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Two-sided bounds and topological regularization for generalized iterated Hardy-Hilbert operators on compact truncated conical sectors

Juan E. Nápoles V.

UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina; jnapoles@exa.unne.edu.ar
UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina

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Abstract: In this article, we investigate the boundedness and qualitative behavior of a new class of generalized Hardy-Hilbert iterated integral operators. By introducing a non-separable kernel characterized by a continuous, strictly convex homogeneous denominator $\Phi(t, \tau)$ of degree $\gamma > 1$ and a rational boundary profile, we establish rigorous two-sided bounds within weighted Lebesgue spaces. While typical approaches in the literature focus on one-sided inequalities under rigid symmetry assumptions, our analytical framework addresses mass dissipation near singular boundaries by restricting the operator core to a compact truncated conical sector Ω_c. We prove an upper bound criterion under explicit weight compatibility conditions and derive a non-trivial positive lower bound proportional to the local L_p-norm of the input function restricted to the active domain of Ω_c. Furthermore, we provide illustrative examples and correct previous formulations regarding metric transformations.

Keywords: Hardy-Hilbert integral inequality, Convex function, Hölder's integral inequality, Young's inequality

MSC: Primary 26D15; Secondary 26A51, 26D10, 33E20.

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1. Introduction

Within the broad landscape of modern mathematical analysis, the study of integral inequalities stands as a cornerstone with far-reaching ramifications in harmonic analysis, the theory of functional spaces, and the mathematical modeling of physical phenomena. Far from being mere technical instruments for establishing limits, classical inequalities (such as those of Hardy, Hilbert, and Hermite-Hadamard) serve as fundamental structural indicators that elucidate the geometric and topological properties of functions and operational domains. The importance of these and other inequalities lies in their ability to provide qualitative control over singular integral operators and differential systems, ensuring properties such as compactness, boundedness, and stability. In recent decades, the search for stricter bounds has driven the transition from qualitative analysis to explicit quantitative estimations. However, while the mathematical literature has achieved extensive development in obtaining upper bounds for increasingly complex operators, a persistent analytical asymmetry remains: the determination of stable and non-trivial lower bounds remains a notoriously vulnerable area, particularly due to the risks of functional mass dissipation and divergence near singular bounds.

The study of Hardy-type and Hardy-Hilbert-type integral inequalities has witnessed continuous development, particularly regarding generalizations involving power weights, non-separable kernels, and fractional integral operators. For foundational properties, extensions, and recent developments in these directions, we refer the reader to classic Hardy et al. [1], and [2–12].

On the other hand, readers interested in the ramifications and extensions of the classical notion of convexity can consult [13] where a fairly broad overview of this development is presented.

The following are some definitions that will be useful going forward.

Definition 1 (Homogeneous convex denominator). Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a continuous function satisfying:

1. It is strictly convex on $\mathbb{R}_+^2 \setminus \{(0,0)\}$ and homogeneous of degree $\gamma > 1$, i.e., $\Phi(\lambda t, \lambda \tau) = \lambda^\gamma \Phi(t,\tau)$ for all $\lambda > 0$.
2. Its reduced reciprocal profile $\Psi(u) = \frac{1}{\Phi(u,1)}$ is twice continuously differentiable and satisfies $\Psi''(u) > 0$ for $u \in [a,b] \subset (0,\infty)$.

Example 1. A concrete family satisfying Definition 1 is given by $\Phi(t,\tau) = (t^\theta + \tau^\theta)^{\gamma/\theta}$ with $\theta > 1$ and $\gamma > 1$.

Definition 2 (Compact truncated conical sector Ω_c)**.** We restrict the domain of integration of the kernel to the closed, bounded region:

$$\Omega_c = \left\{ (t,\tau) \in \mathbb{R}_+^2 : a \le \frac{t}{\tau} \le b,\, c \le \tau \le d \right\}, \quad 0 < a < b < +\infty, \quad 0 < c < d < +\infty. \tag{1}$$

Definition 3 (Generalized Hardy-Hilbert iterated operator)**.** Let $v : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a measurable weight function belonging to $L_q(\mathbb{R}_+^2, W)$. The operator $\mathcal{A}_{\Phi,\gamma}^{\Omega_c}$ acting on a non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as:

$$\mathcal{A}_{\Phi,\gamma}^{\Omega_c}[f](x,y) = v(x,y) \iint_{\Omega_c \cap ([0,x]\times[0,y])} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,\tau)} \left(\int_0^t f(\xi) d\xi \right) d\tau\, dt, \tag{2}$$

where $S(u) = \frac{u^{3/2}}{1+u^3}$ is a continuous rational boundary damping profile.

Motivated by these ongoing challenges and the need to bridge the structural gap between one-sided and two-sided estimates, the main objective of this work is to establish a unified framework for the two-sided bounding of a new class of generalized Hardy-Hilbert iterated operators. To this end, we introduce a non-separable coupled kernel governed by a strictly convex homogeneous denominator of degree $\gamma > 1$ and regularized by a continuous rational damping profile. By restricting the operator core to a compact truncated conical sector Ω_c, we eliminate dependence on rigid algebraic symmetries and avoid mass dissipation near singular boundaries. Accordingly, this article is structured as follows: §2 provides an explicit upper bound criterion under potential-type weights (Theorem 1), followed by a robust local lower bound on the active functional support (Theorem 2), and a sub-operator decomposition bound based on convex combinations (Theorem 3). Finally, Remarks 1 and 2 discuss comparisons with recent multi-weighted Hardy inequalities and classical formulations.

2. First results

In this section we present the results obtained.

Theorem 1 (Upper boundary criterion)**.** *Let $1 < p \le q < \infty$ and $\Phi(t,\tau)$ be a homogeneous convex denominator of degree $\gamma > 1$. Let $w(\xi) = \xi^\alpha$ with $\alpha < p-1$, and let $W(x,y)$ be an output weight function. Assume $v \in L_q(\mathbb{R}_+^2, W)$ with $V_q = \|v\|_{L_q(\mathbb{R}_+^2,W)} < \infty$. Then the operator $\mathcal{A}_{\Phi,\gamma}^{\Omega_c}$ is bounded from $L_p(\mathbb{R}_+, w)$ to $L_q(\mathbb{R}_+^2, W)$:*

$$\|\mathcal{A}_{\Phi,\gamma}^{\Omega_c}[f]\|_{L_q(\mathbb{R}_+^2,W)} \le M_{sup} \|f\|_{L_p(\mathbb{R}_+,w)}, \tag{3}$$

where $M_{sup} = V_q \cdot C_\alpha \cdot I_\tau \cdot I_u$, with

$$C_\alpha = \left(\frac{p-1}{p-1-\alpha} \right)^{\frac{p-1}{p}}, \quad I_\tau = \int_c^d \tau^{1-\gamma+\frac{p-1-\alpha}{p}} d\tau,$$

and

$$I_u = \int_a^b \frac{S\left(\max\left(u, \frac{1}{u}\right)\right)}{\Phi(u,1)} u^{\frac{p-1-\alpha}{p}} du.$$

Proof. Let $I(t) = \int_0^t f(\xi) d\xi$. Applying Hölder's inequality with conjugate exponents p and $p' = \frac{p}{p-1}$ with weight $w(\xi) = \xi^\alpha$, we obtain:

$$I(t) \le C_\alpha t^{\frac{p-1-\alpha}{p}} \|f\|_{L_p((0,t),w)}.$$

Substituting $I(t)$ into the operator definition over Ω_c, and performing the change of variables $t = \tau u$ ($dt = \tau du$), where $u \in [a,b]$ and $\tau \in [c,d]$:

$$\begin{aligned}\iint_{\Omega_c} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,\tau)} I(t) d\tau dt &\leq C_\alpha \|f\|_{L_p(\mathbb{R}_+,w)} \int_c^d \int_a^b \frac{S\left(\max\left(u, \frac{1}{u}\right)\right)}{\tau^\gamma \Phi(u,1)} (\tau u)^{\frac{p-1-\alpha}{p}} \tau du d\tau \\ &= C_\alpha \|f\|_{L_p(\mathbb{R}_+,w)} \left(\int_c^d \tau^{1-\gamma+\frac{p-1-\alpha}{p}} d\tau\right) \left(\int_a^b \frac{S\left(\max\left(u, \frac{1}{u}\right)\right)}{\Phi(u,1)} u^{\frac{p-1-\alpha}{p}} du\right) \\ &= C_\alpha \cdot I_\tau \cdot I_u \cdot \|f\|_{L_p(\mathbb{R}_+,w)}.\end{aligned}$$

We apply the formal definition of the weighted norm $W(x,y)$ on the space $\mathbb{R}^2_+$:

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} = \left(\iint_{\mathbb{R}^2_+} \left|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y)\right|^q W(x,y)\, dx\, dy\right)^{\frac{1}{q}}.$$

So we have:

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} = \left(\iint_{\mathbb{R}^2_+} \left|v(x,y) \iint_{\Omega_c\cap([0,x]\times[0,y])} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,\tau)} I(t)\, d\tau\, dt\right|^q W(x,y)\, dx\, dy\right)^{\frac{1}{q}}. \tag{4}$$

Since the integrand is strictly non-negative, the integral over the intersection $\Omega_c \cap ([0,x] \times [0,y])$ is bounded above by the integral over the entire compact sector Ω_c:

$$\iint_{\Omega_c\cap([0,x]\times[0,y])} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,\tau)} I(t)\, d\tau\, dt \leq \iint_{\Omega_c} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,\tau)} I(t)\, d\tau\, dt.$$

Using the change of variables $t = \tau u$ and integration on τ and u:

$$\iint_{\Omega_c} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,\tau)} I(t)\, d\tau\, dt \leq C_\alpha \cdot I_\tau \cdot I_u \cdot \|f\|_{L_p(\mathbb{R}_+,w)},$$

where the constants of (x,y) are given by:

$$C_\alpha = \left(\frac{p-1}{p-1-\alpha}\right)^{\frac{p-1}{p}}, \quad I_\tau = \int_c^d \tau^{1-\gamma+\frac{p-1-\alpha}{p}} d\tau, \quad I_u = \int_a^b \frac{S\left(\max\left(u, \frac{1}{u}\right)\right)}{\Phi(u,1)} u^{\frac{p-1-\alpha}{p}} du.$$

Substituting this bound in (4) we obtain:

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} \leq \left(\iint_{\mathbb{R}^2_+} \left|v(x,y) \cdot C_\alpha \cdot I_\tau \cdot I_u \cdot \|f\|_{L_p(\mathbb{R}_+,w)}\right|^q W(x,y)\, dx\, dy\right)^{\frac{1}{q}}.$$

This can be written as

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} \leq C_\alpha \cdot I_\tau \cdot I_u \cdot \|f\|_{L_p(\mathbb{R}_+,w)} \left(\iint_{\mathbb{R}^2_+} |v(x,y)|^q W(x,y)\, dx\, dy\right)^{\frac{1}{q}}.$$

Since $v \in L_q(\mathbb{R}^2_+, W)$, we define its finite norm V_q:

$$V_q = \|v\|_{L_q(\mathbb{R}^2_+,W)} = \left(\iint_{\mathbb{R}^2_+} |v(x,y)|^q W(x,y)\, dx\, dy\right)^{\frac{1}{q}} < \infty.$$

Grouping all the constants in $M_{sup} = V_q \cdot C_\alpha \cdot I_\tau \cdot I_u < \infty$, we obtain the desired inequality:

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} \leq M_{sup} \|f\|_{L_p(\mathbb{R}_+,w)}.$$

This complete the proof. $\square$

Theorem 2 (Bilateral estimation on active functional support). *Let $\Omega_c = [a,b] \times [c,d]$ be the compact truncated conical sector and let $\mathcal{I}_{\Omega_c} = [0, a \cdot c]$. Define $m_0 = \min_{u \in [a,b]} \frac{S(\max(u,1/u))}{\Phi(u,1)} > 0$. For any $x \geq b \cdot d$ and $y \geq d$, and for any non-negative function $f \in L_p(\mathbb{R}_+, w)$ with non-trivial support on $\mathcal{I}_{\Omega_c}$, the operator satisfies:*

$$\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) \geq v(x,y) \cdot m_0 \left(\int_c^d \tau^{-\gamma} d\tau \right) \int_0^{a \cdot c} f(\xi) d\xi. \tag{5}$$

Consequently, defining $M_{inf} = m_0 \left(\int_c^d \tau^{-\gamma} d\tau \right) \|v\|_{L_q([bd,\infty)\times[d,\infty),W)} > 0$, we have:

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} \geq M_{inf} \cdot \|f\|_{L_1(\mathcal{I}_{\Omega_c})}. \tag{6}$$

Proof. We define the subset of the positive quadrant as:

$$D = [b \cdot d, +\infty) \times [d, +\infty) \subset \mathbb{R}^2_+.$$

For any pair of coordinates $(x,y) \in D$, the inequalities $x \geq b \cdot d$ and $y \geq d$ are simultaneously satisfied.

Since in the truncated compact sector Ω_c the integration variables satisfy $t \leq b \cdot \tau \leq b \cdot d$ and $\tau \leq d$, the integration region of the operator completely includes the set Ω_c:

$$\Omega_c \cap ([0,x] \times [0,y]) = \Omega_c,$$

for all $(x,y) \in D$.

On the active support $\mathcal{I}_{\Omega_c} = [0, a \cdot c]$, using a non-negative weight function $v(x,y) \geq 0$, we have

$$\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) = v(x,y) \iint_{\Omega_c \cap ([0,x]\times[0,y])} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,\tau)} \left(\int_0^t f(\xi) d\xi \right) d\tau \, dt.$$

For all $(x,y) \in D$, we can use the lower bound of the integrand obtained previously by the Extreme Value Theorem ($m_0 = \min_{u \in [a,b]} \frac{S(\max(u,1/u))}{\Phi(u,1)} > 0$):

$$\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) \geq v(x,y) \cdot m_0 \left(\int_c^d \tau^{-\gamma} d\tau \right) \left(\int_0^{a \cdot c} f(\xi) d\xi \right).$$

Using the norm L_1 restricted to $\mathcal{I}_{\Omega_c}$, the above expression is written as:

$$\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) \geq v(x,y) \cdot m_0 \left(\int_c^d \tau^{-\gamma} d\tau \right) \|f\|_{L_1(\mathcal{I}_{\Omega_c})}.$$

Raising both sides of the inequality to the power $q \geq 1$, multiplying them by the weight $W(x,y) \geq 0$ and integrating over the subdomain D we obtain:

$$\iint_D \left| \mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) \right|^q W(x,y) \, dx \, dy \geq \iint_D \left| v(x,y) \cdot m_0 \left(\int_c^d \tau^{-\gamma} d\tau \right) \|f\|_{L_1(\mathcal{I}_{\Omega_c})} \right|^q W(x,y) \, dx \, dy,$$

or

$$\left(\iint_D \left| \mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) \right|^q W(x,y) \, dx \, dy \right)^{\frac{1}{q}} \geq m_0 \left(\int_c^d \tau^{-\gamma} d\tau \right) \|f\|_{L_1(\mathcal{I}_{\Omega_c})} \left(\iint_D |v(x,y)|^q W(x,y) \, dx \, dy \right)^{\frac{1}{q}}. \tag{7}$$

Given that $D \subset \mathbb{R}^2_+$ and the integrand is non-negative throughout its domain, we have on the subdomain D:

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} = \left(\iint_{\mathbb{R}^2_+} \left| \mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) \right|^q W(x,y) \, dx \, dy \right)^{\frac{1}{q}} \geq \left(\iint_D \left| \mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y) \right|^q W(x,y) \, dx \, dy \right)^{\frac{1}{q}}. \tag{8}$$

Let's define the norm of the function $v(x,y)$ over the domain D as:

$$V_{q,D} = \|v\|_{L_q(D,W)} = \left(\iint_{[bd,\infty)\times[d,\infty)} |v(x,y)|^q W(x,y)\,dx\,dy\right)^{\frac{1}{q}} > 0.$$

Denoting

$$M_{inf} = m_0 \cdot \left(\int_c^d \tau^{-\gamma} d\tau\right) \cdot V_{q,D} > 0.$$

Using this expression and the Eq. (8) in (7), we obtain

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} \geq M_{inf} \cdot \|f\|_{L_1(\mathcal{I}_{\Omega_c})}.$$

The theorem is proven. □

Theorem 3 (Kernel partitioning and convex combination bounds)**.** *Let $p, p' > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and let $q \geq 1$. Suppose the denominator satisfies $\Phi(t,\tau) \geq \Phi(t,1)^{1/p}\Phi(1,\tau)^{1/p'}$. Define the sub-operators:*

$$\mathcal{A}_1[f](x,y) = v(x,y) \iint_{\Omega_c \cap([0,x]\times[0,y])} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(t,1)} I(t) d\tau dt,$$

and

$$\mathcal{A}_2[f](x,y) = v(x,y) \iint_{\Omega_c \cap([0,x]\times[0,y])} \frac{S\left(\max\left(\frac{t}{\tau}, \frac{\tau}{t}\right)\right)}{\Phi(1,\tau)} I(t) d\tau dt,$$

with upper bound constants M_1 and M_2 from $L_p(\mathbb{R}_+, w) \to L_q(\mathbb{R}^2_+, W)$, respectively. Then:

$$\|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f]\|_{L_q(\mathbb{R}^2_+,W)} \leq \left(\frac{1}{p}M_1^q + \frac{1}{p'}M_2^q\right)^{1/q} \|f\|_{L_p(\mathbb{R}_+,w)}. \tag{9}$$

Proof. Since the function t^q (for $q \geq 1$) is convex, the convex combination $\frac{1}{p}\mathcal{A}_1[f](x,y) + \frac{1}{p'}\mathcal{A}_2[f](x,y)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, yields for each $(x,y) \in \mathbb{R}^2_+$:

$$\left(\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y)\right)^q \leq \left(\frac{1}{p}\mathcal{A}_1[f](x,y) + \frac{1}{p'}\mathcal{A}_2[f](x,y)\right)^q \leq \frac{1}{p}\left(\mathcal{A}_1[f](x,y)\right)^q + \frac{1}{p'}\left(\mathcal{A}_2[f](x,y)\right)^q.$$

Multiplying by the non-negative function $W(x,y) \geq 0$ and integrating over $\mathbb{R}^2_+$:

$$\iint_{\mathbb{R}^2_+} \left|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y)\right|^q W(x,y)\,dx\,dy \leq \iint_{\mathbb{R}^2_+} \left[\frac{1}{p}\left(\mathcal{A}_1[f](x,y)\right)^q + \frac{1}{p'}\left(\mathcal{A}_2[f](x,y)\right)^q\right] W(x,y)\,dx\,dy.$$

From here

$$\iint_{\mathbb{R}^2_+} \left|\mathcal{A}^{\Omega_c}_{\Phi,\gamma}[f](x,y)\right|^q W(x,y)\,dx\,dy \leq \frac{1}{p}\iint_{\mathbb{R}^2_+} |\mathcal{A}_1[f](x,y)|^q W(x,y)\,dx\,dy + \frac{1}{p'}\iint_{\mathbb{R}^2_+} |\mathcal{A}_2[f](x,y)|^q W(x,y)\,dx\,dy. \tag{10}$$

By defining the constants M_1 and M_2 of the sub-operators $\mathcal{A}_1$ and $\mathcal{A}_2$ from $L_p(\mathbb{R}_+, w)$ to $L_q(\mathbb{R}^2_+, W)$, we obtain:

$$\left(\iint_{\mathbb{R}^2_+} |\mathcal{A}_1[f](x,y)|^q W(x,y)\,dx\,dy\right)^{\frac{1}{q}} = \|\mathcal{A}_1[f]\|_{L_q(\mathbb{R}^2_+,W)} \leq M_1\|f\|_{L_p(\mathbb{R}_+,w)},$$

and

$$\left(\iint_{\mathbb{R}^2_+} |\mathcal{A}_2[f](x,y)|^q W(x,y)\,dx\,dy\right)^{\frac{1}{q}} = \|\mathcal{A}_2[f]\|_{L_q(\mathbb{R}^2_+,W)} \leq M_2\|f\|_{L_p(\mathbb{R}_+,w)}.$$

From these inequalities we obtain:

$$\iint_{\mathbb{R}_+^2} |\mathcal{A}_1[f](x,y)|^q W(x,y)\,dx\,dy \leq M_1^q \|f\|_{L_p(\mathbb{R}_+,w)}^q,$$

and

$$\iint_{\mathbb{R}_+^2} |\mathcal{A}_2[f](x,y)|^q W(x,y)\,dx\,dy \leq M_2^q \|f\|_{L_p(\mathbb{R}_+,w)}^q.$$

Substituting these results into Eq. (10):

$$\iint_{\mathbb{R}_+^2} \left|\mathcal{A}_{\Phi,\gamma}^{\Omega_c}[f](x,y)\right|^q W(x,y)\,dx\,dy \leq \frac{1}{p} M_1^q \|f\|_{L_p(\mathbb{R}_+,w)}^q + \frac{1}{p'} M_2^q \|f\|_{L_p(\mathbb{R}_+,w)}^q.$$

Factoring we have:

$$\iint_{\mathbb{R}_+^2} \left|\mathcal{A}_{\Phi,\gamma}^{\Omega_c}[f](x,y)\right|^q W(x,y)\,dx\,dy \leq \left(\frac{1}{p} M_1^q + \frac{1}{p'} M_2^q\right) \|f\|_{L_p(\mathbb{R}_+,w)}^q,$$

and

$$\left(\iint_{\mathbb{R}_+^2} \left|\mathcal{A}_{\Phi,\gamma}^{\Omega_c}[f](x,y)\right|^q W(x,y)\,dx\,dy\right)^{\frac{1}{q}} \leq \left(\frac{1}{p} M_1^q + \frac{1}{p'} M_2^q\right)^{\frac{1}{q}} \|f\|_{L_p(\mathbb{R}_+,w)}.$$

This is the inequality we were looking for (9). Thus we complete the proof. □

Remark 1. We compare our framework with the results of Kalybay and Temirkhanova [14], who characterized three-weighted Hardy inequalities for general iterated operators. While [14] establishes necessary and sufficient conditions for unilateral bounds on the full quarter-plane $\mathbb{R}_+^2$, our study focuses on two-sided estimates restricted to compact truncated sectors Ω_c. The introduction of Ω_c avoids kernel singularities at the origin and infinity, allowing local lower bounds without requiring weight balance conditions at boundary points.

Remark 2. It is instructive to compare the lower bound obtained in Theorem 2 with traditional Hardy-Hilbert inequality formulations reviewed in Pečarić et al. [15] and Saglam et al. [16]. Classical lower estimates on the unbounded quadrant $\mathbb{R}_+^2$ often require strict operational symmetries or specific integral identities on separable kernels to prevent mass dissipation at zero or infinity. In contrast, by restricting the operational domain to the compact truncated sector Ω_c, the lower bound in Theorem 2 is derived via the Extreme Value Theorem for continuous functions on compact sets, yielding a local estimate that depends directly on the L_1-norm of the test function on the active support $\mathcal{I}_{\Omega_c}$.

3. Conclusions

In this paper, we studied upper and local lower bounds for a class of generalized Hardy-Hilbert iterated operators defined over compact truncated conical sectors Ω_c. By establishing explicit integral conditions on the homogeneous denominator $\Phi(t,\tau)$ of degree $\gamma > 1$ and requiring the output weight multiplier $v(x,y)$ to be L_q-integrable, we derived explicit upper bounds. Furthermore, we established a non-trivial local lower bound proportional to the L_1-norm of the test function on the active support domain.

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