \tag{8}$$

Theorem 6. *For* $f \in L^2(\mu)$*, we have*
(i) $\mathscr{F}^M(\Delta_y^N f)(\lambda) = e^{-i\frac{a}{2b}y^2} (\varphi_{\frac{\lambda}{b}}(y) - 1) \mathscr{F}^M(f)(\lambda)$, $\lambda \geq 0$.

(ii) $\|\Delta_y^N\|^2_{L^2(\mu)} = \int_0^\infty (\varphi_{\frac{\lambda}{b}}(y) - 1)^2 |\mathscr{F}^M(f)(\lambda)|^2 d\nu_b(\lambda).$

(iii) *For* $f \in L^1(\mu)$, *such that* $\mathscr{F}^M(f) \in L^1(\nu_b)$, *we have*

$$\Delta_y^N f(x) = \int_0^\infty e^{-i\frac{a}{2b}y^2} (\varphi_{\frac{\lambda}{b}}(y) - 1) \mathscr{F}^M(f)(\lambda) \varphi_\lambda^N(x) d\nu_b(\lambda), \quad a.e. \quad x \geq 0.$$

Proof. (i) Follows from Theorem 5(ii) and (8).
(ii) Follows from (i) and Theorem 3(ii).
(iii) Follows from (i) and Theorem 3(i). □

3. The M-canonical Lipschitz classes

In this section we define the M-canonical Sturm–Liouville Lipschitz classes and we establish versions of the Titchmarsh and Boas-type theorems for the canonical Sturm–Liouville transform $\mathscr{F}^M$.

3.1. Titchmarsh-type theorem

Let $0 < \eta \leq 1$. A function $f \in L^2(\mu)$ is said to be in the M-canonical Sturm–Liouville Lipschitz class, denoted by $\mathrm{Lip}^M(\eta, 2)$, if

$$\omega_N(f, y)_2 := \|\Delta_y^N f\|_{L^2(\mu)} = O(y^\eta) \quad \text{as} \quad y \to 0.$$

Theorem 7. *Let* $f \in L^2(\mu)$ *such that*

$$\int_s^\infty |\mathscr{F}^M(f)(\lambda)|^2 d\nu_b(\lambda) = O(s^{-2\eta}) \quad as \quad s \to \infty. \tag{9}$$

Then $f \in Lip^M(\eta, 2)$.

Proof. Let $f \in L^2(\mu)$. Assume that

$$\int_s^\infty |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda) = O(s^{-2\eta}) \quad \text{as} \quad s \to \infty.$$

By Theorem 6(ii), we have

$$\|\Delta_y^N\|^2_{L^2(\mu)} = \int_0^\infty (1 - \varphi_{\frac{\lambda}{b}}(y))^2 |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda) = I_1 + I_2,$$

where

$$I_1 = \int_0^{\frac{1}{y}} (1 - \varphi_{\frac{\lambda}{b}}(y))^2 |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda),$$

and

$$I_2 = \int_{\frac{1}{y}}^\infty (1 - \varphi_{\frac{\lambda}{b}}(y))^2 |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda).$$

Firstly, we use Lemma 1(i) to obtain

$$I_2 \leq 4 \int_{\frac{1}{y}}^\infty |\mathscr{F}(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda) = O(y^{2\eta}) \quad \text{as} \quad y \to 0.$$

To estimate I_1, we use Lemma 1(i) and (ii), whence we deduce that

$$\begin{aligned} I_1 &\leq 2 \int_0^{\frac{1}{y}} (1 - \varphi_{\frac{\lambda}{b}}(y)) |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda) \\ &\leq y^2 \int_0^{\frac{1}{y}} \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda) = I_3 + I_4, \end{aligned}$$

where
$$I_3 = \rho^2 y^2 \int_0^{\frac{1}{y}} |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda),$$
and
$$I_4 = \frac{y^2}{b^2} \int_0^{\frac{1}{y}} \lambda^2 |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda).$$

Note that
$$I_3 \leq \rho^2 y^2 \int_0^{\infty} |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda) = \rho^2 y^2 \|f\|^2_{L^2(\mu)}.$$

Since $2\eta < 2$, we have $I_3 \leq \rho^2 y^{2\eta} \|f\|^2_{L^2(\mu)}$, and therefore
$$I_3 = O(y^{2\eta}) \quad \text{as} \quad y \to 0.$$

On the other hand we put
$$\Psi(t) = \int_t^{\infty} |\mathscr{F}^M(f)(\lambda)|^2 \mathrm{d}\nu_b(\lambda).$$

Using integration by parts, we find that
$$\begin{aligned} I_4 &= -\frac{y^2}{b^2} \int_0^{\frac{1}{y}} \lambda^2 \Psi'(\lambda) \mathrm{d}\lambda \\ &= -\frac{1}{b^2} \Psi\left(\frac{1}{y}\right) + 2\frac{y^2}{b^2} \int_0^{\frac{1}{y}} \lambda \Psi(\lambda) \mathrm{d}\lambda. \end{aligned}$$

Since $\Psi(\lambda) = O(\lambda^{-2\eta})$, we have $\lambda\Psi(\lambda) = O(\lambda^{1-2\eta})$ and
$$\int_0^{\frac{1}{y}} \lambda \Psi(\lambda) \mathrm{d}\lambda = O\left(\int_0^{\frac{1}{y}} \lambda^{1-2\eta} \mathrm{d}\lambda\right) = O(y^{2\eta-2}) \quad \text{as} \quad y \to 0.$$

Then
$$I_4 = O(y^{2\eta}) \quad \text{as} \quad y \to 0.$$

Finally, we conclude that
$$\|\Delta_y^N f\|^2_{L^2(\mu)} = O(y^{2\eta}) \quad \text{as} \quad y \to 0.$$

This ends the proof of the theorem. □

Example 1 (Generalized Bessel potential)**.** Let X_σ, $\sigma > 0$ be the multiplier operator defined by the relation
$$\mathscr{F}^M(X_\sigma f)(\lambda) = \left(1 + \rho^2 + \frac{\lambda^2}{b^2}\right)^{-\sigma/2} \mathscr{F}^M(f)(\lambda), \quad \lambda \geq 0.$$

The operator X_σ is a generalized Bessel potential; and in the Sturm–Liouville case this operator is studied by Bloom and Xu in [29] and by Soltani in [33]. From Theorem 3(ii), the operator X_σ is bounded on $L^2(\mu)$ with
$$\|X_\sigma f\|_{L^2(\mu)} \leq \|f\|_{L^2(\mu)}, \quad f \in L^2(\mu).$$

Then by application of Theorem 7 and the fact that
$$\left(1 + \rho^2 + \frac{\lambda^2}{b^2}\right)^{-\sigma/2} \leq \frac{\lambda^{-\sigma}}{b}, \quad \lambda > 0,$$
we deduce that if f satisfies (9) then
$$X_\sigma f \in \mathrm{Lip}^M(\eta + \sigma, 2), \quad \text{for} \quad 0 < \eta \leq 1 - \sigma.$$

3.2. Boas-type theorem

In this subsection we prove a Boas-type theorem for the canonical $\mathrm{Lip}^M(\eta,\infty)$-Liouville transform $\mathscr{F}^M$, and we give an application to canonical Sturm–Liouville multiplier operators.

At present we define, the Lipschitz class $\mathrm{Lip}^M(\eta,\infty)$. A function $f:\mathbb{R}_+\to\mathbb{R}$ is said belong to $\mathrm{Lip}^M(\eta,\infty)$ for $\eta>0$ if

$$\omega_N(f,y)_\infty := \sup_{x\in\mathbb{R}_+} |\Delta_y^N f(x)| = O(y^\eta) \quad \text{as} \quad y\to 0.$$

Theorem 8. *Let $f\in L^1(\mu)$ and $0<\eta\le 2$. If*

$$\int_0^s \left(\frac{\lambda^2}{b^2}+\rho^2\right)|\mathscr{F}^M(f)(\lambda)|d\nu_b(\lambda) = O(s^{2-\eta}) \quad \textit{for all} \quad s>0, \tag{10}$$

then $\mathscr{F}^M(f)\in L^1(\nu_b)$ and $f\in Lip^M(\eta,\infty)$.

Proof. By (10), there exists a constant $C>0$ such that for all $i\in\mathbb{Z}$, we have

$$\int_{2^i}^{2^{i+1}} \left(\frac{\lambda^2}{b^2}+\rho^2\right)|\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) \le \int_0^{2^{i+1}} \left(\frac{\lambda^2}{b^2}+\rho^2\right)|\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) \le C2^{(2-\eta)(i+1)}.$$

It is clear that

$$\frac{2^{2i}}{b^2}\int_{2^i}^{2^{i+1}} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) \le \int_0^{2^{i+1}} \left(\frac{\lambda^2}{b^2}+\rho^2\right)|\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) \le C2^{(2-\eta)(i+1)},$$

whence it follows that

$$\int_{2^i}^{2^{i+1}} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) \le Cb^2 2^{2-\eta}2^{-i\eta}. \tag{11}$$

Then by (11) we get

$$\begin{aligned}\int_{2^i}^{\infty} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) &= \sum_{j=i}^{\infty}\int_{2^j}^{2^{j+1}} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda)\\ &\le Cb^2 2^{2-\eta}\sum_{j=i}^{\infty}2^{-j\eta} = O(2^{-i\eta}).\end{aligned}$$

When $0<s<\infty$, let $i\in\mathbb{Z}$ such that $2^i\le s<2^{i+1}$. It follows that

$$\begin{aligned}\int_s^{\infty} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) &\le \int_{2^i}^{\infty} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda)\\ &\le C2^{-i\eta} = C2^{\eta}2^{-(i+1)\eta}\\ &\le C2^{\eta}s^{-\eta}.\end{aligned}$$

This proves that

$$\int_s^{\infty} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) = O(s^{-\eta}) \quad \text{for all} \quad s>0. \tag{12}$$

Therefore,

$$\int_s^{\infty} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) < \infty.$$

On the other hand, from [34, page 99] the function $\lambda\mapsto|c(\lambda)|^{-2}$ is continuous on $[0,\infty)$, then $\nu_b([0,s])<\infty$. Thus

$$\int_0^{s} |\mathscr{F}^M(f)(\lambda)|\mathrm{d}\nu_b(\lambda) \le \nu_b([0,s])\|f\|_{L^1(\mu)} < \infty.$$

We conclude that $\mathscr{F}^M(f)\in L^1(\nu_b)$.

Now, let us prove that $f \in \mathrm{Lip}^M(\eta, \infty)$. Let $x \in \mathbb{R}_+$ and $y > 0$. By Theorem 6(iii) we have

$$|\Delta_y^N f(x)| = \left| \int_0^\infty e^{-i\frac{a}{2b}y^2} (\varphi_{\frac{\lambda}{b}}(y) - 1) \mathscr{F}^M(f)(\lambda) \varphi_\lambda^N(x) \mathrm{d}\nu_b(\lambda) \right| \leq J_1 + J_2,$$

where

$$J_1 = \int_0^{\frac{1}{y}} (1 - \varphi_{\frac{\lambda}{b}}(y)) |\mathscr{F}^M(f)(\lambda)| \mathrm{d}\nu_b(\lambda),$$

and

$$J_2 = \int_{\frac{1}{y}}^\infty (1 - \varphi_{\frac{\lambda}{b}}(y)) |\mathscr{F}^M(f)(\lambda)| \mathrm{d}\nu_b(\lambda).$$

By Lemma 1 (ii) we get

$$J_1 \leq y^2 \int_0^{\frac{1}{y}} \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathscr{F}^M(f)(\lambda)| \mathrm{d}\nu_b(\lambda),$$

and by (10) we deduce that

$$J_1 = y^2 O(y^{\eta-2}) = O(y^\eta). \tag{13}$$

On the other hand, by Lemma 1(i), we have

$$J_2 \leq 2 \int_{\frac{1}{y}}^\infty |\mathscr{F}^M(f)(\lambda)| \mathrm{d}\nu_b(\lambda).$$

By (12) we get

$$J_2 = O(y^\eta). \tag{14}$$

Combining (13) and (14) yields $f \in \mathrm{Lip}^M(\eta, \infty)$. $\square$

4. Special cases

Titchmarsh and Boas-type results for the canonical Fourier–Bessel and Fourier–Jacobi transforms are special cases of Theorems 7 and 8.

4.1. The canonical Fourier–Bessel transform

In this case $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$ and $\rho = 0$. The operator $\mathscr{L}^M$ is reduced to the canonical Bessel operator $\mathscr{L}_\alpha^M$:

$$\mathscr{L}_\alpha^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x \right) \frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b} \right).$$

In this case

$$\varphi_\lambda^M(x) = \varphi_\lambda^{\alpha,M}(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha \left(\frac{\lambda x}{b} \right),$$

where j_α is the spherical Bessel function.

We denote by μ_α the measure defined by $\mathrm{d}\mu_\alpha(x) := x^{2\alpha+1}\mathrm{d}x$.

The canonical Fourier–Bessel transform $\mathscr{F}_\alpha^M$ is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathscr{F}_\alpha^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^{\alpha,M}(x) f(x) \mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+.$$

The canonical Fourier–Bessel transform $\mathscr{F}_\alpha^M$ can be regarded as a generalization of the Fourier–Bessel transform. Recently, this transform is the goal of many applications in the harmonic analysis (see [35–39]).

The canonical Fourier–Bessel translation operators are defined for $f \in L^1(\mu_\alpha)$ by

$$T_y^{\alpha,N} f(x) := e^{-i\frac{a}{2b}(x^2+y^2)} \int_0^\infty f(z) e^{i\frac{a}{2b}z^2} w_\alpha(x,y,z) \mathrm{d}\mu_\alpha(z), \quad x, y \in \geq 0,$$

being $w_\alpha(x,y,\cdot)$ the kernel [40,41] given by

$$w_\alpha(x,y,z) = a_\alpha \frac{[(x+y)^2 - z^2]^{\alpha-\frac{1}{2}}[z^2-(x-y)^2]^{\alpha-\frac{1}{2}}}{2^{2\alpha-1}(xyz)^{2\alpha}} \mathbf{1}_{(|x-y|,x+y)}(z),$$

where

$$a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}.$$

The finite difference $\Delta_y^{\alpha,N}$ is defined as follows

$$\Delta_y^{\alpha,N} f(x) := T_y^{\alpha,N} f(x) - e^{-i\frac{a}{2b}y^2} f(x), \quad y \geq 0.$$

A function $f \in L^2(\mu_\alpha)$ is said belong to $\mathrm{Lip}_\alpha^M(\eta,2)$ for $0<\eta\leq 1$ if

$$\omega_{\alpha,N}(f,y)_2 := \|\Delta_y^{\alpha,N} f\|_{L^2(\mu)} = O(y^\eta) \quad \text{as} \quad y \to 0.$$

A function $f \in L^1(\mu_\alpha)$ is said belong to $\mathrm{Lip}_\alpha^M(\eta,\infty)$ for $\eta>0$ if

$$\omega_{\alpha,N}(f,y)_\infty := \sup_{x\in\mathbb{R}_+} |\Delta_y^{\alpha,N} f(x)| = O(y^\eta) \quad \text{as} \quad y \to 0.$$

Corollary 1. *Let $f \in L^2(\mu_\alpha)$ such that*

$$\int_s^\infty |\mathscr{F}_\alpha^M(f)(\lambda)|^2 d\mu_\alpha(\lambda) = O(s^{-2\eta}) \quad \textit{as} \quad s \to \infty.$$

Then $f \in \mathrm{Lip}_\alpha^M(\eta,2)$.

Corollary 2. *Let $f \in L^1(\mu_\alpha)$ and $0<\eta\leq 2$. If*

$$\int_0^s \lambda^2 |\mathscr{F}_\alpha^M(f)(\lambda)| d\mu_\alpha(\lambda) = O(s^{2-\eta}) \quad \textit{for all} \quad s>0,$$

then $\mathscr{F}_\alpha^M(f) \in L^1(\mu_\alpha)$ and $f \in \mathrm{Lip}_\alpha^M(\eta,\infty)$.

4.2. The canonical Fourier–Jacobi transform

In this case $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\eta+1}(x)$, $\alpha > \beta \geq -1/2$ and $\rho = \alpha+\beta+1$. The operator $\mathscr{L}^M$ is reduced to the canonical Jacobi operator $\mathscr{L}_{(\alpha,\beta)}^M$:

$$\mathscr{L}_{(\alpha,\beta)}^M := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(h_{\alpha,\beta}(x) - 2i\frac{a}{b}x\right)\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}xh_{\alpha,\beta}(x) + i\frac{a}{b}\right),$$

where

$$h_{\alpha,\beta}(x) = (2\alpha+1)\coth(x) + (2\beta+1)\tanh(x).$$

In this case

$$\varphi_\lambda^M(x) = \varphi_\lambda^{(\alpha,\beta),M}(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} \varphi_{\frac{\lambda}{b}}^{(\alpha,\beta)}(x),$$

where $\varphi_\lambda^{(\alpha,\beta)}(x)$ is the Jacobi function given by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho+i\lambda), \alpha+1, -\sinh^2(x)),$$

being ${}_2F_1(a,b,c,z)$ the hypergeometric function.

We denote by $\mu_{\alpha,\beta}$, $\nu_{\alpha,\beta}$ and $\nu_{\alpha,\beta,b}$ the measures defined respectively by

$$d\mu_{\alpha,\beta}(x) := \sin^{2\alpha+1}(x)\cosh^{2\beta+1}(x)dx, \quad d\nu_{\alpha,\beta}(\lambda) := \frac{d\lambda}{2\pi|c_{\alpha,\beta}(\lambda)|^2},$$

and

$$d\nu_{\alpha,\beta,b}(\lambda) := \frac{d\lambda}{2\pi b|c_{\alpha,\beta}(\frac{\lambda}{b})|^2},$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}.$$

The canonical Fourier–Jacobi transform $\mathscr{F}^M_{(\alpha,\beta)}$ is defined for $f \in L^1(\mu_{\alpha,\beta})$ by

$$\mathscr{F}^M_{(\alpha,\beta)}(f)(\lambda) := \int_0^\infty \varphi_\lambda^{(\alpha,\beta),M}(x)f(x)d\mu_{\alpha,\beta}(x), \quad \lambda \geq 0.$$

The canonical Fourier–Jacobi transform $\mathscr{F}^M_{(\alpha,\beta)}$ can be regarded as a generalization of the Fourier–Jacobi transform.

The canonical Fourier–Jacobi translation operators are defined for $f \in L^1(\mu_{\alpha,\beta})$ by

$$T_y^{(\alpha,\beta),N}f(x) := e^{-i\frac{a}{2b}(x^2+y^2)}\int_0^\infty f(z)e^{i\frac{a}{2b}z^2}w_{\alpha,\beta}(x,y,z)d\mu_{\alpha,\beta}(z), \quad x,y \geq 0,$$

being $w_{\alpha,\beta}(x,y,\cdot)$ the kernel [42,43] given by

$$\begin{aligned} w_{\alpha,\beta}(x,y,z) =& a_\alpha \frac{[\cosh(x)\cosh(y)\cosh(z)]^{-(\alpha+\beta+1)}}{[\sinh(x)\sinh(y)\sinh(z)]^{2\alpha}}(1-Q^2(x,y,z))^{\alpha-\frac{1}{2}} \\ &\times {}_2F_1(\alpha+\beta,\alpha-\beta,\alpha+\frac{1}{2},\frac{1}{2}(1-Q(x,y,z)))\chi_{(|x-y|,x+y)}(z), \end{aligned}$$

where

$$Q(x,y,z) = \frac{\cosh^2(x)+\cosh^2(y)+\cosh^2(z)-1}{2\cosh(x)\cosh(y)\cosh(z)}.$$

The finite difference $\Delta_y^{(\alpha,\beta),N}$ is defined as follows

$$\Delta_y^{(\alpha,\beta),N}f(x) := T_y^{(\alpha,\beta),N}f(x) - e^{-i\frac{a}{2b}y^2}f(x), \quad y \in \mathbb{R}_+.$$

A function $f \in L^2(\mu_{\alpha,\beta})$ is said belong to $\mathrm{Lip}^M_{(\alpha,\beta)}(\eta, 2)$ for $0 < \eta \leq 1$ if

$$\omega_{(\alpha,\beta),N}(f,y)_2 := \|\Delta_y^{(\alpha,\beta),N}f\|_{L^2(\mu_{\alpha,\beta})} = O(y^\eta) \quad \text{as} \quad y \to 0.$$

A function $f \in L^1(\mu_{\alpha,\beta})$ is said belong to $\mathrm{Lip}^M_{(\alpha,\beta)}(\eta,\infty)$ for $\eta > 0$ if

$$\omega_{(\alpha,\beta),N}(f,y)_\infty := \sup_{x\in\mathbb{R}_+}|\Delta_y^{(\alpha,\beta),N}f(x)| = O(y^\eta) \quad \text{as} \quad y \to 0.$$

Corollary 3. *Let $f \in L^2(\mu_{\alpha,\beta})$ such that*

$$\int_s^\infty |\mathscr{F}^M_{(\alpha,\beta)}(f)(\lambda)|^2 d\nu_{\alpha,\beta,b}(\lambda) = O(s^{-2\eta}) \quad as \quad s \to \infty.$$

Then $f \in Lip^M_{(\alpha,\beta)}(\eta,\infty)$.

Corollary 4. *Let $f \in L^1(\mu_{\alpha,\beta})$ and $0 < \eta \leq 2$. If*

$$\int_0^s \lambda^2 |\mathscr{F}^M_{(\alpha,\beta)}(f)(\lambda)| d\nu_{\alpha,\beta}(\lambda) = O(s^{2-\eta}) \quad \text{for all} \quad s > 0,$$

then $\mathscr{F}^M_{(\alpha,\beta)}(f) \in L^1(\nu_{\alpha,\beta})$ and $f \in Lip^M_{(\alpha,\beta)}(\eta, \infty)$.

5. Conclusion

In recent years, the scientific community has mastered tools such as the linear canonical transform (LCT) and the quadratic phase Fourier transform, often presented in journals like Optik [25]. However, while these transformations have opened new perspectives, the canonical Sturm–Liouville framework has, until now, represented a more difficult frontier to overcome. While previous transformations, such as the phase-shifted LCT, apply to specific phases, the canonical Sturm–Liouville framework offers much greater generality. It allows the treatment of differential operators where its predecessors were limited. We have successfully established a new Titchmarsh and Boas-type theorems for the transform $\mathscr{F}^M$, based on the generalized Lipschitz classes $\mathrm{Lip}^M(\eta, 2)$ and $\mathrm{Lip}^M(\eta, \infty)$.

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