Open Journal of
Mathematical Sciences

PSR Press

Article

Using Pell equation solutions to find all triangular numbers multiple of other triangular numbers

Vladimir Pletser

European Space Agency (ret.), Noordwijk, The Netherlands; Pletservladimir@gmail.com

Received: 31 Aug 2023; Revised: 19 Sep 2025; Accepted: 01 Mar 2026; Published: 06 May 2026

Abstract: For all positive non-square integer multipliers, there are infinitely many triangular numbers that are multiples of other triangular numbers. With a simple change of variables, one obtains a Pell equation, whose odd solutions provide the indices of the many infinitely triangular numbers multiple of other triangular numbers. General algebraic expressions of fundamental solutions of Pell equations are found for the multiplier expressed in function of the closest integer square. Finally, recurrent relations yielding the triangular numbers and their multiples and indices are calculated for non-square multipliers.

Keywords: triangular number, multiple of triangular number, recurrent relation, pell equation, fundamental solution

MSC: Primary 11D09; Secondary 11A25, 11B37, 11D45.

1. Introduction

Let $T_\xi = \frac{\xi(\xi+1)}{2}$ denote the triangular number [1,2] of ξ. Several authors [3–8] have investigated the Diophantine equation

$$T_\xi = kT_t, \tag{1}$$

searching for triangular numbers that are multiples of other triangular numbers. Historical accounts can also be found in [9](p. 585). Recently, Pletser showed [10] that, for non-square integers k, the four variables t, ξ, T_t, and T_ξ can be represented by recurrent relations involving a rank r and two parameters κ and γ. The rank r is defined as the number of successive solutions t of (1)such that their ratios are slowly decreasing without jumps. The parametersκ and γ are, respectively, the sum and the product of the $(r-1)^{\text{th}}$and r^{th} solutions t of (1).

We only consider solutions of (1) for $k > 1$, as for $k = 0$ and $k = 1$, the solutions are trivial, respectively $\xi = 0$ and $\xi = t$ for any positive integer t.

Triangular numbers T_t are defined for positive integers t. This definition can be extended to negative integers, yielding $T_{-t} = T_{t-1}$.

In this paper we investigate how to find all solutions of (1) using the method of resolution of Pell equations associated with (1). We show that the rank r and parameters κ and γ of recurrent relations can be deduced from fundamental solutions of Pell equations. §2 introduces the rank r and recurrent relations. §3 gives a short reminder on how to find solutions of Pell equations. In §4, Pell equation methods are applied to find all multiples of triangular numbers that are triangular numbers. In certain cases, general expressions of fundamental solutions of the Pell equations associated with (1) are given for the multiplier k expressed as $k = s^2 + \sigma$, with σ either an integer or an integer function of s. Recurrent relations yielding t, ξ, T_t, and T_ξ are deduced for non-square k.

2. Rank and recurrent relations

Sequences of t, ξ, T_t, and T_ξ solutions of (1) for $k = 2, 3, 5, 6, 7, 8, 10$ can be found in the Online Encyclopedia of Integer Sequences (OEIS), as shown in Table 1.

Note that $(t_0, \xi_0) = (0, 0)$ is always a first solution of (1) for all non-square k.

Table 1. OEIS [11] references of sequences of integer solutions of (1) for $k = 2, 3, 5, 6, 7, 8, 10$

k	2	3	5	6	7	8	10
t	A053141	A061278	A077259	A077288	A077398	A336623	A341893
ξ	A001652	A001571	A077262	A077291	A077401	A336625	A341895
T_t	A075528	A076139	A077260	A077289	A077399	A336624	A068085
T_ξ	A029549	A076140	A077261	A077290	A077400	A336626	—

Let us consider the two cases $k = 3$ and $k = 6$, yielding the successive solution pairs (t_n, ξ_n) of (1) as shown in Table 2.

Table 2. Solution pairs (t_n, ξ_n) of (1) and ratios for $k = 3$ and 6

	$k = 3$			$k = 6$			
n	t_n	ξ_n	t_n/t_{n-1}	t_n	ξ_n	t_n/t_{n-1}	t_n/t_{n-2}
0	0	0		0	0		
1	1	2	-	1	3	-	-
2	5	9	5	3	8	3	-
3	20	35	4	14	35	4.66667	14
4	76	132	3.8	34	84	2.42857	11.33333
5	285	494	3.75	143	351	4.20588	10.21429
6	1065	1845	3.73684	341	836	2.38461	10.02941

We indicate the ratios t_n/t_{n-1} for both cases and t_n/t_{n-2} for $k = 6$. For $k = 3$, the ratio t_n/t_{n-1} varies between close values, from 5 down to 3.737, while for $k = 6$, the ratio t_n/t_{n-1} alternates between values 3 ... 2.385 and 4.667 ... 4:206, while the ratio t_n/t_{n-2} decreases more regularly from 14 to 10.029 (corresponding approximately to the product of the alternating values of the ratio t_n/t_{n-1}). We call *rank* r the integer such that the ratio t_n/t_{n-r} is approximately constant or, better, decreases regularly and monotonically (more precise definitions are given further).

So, here, cases $k = 3$ and $k = 6$ have, respectively, ranks $r = 1$ and 2.

The rank r is the index of t_r and ξ_r solutions of (1) such that

$$\kappa = t_r + t_{r-1} = \xi_r - \xi_{r-1} - 1, \tag{2}$$

where κ is a constant, different for each k [10]. The *rank* r is also such that

$$\frac{t_{2r} - t_{r-1}}{t_r} = 2\kappa + 3. \tag{3}$$

For $k = 6$ and $r = 2$, $\kappa = t_2 + t_1 = 3 + 1 = 4$, and $\kappa = \xi_2 - \xi_1 - 1 = 8 - 3 - 1 = 4$, yielding $2\kappa + 3 = 11$. The ranks r are given in the sequence A341894 in OEIS [11].

Pletser deduced [10] four recurrent equations for t_n, ξ_n, T_{t_n}, and T_{ξ_n}:

$$\begin{aligned}
t_n &= 2(\kappa+1)t_{n-r} - t_{n-2r} + \kappa && (4)\\
\xi_n &= 2(\kappa+1)\xi_{n-r} - \xi_{n-2r} + \kappa && (5)\\
T_{t_n} &= \left(4(\kappa+1)^2 - 2\right) T_{t_{n-r}} - T_{t_{n-2r}} + (T_\kappa - \gamma) && (6)\\
T_{\xi_n} &= \left(4(\kappa+1)^2 - 2\right) T_{\xi_{n-r}} - T_{\xi_{n-2r}} + k(T_\kappa - \gamma), && (7)
\end{aligned}$$

where coefficients are functions of the two constants κ and γ, respectively, the sum (2) and the product $\gamma = t_{r-1}t_r$. Note that the first three recurrence relations (4) to (6) are independent of k.

3. Pell equations: A reminder

The Diophantine bivariate quadratic equation

$$X^2 - DY^2 = N, \tag{8}$$

with integers X, Y, D, N and square-free D is called the Pell equation. Several mathematicians have investigated this equation; see historical accounts in [9] (p. 340) and [12–15]. Treatments and solutions are described in several classical textbooks, e.g., [16–19] and references therein. We remind here some general formulas and how to calculate solutions. Details can be found in references.

For $N = 1$ we call (8) the simple Pell equation

$$x^2 - Dy^2 = 1. \tag{9}$$

This equation admits the obvious trivial solution $(x_0, y_0) = (1, 0)$ and infinitely many solutions given by

$$(x_n, y_n) = \left(\frac{(x_f + \sqrt{D}y_f)^n + (x_f - \sqrt{D}y_f)^n}{2}, \frac{(x_f + \sqrt{D}y_f)^n - (x_f - \sqrt{D}y_f)^n}{2\sqrt{D}} \right), \tag{10}$$

where n are positive integers and (x_f, y_f) is the least solution to (9), i.e., the smallest integer solution different from the trivial solution, $x_f > 1, y_f > 0$. We call this least solution the fundamental solution (x_f, y_f), which yields directly three other solutions $(-x_f, y_f)$, $(x_f, -y_f)$ and $(-x_f, -y_f)$. Lagrange devised a method to find the fundamental solution, based on the continued fraction expansion of the quadratic irrational $\sqrt{D}$, that can be summarized as follows. One computes the j^{th} convergent (p_j/q_j) of the continued fraction $[\alpha_0; \alpha_1, ..., \alpha_j, \alpha_{j+1}, ...]$ of $\sqrt{D}$, with $\alpha_0 = \left\lfloor \sqrt{D} \right\rfloor$, i.e., the greatest integer $\leq \sqrt{D}$. This continued fraction becomes periodic after the following term, $\alpha_{j+1} = 2\alpha_0$ if $\sqrt{D}$ is a quadratic irrational. The recurrence relations

$$p_i = \alpha_i p_{i-1} + p_{i-2} \ ; \ \ q_i = \alpha_i q_{i-1} + q_{i-2},$$

yield the terms p_i and q_i of the convergent, with $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$ and $q_{-1} = 0$. The fundamental solution is then $(x_f, y_f) = (p_j, q_j)$ if j is odd, or $(x_f, y_f) = (p_{2j+1}, q_{2j+1})$ if j is even.

For $N \neq 1$, (8) is called the generalized Pell equation[19], which can have either no solution, or one, or several fundamental solutions $\left(X_{fi}, Y_{fi}\right)$, with positive integers i such that $1 \leq i \leq \rho$, where ρ is the total number of fundamental solutions admitted by (8). All integer solutions, if they exist, are found on double infinite branches that can be expressed in terms of the fundamental solution(s) $\left(X_{fi}, Y_{fi}\right)$ and $\left(-X_{fi}, Y_{fi}\right)$. Methods to calculate the fundamental solution(s) of the generalized Pell equation (see, e.g., [16,18–24] and references therein) are all based on Lagrange's method of continued fractions, sometimes adapted [25]. The nearest integer continued fraction method and the Lagrange-Mollin-Matthews method [24] are used further to calculate the fundamental solutions of, respectively, the simple and the generalized Pell equations.

Once fundamental solutions are known, the other solutions (X_n, Y_n) of (8) are calculated by

$$X_n + \sqrt{D}Y_n = \pm(X_{fi} + \sqrt{D}Y_{fi})(x_f + \sqrt{D}y_f)^n, \tag{11}$$

for a proper choice of sign [19], yielding respectively, for $n = 0, 1, 2$ (assuming a $+$ sign),

$$(X_0, Y_0) = \left(X_{fi}, Y_{fi}\right), \tag{12}$$

$$(X_1, Y_1) = \left(\left(X_{fi}x_f + DY_{fi}y_f\right), \left(X_{fi}y_f + Y_{fi}x_f\right)\right), \tag{13}$$

$$(X_2, Y_2) = \left(\left(X_{fi}\left(x_f^2 + Dy_f^2\right) + 2DY_{fi}x_f y_f\right), \left(Y_{fi}\left(x_f^2 + Dy_f^2\right) + 2X_{fi}x_f y_f\right)\right). \tag{14}$$

Note that, for each n, one can have several (up to ρ) solutions depending on the different generalized fundamental solutions $\left(X_{fi}, Y_{fi}\right)$.

The other solutions (X_n, Y_n) of (8) may also be represented by recurrence relations

$$(X_n, Y_n) = \left(\left(x_f X_{n-1} + D y_f Y_{n-1}\right), \left(x_f Y_{n-1} + y_f X_{n-1}\right)\right), \tag{15}$$

that can also be written as

$$(X_n, Y_n) = \left(\left(2x_f X_{n-1} - X_{n-2}\right), \left(2x_f Y_{n-1} - Y_{n-2}\right)\right), \tag{16}$$

or by Chebyshev polynomials of the first kind $T_{n-1}\left(x_f\right)$ and of the second kind $U_{n-2}\left(x_f\right)$, evaluated at x_f [26],

$$(X_n, Y_n) = \left(\left(X_{fi} T_{n-1}\left(x_f\right) + D Y_{fi} y_f U_{n-2}\left(x_f\right)\right), \left(X_{fi} y_f U_{n-2}\left(x_f\right) + Y_{fi} T_{n-1}\left(x_f\right)\right)\right). \tag{17}$$

The second recurrent relations (16) are similar in form to above recurrent relations (4) and (5), and already given in [10].

4. Pell equations and multiples of triangular numbers

4.1. Solutions of Pell equations

For non-square integers k and with the change of variables

$$(X, Y) = (2\xi + 1,\ 2t + 1), \tag{18}$$

relation (1) becomes

$$X^2 - kY^2 = 1 - k, \tag{19}$$

and the associated simple Pell equation reads

$$x^2 - ky^2 = 1. \tag{20}$$

Odd solutions (X, Y) of (19) provide then pairs (ξ, t), solutions of (1). Following the procedure of §3, the fundamental solutions of the simple and generalized Pell equations are calculated and shown in Table 3 for non-square k between 2 and 102. The second and third columns give, respectively, the rank r (Sequence A341894 in the OEIS [11]) and the total number of fundamental solutions of generalized Pell equations. The fourth column shows the single fundamental solution of simple Pell equations. The fifth and sixth columns give the fundamental solutions of generalized Pell equations. The fifth column shows those solutions with both X_{fi} and Y_{fi} odd or having different parities. The sixth column gives those solutions with both X_{fi} and Y_{fi} even (except for $k = 56$, see discussion further).

Table 3. Fundamental solutions of simple (19) and generalized (20) Pell equations for $2 \leq k \leq 102$

k	r	ρ	(x_f, y_f)	(X_{fi}, Y_{fi})	
2	1	1	$(3,2)$	$(1,1)$	
3	1	1	$(2,1)$	$(1,1)$	
5	2	3	$(9,4)$	$(\pm1,1)$	$(4,2)$
6	2	2	$(5,2)$	$(\pm1,1)$	
7	2	2	$(8,3)$	$(\pm1,1)$	
8	2	2	$(3,1)$	$(\pm1,1)$	
10	3	3	$(19,6)$	$(\pm1,1), (9,3)$	
11	2	2	$(10,3)$	$(\pm1,1)$	
12	2	2	$(7,2)$	$(\pm1,1)$	
13	4	6	$(649,180)$	$(\pm1,1), (\pm25,7)$	$(\pm14,4)$
14	2	2	$(15,4)$	$(\pm1,1)$	

15	2	2	$(4,1)$	$(\pm1,1)$	
17	2	3	$(33,8)$	$(\pm1,1)$	$(16,4)$
18	2	2	$(17,4)$	$(\pm1,1)$	
19	3	3	$(170,39)$	$(\pm1,1),(39,9)$	
20	2	2	$(9,2)$	$(\pm1,1)$	
21	4	6	$(55,12)$	$(\pm1,1),(\pm13,3)$	$(\pm8,2)$
22	4	4	$(197,42)$	$(\pm1,1),(\pm23,5)$	
23	2	2	$(24,5)$	$(\pm1,1)$	
24	2	2	$(5,1)$	$(\pm1,1)$	
26	3	3	$(51,10)$	$(\pm1,1),(25,5)$	
27	2	2	$(26,5)$	$(\pm1,1)$	
28	4	4	$(127,24)$	$(\pm1,1),(\pm15,3)$	
29	4	6	$(9801,1820)$	$(\pm1,1),(\pm59,11)$	$(\pm86,16)$
30	2	2	$(11,2)$	$(\pm1,1)$	
31	4	4	$(1520,273)$	$(\pm1,1),(\pm61,11)$	
32	2	2	$(17,3)$	$(\pm1,1)$	
33	2	4	$(23,4)$	$(\pm1,1)$	$(\pm10,2)$
34	2	2	$(35,6)$	$(\pm1,1)$	
35	2	2	$(6,1)$	$(\pm1,1)$	
37	2	3	$(73,12)$	$(\pm1,1)$	$(36,6)$
38	2	2	$(37,6)$	$(\pm1,1)$	
39	2	2	$(25,4)$	$(\pm1,1)$	
40	4	4	$(19,3)$	$(\pm1,1),(\pm11,2)$	
k	r	ρ	(x_f,y_f)	(X_{fi},Y_{fi})	
41	4	4	$(2049,320)$	$(\pm1,1),(\pm83,13)$	
42	2	2	$(13,2)$	$(\pm1,1)$	
43	4	4	$(3482,531)$	$(\pm1,1),(\pm85,13)$	
44	2	2	$(199,30)$	$(\pm1,1)$	
45	4	6	$(161,24)$	$(\pm1,1),(\pm19,3)$	$(\pm26,4)$
46	6	6	$(24335,3588)$	$(\pm1,1),(\pm47,7),(\pm183,27)$	
47	2	2	$(48,7)$	$(\pm1,1)$	
48	2	2	$(7,1)$	$(\pm1,1)$	
50	3	3	$(99,14)$	$(\pm1,1),(49,7)$	
51	3	3	$(50,7)$	$(\pm1,1),(35,5)$	
52	4	4	$(649,90)$	$(\pm1,1),(\pm79,11)$	
53	4	6	$(66249,9100)$	$(\pm1,1),(\pm211,29)$	$(\pm160,22)$
54	2	2	$(485,66)$	$(\pm1,1)$	
55	4	4	$(89,12)$	$(\pm1,1),(\pm21,3)$	
56	2	4	$(15,2)$	$(\pm1,1)$	$(\pm13,2)$
57	4	4	$(151,20)$	$(\pm1,1),(\pm37,5)$	
58	4	4	$(19603,2574)$	$(\pm1,1),(\pm175,23)$	
59	2	2	$(530,69)$	$(\pm1,1)$	
60	2	2	$(31,4)$	$(\pm1,1)$	
61	8	12	$(1766319049, 226153980)$	$(\pm1,1),(\pm367,47), (\pm6709,859),(\pm94793,12137)$	$(\pm62,8),(\pm5186,664)$
62	2	2	$(63,8)$	$(\pm1,1)$	
63	2	2	$(8,1)$	$(\pm1,1)$	
65	2	5	$(129,16)$	$(\pm1,1)$	$(\pm14,2),(64,8)$
66	4	4	$(65,8)$	$(\pm1,1),(\pm23,3)$	
67	4	4	$(48842,5967)$	$(\pm1,1),(\pm401,49)$	

68	2	2	$(33,4)$	$(\pm1,1)$	
69	4	6	$(7775,936)$	$(\pm1,1),(\pm91,11)$	$(\pm116,14)$
70	4	4	$(251,30)$	$(\pm1,1),(\pm41,5)$	
71	4	4	$(3480,413)$	$(\pm1,1),(\pm143,17)$	
72	2	2	$(17,2)$	$(\pm1,1)$	
73	6	6	$(2281249,267000)$	$(\pm1,1),(\pm145,17),(\pm1461,171)$	
74	2	2	$(3699,430)$	$(\pm1,1)$	
75	2	2	$(26,3)$	$(\pm1,1)$	
k	r	ρ	(x_f,y_f)	(X_{fi},Y_{fi})	
76	6	6	$(57799,6630)$	$(\pm1,1),(\pm113,13),(\pm305,35)$	
77	4	6	$(351,40)$	$(\pm1,1),(\pm43,5)$	$(\pm34,4)$
78	4	4	$(53,6)$	$(\pm1,1),(\pm25,3)$	
79	2	2	$(80,9)$	$(\pm1,1)$	
80	2	2	$(9,1)$	$(\pm1,1)$	
82	3	3	$(163,18)$	$(\pm1,1),(81,9)$	
83	2	2	$(82,9)$	$(\pm1,1)$	
84	2	2	$(55,6)$	$(\pm1,1)$	
85	8	12	$(285769,30996)$	$(\pm1,1),(\pm101,11),\ (\pm341,37),(\pm1429,155)$	$(\pm16,2),(\pm424,46)$
86	4	4	$(10405,1122)$	$(\pm1,1),(\pm343,37)$	
87	2	2	$(28,3)$	$(\pm1,1)$	
88	4	4	$(197,21)$	$(\pm1,1),(\pm65,7)$	
89	4	4	$(500001,53000)$	$(\pm1,1),(\pm179,19)$	
90	2	2	$(19,2)$	$(\pm1,1)$	
91	6	6	$(1574,165)$	$(\pm1,1),(\pm27,3),(\pm181,19)$	
92	4	4	$(1151,120)$	$(\pm1,1),(\pm47,5)$	
93	4	6	$(12151,1260)$	$(\pm1,1),(\pm125,13)$	$(\pm154,16)$
94	4	4	$(2143295,221064)$	$(\pm1,1),(\pm281,29)$	
95	2	2	$(39,4)$	$(\pm1,1)$	
96	4	4	$(49,5)$	$(\pm1,1),(\pm17,2)$	
97	4	8	$(62809633,6377352)$	$(\pm1,1),(\pm581,59)$	$(\pm98,10),(\pm12902,1310)$
98	2	2	$(99,10)$	$(\pm1,1)$	
99	2	2	$(10,1)$	$(\pm1,1)$	
101	2	3	$(201,20)$	$(\pm1,1)$	$(100,10)$
102	2	2	$(101,10)$	$(\pm1,1)$	

From these Tables, we deduce the following.

First, the rank of solutions of (1) is equal to, or less than, the total number of fundamental solutions of the generalized Pell equations, $r \leq \rho$, as was expected.

Second, for all single fundamental solutions (x_f, y_f) of the simple Pell equation, x_f and y_f are of different parities, i.e., one is odd, the other even (except for some cases of $k \equiv 0 \pmod 8$, where both x_f and y_f are odd; see further). It is easy to see why: for (20) to hold, the following three conditions must hold:

C1: x_f and y_f cannot be simultaneously even, whatever the value of k is;

C2: if k is even, x_f must necessarily be odd and y_f can be either even or odd;

C3: if k is odd, x_f and y_f must have different parities, one odd and the other even.

Third, the sets of fundamental solutions of the generalized Pell equation always include $\left(X_{f_1}, Y_{f_1}\right) = (1,1)$ and $\left(X_{f_2}, Y_{f_2}\right) = (-1,1)$, which is obvious from (19). The only two exceptions are for the cases $k = 2$ and 3. For these two cases, although $(-1,1)$ is also a solution to (19), it does not bring a new branch of solutions calculated by (11) to (13) different from the one obtained with $(1,1)$. Therefore, there is only one fundamental solution, i.e., $\rho = 1$ for these two cases. Furthermore, the two pairs $(1,-1)$ and $(-1,-1)$ are also solutions of (19), but they do not yield new branches of solutions different from those obtained with $(1,1)$ and $(-1,1)$.

Fourth, all generalized fundamental solutions $\left(X_{fi}, Y_{fi}\right)$ with $i > 2$, i.e., other than $(1,1)$, have both X_{fi} and Y_{fi} odd, except for$k = 40, 96, 208, ...$ where Y_{fi} is even (see further).

Fifth, there are $(\rho - r)$ generalized fundamental solutions with both X_{fi} and Y_{fi} even. These are shown separately as they do not yield any solutions to (1).

4.2. Solutions in (ξ, t)

The two generalized fundamental solutions $\left(X_{f_{1,2}}, Y_{f_{1,2}}\right) = (\pm 1, 1)$ and using $\left(X_{0_{1,2}}, Y_{0_{1,2}}\right)$ from(12) yield the two trivial solutions $\left(\xi_{0_{1,2}}, t_{0_{1,2}}\right) = \left(\frac{\pm1-1}{2}, \frac{1-1}{2}\right) = (0,0)$ and $(-1,0)$ to (1). The next generalized solution (13) reads

$$\left(X_{1_{1,2}}, Y_{1_{1,2}}\right) = \left(\left(\pm x_f + k y_f\right), \left(\pm y_f + x_f\right)\right),$$

yielding, from (18),

$$(\xi_{1,2}, t_{1,2}) = \left(\frac{\pm x_f + k y_f - 1}{2}, \frac{\pm y_f + x_f - 1}{2}\right), \tag{21}$$

with both terms integers under the three conditions C1 to C3 above.

For other generalized fundamental solutions $\left(X_{fi}, Y_{fi}\right)$ with $i > 2$, different from $(\pm 1, 1)$, one has from (12), $(X_{0_i}, Y_{0_i}) = \left(X_{fi}, Y_{fi}\right)$, yielding

$$\left(\xi_{0_{1,2}}, t_{0_{1,2}}\right) = \left(\frac{X_{fi} - 1}{2}, \frac{Y_{fi} - 1}{2}\right), \tag{22}$$

integer solutions of (1) if X_{fi} and Y_{fi} are both odd. The next generalized solution (13) reads

$$(X_{1_i}, Y_{1_i}) = \left(\left(X_{fi} x_f + k Y_{fi} y_f\right), \left(X_{fi} y_f + Y_{fi} x_f\right)\right), \tag{23}$$

yielding

$$(\xi_{1_i}, t_{1_i}) = \left(\frac{X_{fi} x_f + k Y_{fi} y_f - 1}{2}, \frac{X_{fi} y_f + Y_{fi} x_f - 1}{2}\right). \tag{24}$$

One sees that X_{fi} and Y_{fi} cannot be simultaneously even for ξ_{1_i} and t_{1_i} to be integers. For both X_{fi} and Y_{fi} odd, the above three conditions C1 to C3 on x_f and y_f ensure that ξ_{1_i} and t_{1_i} are integers.

For the cases of X_{fi} odd and Y_{fi} even, e.g., for $k = 40$ and 96 in Table 3, as k is even, x_f and y_f must be simultaneously odd by condition C2 and for (24) to provide integer solutions.

Finally, simple Pell equation fundamental solutions (x_f, y_f) with both x_f and y_f odd appear for most cases of $k \equiv 0 \,(\mathrm{mod}\, 8)$. Exceptions are for $k = 56$, 72, 112, 184, 240, 248, 264, 272, 376, ..., i.e., for some (but not all) cases of $k \equiv \pm 8, \pm 16 \,(\mathrm{mod}\, 64)$ where y_f is even. In these cases, as both k and y_f are even, Y_{fi} cannot be even for (24) to provide integer solutions and these generalized fundamental solutions $\left(X_{fi}, Y_{fi}\right)$ must be discarded.

For the general case of $k \equiv 0 \,(\mathrm{mod}\, 8)$, the fact that y_f is not odd can be explained as follows. As $k \equiv 0 \,(\mathrm{mod}\, 8)$ is not square-free, the simple Pell Eq. (20) can be simplified posing $k = c^2 k'$, with k' square-free, yielding

$$x^2 - k' y^2 = 1, \tag{25}$$

with $y' = cy$. The fundamental solution (x_f, y'_f) of (25) yields then the fundamental solution $(x_f, y_f) = (x_f, \frac{y'_f}{c})$ of (20).

For example, for $k = 8$, let $k' = 2$ and $c = 2$; (25) yields $(x_f, y'_f) = (3,2)$ and $(x_f, \frac{y'_f}{c}) = (x_f, y_f) = (3,1)$.

For most cases of $k \equiv 0 \,(\mathrm{mod}\, 8)$, y'_f is divisible by c such that $\frac{y'_f}{c}$ is odd yielding then y_f odd. For the exceptions of some cases of $k \equiv \pm 8, \pm 16 \,(\mathrm{mod}\, 64)$, this procedure does not lead to an odd value of $\frac{y'_f}{c}$. For example, for $k = 56$, let $k' = 14$ and $c = 2$, yielding $(x_f, y'_f) = (15, 4)$ and $y_f = \frac{y'_f}{c} = 2$. For $k = 72$, let $k' = 2$ and $c = 6$, yielding $(x_f, y'_f) = (3, 2)$. However, y'_f is not divisible by $c = 6$ and one must consider not the first fundamental solution of the simple Pell equation for $k' = 2$, but the second solution given by (10) for $n = 2$, yielding $(x_2, y'_2) = (17, 12)$ that gives $y_f = \frac{y'_f}{c} = 2$ and finally $(x_f, y_f) = (17, 2)$.

4.3. First r solutions in (ξ, t)

In order to calculate all solutions of (1), we have to find the first r solutions (ξ_i, t_i) (with $0 \le i \le r$) of (1), arranged in increasing order,

$$\xi_0 = 0 < \xi_1 < ... < \xi_i < ... < \xi_r \quad , \quad t_0 = 0 < t_1 < ... < t_i < ... < t_r,$$

and that correspond to the r fundamental solutions $\left(X_{fi}, Y_{fi}\right)$ of the generalized Pell Eq. (19), with both X_{fi} and Y_{fi} odd or of different parities.

The generalized fundamental solutions $\left(X_{f_1}, Y_{f_1}\right) = (1, 1)$ and $\left(X_{f_2}, Y_{f_2}\right) = (-1, 1)$ provide respectively, the solutions (ξ_r, t_r) and (ξ_{r-1}, t_{r-1}) of (1) from (13).

This yields successively

$$\begin{aligned}(X_{1_1}, Y_{1_1}) &= (X_{f_1} x_f + kY_{f_1} y_f,\ X_{f_1} y_f + Y_{f_1} x_f) = (x_f + ky_f,\ -y_f + x_f),\\ (X_{1_2}, Y_{1_2}) &= (X_{f_2} x_f + kY_{f_2} y_f,\ X_{f_2} y_f + Y_{f_2} x_f) = (-x_f + ky_f,\ -y_f + x_f),\end{aligned}$$

and

$$(\xi_r, t_r) = \left(\frac{X_{1_1} - 1}{2}, \frac{Y_{1_1} - 1}{2}\right) = \left(\frac{x_f + ky_f - 1}{2}, \frac{y_f + x_f - 1}{2}\right), \tag{26}$$

$$(\xi_{r-1}, t_{r-1}) = \left(\frac{X_{1_2} - 1}{2}, \frac{Y_{1_2} - 1}{2}\right) = \left(\frac{-x_f + ky_f - 1}{2}, \frac{-y_f + x_f - 1}{2}\right). \tag{27}$$

For $r > 2$, the next two generalized fundamental solutions (X_{f3}, Y_{f3}) and $(X_{f4}, Y_{f4}) = (-X_{f3}, Y_{f3})$ yield respectively (ξ_1, t_1) and (ξ_2, t_2). If both X_{f3} and Y_{f3} are odd, then (12) (for $n = 0$) can be used for (ξ_1, t_1), yielding

$$(\xi_1, t_1) = \left(\frac{X_{f_3} - 1}{2}, \frac{Y_{f_3} - 1}{2}\right). \tag{28}$$

Relation (12)could also be used for (ξ_2, t_2) with $(-X_{f_3}, Y_{f_3})$, but it would provide a negative value for ξ_2. Instead, we use (13) (for $n = 1$), giving

$$(\xi_2, t_2) = \left(\frac{-X_{f_3} x_f + kY_{f_3} y_f - 1}{2}, \frac{-X_{f_3} y_f + Y_{f_3} x_f - 1}{2}\right), \tag{29}$$

The next two generalized fundamental solutions (X_{f_5}, Y_{f_5}) and $(X_{f_6}, Y_{f_6}) = (-X_{f_5}, Y_{f_5})$ yield similarly the next two solutions (ξ_i, t_i), which are then arranged in increasing order.

For example, for $k = 13$, we have $r = 4$, $(x_f, y_f) = (649, 180)$, and generalized solutions $(X_{f_i}, Y_{f_i}) = (\pm 1, 1)$ and $(\pm 25, 7)$. Relations (26) and (27) yield respectively, $(\xi_4, t_4) = (1494, 414)$ and $(\xi_3, t_3) = (845, 234)$. Relations (28) and (29) yield respectively, $(\xi_1, t_1) = (12, 3)$, $(\xi_2, t_2) = (77, 21)$.

Another example, for $k = 46$, we have $r = 6$, $(x_f, y_f) = (24335, 3588)$, $(X_{fi}, Y_{fi}) = (\pm 1, 1)$, $(\pm 47, 7)$ and $(\pm 183, 27)$. With $(X_{f_{1,2}}, Y_{f_{1,2}}) = (\pm 1, 1)$, (26) and (27) yield respectively, $(\xi_6, t_6) = (94691, 13961)$, $(\xi_5, t_5) = (70356, 10373)$. With $(X_{f_{3,4}}, Y_{f_{3,4}}) = (\pm 47, 7)$ and $(X_{f_{5,6}}, Y_{f_{5,6}}) = (\pm 183, 27)$, (28) and (29) yield respectively and successively, $(\xi_1, t_1) = (23, 3)$, $(\xi_4, t_4) = (5795, 854)$ and $(\xi_2, t_2) = (91, 13)$, $(\xi_3, t_3) = (1495, 220)$.

For the cases where Y_{f_i} is even, i.e., $k = 40, 96, 120, ...$, (26) to (28) cannot be used with $(X_{f_{1,2}}, Y_{f_{1,2}}) = (\pm 1, 1)$ as both k and Y_{f_i} are even, yielding non-integer solutions for ξ and t. Instead, the other generalized fundamental solutions have to be used with (13) (for $n = 1$) and (14) (for $n = 2$). For $r = 4$, one has then, with $(X_{f_1}, Y_{f_1}) = (1, 1)$ and $(X_{f_2}, Y_{f_2}) = (-1, 1)$,

$$(X_{2_1}, Y_{2_1}) = (x_f^2 + ky_f^2 + 2kx_f y_f,\ x_f^2 + ky_f^2 + 2x_f y_f), \tag{30}$$

$$(X_{2_2}, Y_{2_2}) = (-(x_f^2 + ky_f^2) + 2kx_f y_f,\ x_f^2 + ky_f^2 - 2x_f y_f), \tag{31}$$

yielding

$$(\xi_4, t_4) = \left(\frac{x_f^2 + ky_f^2 + 2kx_f y_f - 1}{2}, \frac{x_f^2 + ky_f^2 + 2x_f y_f - 1}{2} \right), \tag{32}$$

$$(\xi_3, t_3) = \left(\frac{-(x_f^2 + ky_f^2) + 2kx_f y_f - 1}{2}, \frac{x_f^2 + ky_f^2 - 2x_f y_f - 1}{2} \right), \tag{33}$$

and with (X_{f_3}, Y_{f_3}) and $(X_{f_4}, Y_{f_4}) = (-X_{f_3}, Y_{f_3})$,

$$(X_{1_3}, Y_{1_3}) = (X_{f_3} x_f + kY_{f_3} y_f,\ X_{f_3} y_f + Y_{f_3} x_f), \tag{34}$$

$$(X_{1_4}, Y_{1_4}) = (X_{f_4} x_f + kY_{f_4} y_f,\ X_{f_4} y_f + Y_{f_4} x_f), \tag{35}$$

yielding

$$(\xi_2, t_2) = \left(\frac{X_{f_3} x_f + kY_{f_3} y_f - 1}{2}, \frac{X_{f_3} y_f + Y_{f_3} x_f - 1}{2} \right), \tag{36}$$

$$(\xi_1, t_1) = \left(\frac{-X_{f_3} x_f + kY_{f_3} y_f - 1}{2}, \frac{-X_{f_3} y_f + Y_{f_3} x_f - 1}{2} \right). \tag{37}$$

For $k = 40$, $r = 4$, $(x_f, y_f) = (19, 3)$, $(X_{fi}, Y_{fi}) = (\pm 1, 1)$, $(\pm 11, 2)$, (30) to (37) yield successively $(X_{2_1}, Y_{2_1}) = (5281, 835)$, $(X_{2_2}, Y_{2_2}) = (3839, 697)$, $(\xi_4, t_4) = (2640, 417)$, $(\xi_3, t_3) = (1919, 303)$, $(X_{1_3}, Y_{1_3}) = (449; 71)$, $(X_{1_4}, Y_{1_4}) = (31, 5)$, $(\xi_2, t_2) = (224, 35)$, and $(\xi_1, t_1) = (15, 2)$.

4.4. All solutions in (ξ, t)

After having found the first r values of (ξ_i, t_i), each corresponding to one of the r generalized fundamental solutions (X_{fi}, Y_{fi}), the r branches of infinitely many other solutions can be calculated using any of the four following methods.

1) General Pell equation solutions

The r general solutions (11) of the Pell equation yield (assuming a + sign)

$$\xi_n + \sqrt{k} t_n = \left(\xi_j + \sqrt{k} t_j + \left(\frac{1 + \sqrt{k}}{2} \right) \right) (x_f + \sqrt{k} y_f)^n - \left(\frac{1 + \sqrt{k}}{2} \right), \tag{38}$$

where (ξ_j, t_j) must be replaced successively by the first r values (ξ_i, t_i).

2) First recurrence relation

The first recurrence relation (15) yields

$$(\xi_n, t_n) = \left(x_f \xi_{n-r} + ky_f t_{n-r} + \frac{x_f + ky_f - 1}{2},\ x_f t_{n-r} + y_f \xi_{n-r} + \frac{x_f + y_f - 1}{2} \right), \tag{39}$$

where indices of ξ_{n-r} and t_{n-r} (instead of ξ_{n-1} and t_{n-1}) refer to preceding values of ξ and t in the same solution branch.

3) Second recurrence relation

The second recurrence relation (16) yields

$$(\xi_n, t_n) = \Big(2x_f\xi_{n-r} - \xi_{n-2r} + (x_f - 1),\ 2x_f t_{n-r} - t_{n-2r} + (x_f - 1)\Big), \tag{40}$$

where indices of ξ_{n-r}, ξ_{n-2r} and t_{n-r}, t_{n-2r} (instead of ξ_{n-1}, ξ_{n-2} and t_{n-1}, t_{n-2}) refer to the preceding and teh one before values of ξ and t in the same solution branch.

4) Chebyshev polynomial formulation

The Chebyshev polynomials (17) yield

$$\begin{aligned}(\xi_n, t_n) = \Bigg(\left(\xi_j + \frac{1}{2}\right) T_{n-1}(x_f) + k\left(t_j + \frac{1}{2}\right) y_f U_{n-2}(x_f) - \frac{1}{2},\\ \left(\xi_j + \frac{1}{2}\right) y_f U_{n-2}(x_f) + \left(t_j + \frac{1}{2}\right) T_{n-1}(x_f) - \frac{1}{2}\Bigg),\end{aligned} \tag{41}$$

where (ξ_j, t_j) must be replaced successively by the first r values of (ξ_i, t_i).

4.5. Relation between Pell equation solutions and recurrent relations

We can now give a new definition of the rank r.

The rank r is the number of fundamental solutions (X_{fi}, Y_{fi}) of the generalized Pell Eq. (19), with X_{fi} odd and Y_{fi} odd or even (if y_f is not even), with $r \leq \rho$, the total number of generalized solutions of (19).

Furthermore, we see that the recurrence relations (40) for both ξ_n and t_n have x_f as the only parameter and that the two relations are independent of k and y_f. This fundamental solution x_f of the simple Pell Eq. (20) acts as a constant of the problem for each k. Note further that summing the expressions of t_r and t_{r-1} in (26) and (27) yields $t_r + t_{r-1} = x_f - 1$. Since this sum was already defined as κ in (2), we obtain

$$\kappa = x_f - 1. \tag{42}$$

Furthermore, (42) also yields that y_f is related to the difference $\delta = t_r - t_{r-1}$ through the simple Pell equation (20), which is verified if $y_f^2 = \kappa^2 - 4t_r t_{r-1} = (t_r - t_{r-1})^2 = \delta^2$, giving (assuming the $+$ sign only)

$$\delta = y_f, \tag{43}$$

for non-square integers k (except for some cases $k \equiv 0 \pmod 8$, see further). Replacing in the simple Pell equation $(\kappa + 1)^2 - k\delta^2 = 1$ yields the condition between the sum and the difference of $t_r$and t_{r-1},

$$\delta = \sqrt{\frac{\kappa(\kappa + 2)}{k}}. \tag{44}$$

For $k \equiv 0 \pmod 8$, (42) and (43) are valid for some cases but for other. Relations (42) and (43) are valid for $k = 56, 72, 112, 184, 240, 248, 264, 272, 376, \ldots$, i.e., for some (but not all) cases of $k \equiv \pm 8, \pm 16 \pmod{64}$. For other cases of $k \equiv 0 \pmod 8$, $\delta > y_f$ and one must find the next pair of solutions to the simple Pell equation by (10) for $n = 2$, i.e., $(x_2, y_2) = \left(x_f^2 + ky_f^2,\ 2x_f y_f\right)$,

$$\kappa = x_f^2 + ky_f^2 - 1 \tag{45}$$

$$\delta = 2x_f y_f. \tag{46}$$

Finally, expressing the coefficients of the four recurrence relations (4) to (7) in terms of the fundamental solutions of the simple Pell equation, one has, in addition to (42), $T_\kappa = \frac{\kappa(\kappa+1)}{2} = \frac{(x_f-1)x_f}{2}$, $\gamma = \frac{(x_f-1)^2 - y_f^2}{4}$ and $T_\kappa - \gamma = \frac{x_f^2 - 1 + y_f^2}{4}$, yielding

$$t_n = 2x_f t_{n-r} - t_{n-2r} + (x_f - 1), \tag{47}$$

$$\xi_n = 2x_f \xi_{n-r} - \xi_{n-2r} + (x_f - 1), \tag{48}$$

$$T_{t_n} = (4x_f^2 - 2)T_{t_{n-r}} - T_{t_{n-2r}} + \frac{x_f^2 - 1 + y_f^2}{4}, \tag{49}$$

$$T_{\xi_n} = (4x_f^2 - 2)T_{\xi_{n-r}} - T_{\xi_{n-2r}} + k\left(\frac{x_f^2 - 1 + y_f^2}{4}\right), \tag{50}$$

and

$$(\xi_r, t_r) = \left(\frac{x_f + ky_f - 1}{2}, \frac{y_f + x_f - 1}{2}\right), \tag{51}$$

$$(\xi_{r-1}, t_{r-1}) = \left(\frac{-x_f + ky_f - 1}{2}, \frac{-y_f + x_f - 1}{2}\right). \tag{52}$$

A simpler method applicable to some k is given in the next section.

4.6. Recurrent relations in function of s for $k = s^2 + \sigma$

Expressing k in function of s^2, with $s = \lfloor\sqrt{k}\rfloor$, the closest integer smaller than $\sqrt{k}$, Tables 4 and 5 show expressions of (x_f, y_f) and, when existing, (X_{fi}, Y_{fi}) for $i > 2$ for several simple expressions of k in function of s^2, $k = s^2 + \sigma$, where σ is either an integer (Table 4) or a function of s (Table 5).

Table 4. Expressions of k, s, r, (x_f, y_f), (X_{fi}, Y_{fi}) for $i > 2$; $(-,-)$: no generalized solutions exist as $r = 2$; (a) except for $k = 51, 66$ $(r = 3, 4)$; (b) except for $k = 40$ $(r = 4)$; (c) except for $k = 85$ $(r = 8)$

k	s	r	$(x_f, y_f) \Vert (X_{fi}, Y_{fi})$
s^2+1	even	2	$(\pm(2s^2+1), 2s) \Vert (-,-)$
	odd	3	$(\pm(2s^2+1), 2s) \Vert (s^2, s)$
s^2+2	any	$2^{(a)}$	$(\pm(s^2+1), s) \Vert (-,-)$
s^2+4	even	$2^{(b)}$	$(\pm(\frac{s^2}{2}+1), \frac{s}{2}) \Vert (-,-)$
	1(mod,4)	$4^{(c)}$	$(\pm(\frac{s^2(s^2+3)^2}{2}+1), \frac{s(s^2+1)(s^2+3)}{2}) \Vert (\pm(s(\frac{s^2-s+4}{2})-1), \frac{s(s-1)}{2}+1)$
	3(mod,4)	4	$(\pm(\frac{(s^2+1)^2(s^2+4)}{2}-1), \frac{s(s^2+1)(s^2+3)}{2}) \Vert (\pm(s(\frac{s^2+s+4}{2})+1), \frac{s(s+1)}{2}+1)$
s^2+8	0(mod,4)	2	$(\pm(\frac{s^2+4}{4}), \frac{s}{4}) \Vert (-,-)$
	2(mod,4)	2	$(\pm(\frac{s^2(s^2+8)}{8}+1), \frac{s(s^2+4)}{8}) \Vert (-,-)$

All these expressions can easily be demonstrated by replacing the appropriate variables in the Pell equations (19) and (20). Note also that these general expressions for the fundamental solutions (x_f, y_f) are valid in all generality for the simple Pell equation (20).

Replacing x_f and y_f in recurrence relations (47) to (50) by their expressions in functions of s yields simplified recurrent relations specific for each k, and the first r expressions of each variable from (51) and (52).

Recall that $(\xi_0, t_0) = (0, 0)$ is always a solution for all non-square k, yielding $T_{t_0} = 0$ and $T_{\xi_0} = 0$, and note that the four variables $t_n, \xi_n, T_{t_n}, T_{\xi_n}$ can be extended to negative integer indices as $t_{-(i+1)} = t_i$, $\xi_{-(i+1)} = -\xi_i - 1$, $T_{t_{-(i+1)}} = T_{t_i}$, and $T_{\xi_{-(i+1)}} = T_{\xi_i}$. Therefore, one would need only to know the first $(r-1)$ values of each variable to determine all solutions and to completely solve the problem.

Tables 6 to 9 show the first three cases of Table 4, for $k = s^2 + 1$ (Table 6), $k = s^2 + 2$ (Table 7) and $k = s^2 + 4$ (Tables 8 and 9). The method can easily be extended to the other cases of Tables 6 and 7.

Note that in Table 6, for $k = s^2 + 1$, one has $(x_f, y_f) = (2s^2 + 1, 2s)$, and for s odd, the additional generalized fundamental solution $(X_{f_3}, Y_{f_3}) = (s^2, s)$, yields, by (28), $(\xi_1, t_1) = \left(\frac{s^2-1}{2}, \frac{s-1}{2}\right)$.

Table 5. Additional expressions of k, s, r, (x_f, y_f) and (X_{fi}, Y_{fi}) for $i > 2$; $(-,-)$: no generalized solutions exist as $r = 2$; $(*,*)$: no apparent pattern; $\hat{\pm}$: plus/minus sign independent from other $\pm$ sign(s); (a) except for $k = \sigma^2 - 1$ with σ even; (b) except for $k = 5, 11, 55, \ldots$ and no $\pm$ for $k = 10$ (as $r = 3$); (c) except for $k = 40$; (d) except for $k = 78$ ($r = 4$); (e) except for $k = 5$ ($r = 2$); (f) except for$k = 96$ ($r = 4$); (g) except for $k = 136$ ($r = 4$)

k	s	r	$(x_f, y_f) \Vert (X_{fi}, Y_{fi})$
$s^2 + s$	any	2	$(\pm(2s+1), 2) \Vert (-,-)$
$s^2 \hat{\pm} \frac{2s}{\sigma}$	$0(\bmod \sigma)$, σ odd	≥ 2	$(\pm(\sigma s \hat{\pm} 1), \sigma) \Vert (*,*)$
	$0(\bmod \frac{\sigma}{2})$, σ even	≥ 2	$(\pm(\sigma s \hat{\pm} 1), \sigma)^{(a)} \Vert (*,*)$
$s^2 + s - 1$	any	≥ 3	$(*,*) \Vert (\pm(\frac{2s^2+4s}{3} - 1), \frac{2s}{3} + 1)^{(c)}$
$s^2 + s - 2$	$0(\bmod 3)$	≥ 4	$(*,*) \Vert (\pm(2s^2 + 2s + 1), 2s + 1)^{(b)}$
	$1(\bmod 3)$	2	$(*,*) \Vert (-,-)$
	$2(\bmod 3)$	4	$(*,*) \Vert (\pm(\frac{2s^2-5}{3}), \frac{2(s-2)}{3} + 1)$
$s^2 + s + 1$	$1(\bmod 3)$	4	$(\pm(\frac{2(2s+1)^2}{3} + 1), 4(\frac{2(s-1)}{3} + 1) \Vert (\pm(\frac{2s^2+2s-1}{3}), \frac{2s+1}{3})$
	$0, 2(\bmod 3)$	≥ 4	$(*,*) \Vert (\pm(2s^2 + 2s + 1), 2s + 1)$
$s^2 + 2s$	any	2	$(\pm(s+1), 1) \Vert (-,-)$
$s^2 + 2s - 1$	any	2	$(\pm(s^2 + 2s), s + 1) \Vert (-,-)$
$s^2 + 2s - 2$	$2(\bmod 3)$	$2^{(d)}$	$(\pm \frac{2s^2+4s-1}{3}, \frac{2(s+1)}{3}) \Vert (-,-)$
$s^2 + 2s - 3$	$0(\bmod 4)$	4	$(\pm \frac{(s+1)(s^2+2s-2)}{2}, \frac{s(s+2)}{2}) \Vert (\pm \frac{s^2+3s-2}{2}, \frac{s+2}{2})$
	$2(\bmod 4)$	$4^{(e)}$	$(\pm \frac{(s+1)(s^2+2s-2)}{2}, \frac{s(s+2)}{2}) \Vert (\pm \frac{s^2+s-4}{2}, \frac{s}{2})$
	odd	$2^{(f)}$	$(\pm \frac{s^2+2s-3}{4}, \frac{s+1}{4}) \Vert (-,-)$
$s^2 + 2s - 7$	$3(\bmod 4)$	$2^{(g)}$	$(\pm \frac{2s^2+4s-1}{3}, \frac{2(s+1)}{3}) \Vert (-,-)$
	$1(\bmod 4)$	4	$(\pm \frac{s^2(s^2-2)+4s(s^2-3)+1}{8}, \frac{(s^2-1)(s+3)}{8}) \Vert (\pm \frac{2s^2+3s-5}{4}, \frac{s+1}{2})$
$s^2 + \frac{3s+1}{2}$	odd	2	$(\pm(4s+3), 4) \Vert (-,-)$

Table 6. Expressions of $t_n, \xi_n, T_{t_n}, T_{\xi_n}$ in function of s for $k = s^2 + 1$

$k = s^2 + 1$, s even, $r = 2$
$t_n = 2(2s^2 + 1)t_{n-2} - t_{n-4} + 2s^2$,$t_2 = s(s+1)$,$t_1 = s(s-1)$
$\xi_n = 2(2s^2 + 1)\xi_{n-2} - \xi_{n-4} + 2s^2$, $\xi_2 = s(s^2 + s + 1)$, $\xi_1 = (s-1)(s^2 + 1)$
$T_{t_n} = 2(8s^2(s^2 + 1) + 1)T_{t_{n-2}} - T_{t_{n-4}} + s^2(s^2 + 2)$,
$T_{t_2} = \frac{s(s+1)(s(s+1)+1)}{2}$,$T_{t_1} = \frac{s(s-1)(s(s-1)+1)}{2}$
$T_{\xi_n} = 2(8s^2(s^2 + 1) + 1)T_{\xi_{n-2}} - T_{\xi_{n-4}} + s^2(s^2 + 1)(s^2 + 2)$,
$T_{\xi_2} = \frac{s(s+1)(s^2+1)(s(s+1)+1)}{2}$, $T_{\xi_1} = \frac{s(s-1)(s^2+1)(s(s-1)+1)}{2}$
$k = s^2 + 1$, s odd, $r = 3$
$t_n = 2(2s^2 + 1)t_{n-3} - t_{n-6} + 2s^2$,$t_3 = s(s+1)$,$t_2 = s(s-1)$, $t_1 = \frac{s-1}{2}$
$\xi_n = 2(2s^2 + 1)\xi_{n-3} - \xi_{n-6} + 2s^2$, $\xi_3 = s(s^2 + s + 1)$,
$\xi_2 = (s-1)(s^2 + 1)$, $\xi_1 = \frac{s^2-1}{2}$
$T_{t_n} = 2(8s^2(s^2 + 1) + 1)T_{t_{n-3}} - T_{t_{n-6}} + s^2(s^2 + 2)$,
$T_{t_3} = \frac{s(s+1)(s(s+1)+1)}{2}$,$T_{t_2} = \frac{s(s-1)(s(s-1)+1)}{2}$, $T_{t_1} = \frac{s^2-1}{8}$
$T_{\xi_n} = 2(8s^2(s^2 + 1) + 1)T_{\xi_{n-3}} - T_{\xi_{n-6}} + s^2(s^2 + 1)(s^2 + 2)$,
$T_{\xi_3} = \frac{s(s+1)(s^2+1)(s(s+1)+1)}{2}$, $T_{\xi_2} = \frac{s(s-1)(s^2+1)(s(s-1)+1)}{2}$, $T_{\xi 1} = \frac{s^4-1}{8}$

In Table 7, for $k = s^2 + 2$, one has $(x_f, y_f) = (s^2 + 1, s)$ and $r = 2$, except for some cases $k = 51, 66, \ldots$ where $r = 3, 4, \ldots$, for which additional generalized fundamental solution(s) must be considered to calculate smaller values of (ξ_i, t_i).

In Table 8, for $k = s^2 + 4$ and s even, one has $(x_f, y_f) = \left(\frac{s^2+2}{2}, \frac{s}{2}\right)$ and $r = 2$, except for $k = 40, \ldots$ where $r = 4$; while for $s \equiv 1 \pmod 4$, one has $(x_f, y_f) = \left(\frac{s^2(s^2+3)^2}{2} + 1, \frac{s(s^2+1)(s^2+3)}{2}\right)$, $r = 4$, except for $k = 85$ where $r = 8$, and $(X_{f_{3,4}}, Y_{f_{3,4}}) = \left(\pm(s(\frac{s^2-s+4}{2}) - 1), \frac{s(s-1)}{2} + 1\right)$ yields, by (28), $(\xi_1, t_1) = \left(\frac{(s-1)(s^2+4)}{4}, \frac{s(s-1)}{4}\right)$ and, by (29), $(\xi_2, t_2) = \left(\frac{s(s^2+4)(s^2+s+2)}{4}, \frac{s(s^3+s^2+4s+2)}{4}\right)$.

Table 7. Expressions of $t_n, \xi_n, T_{t_n}, T_{\xi_n}$ in function of s for $k = s^2 + 2$

$k = s^2 + 2, r = 2$
$t_n = 2(s^2+1)t_{n-2} - t_{n-4} + s^2, t_2 = \frac{s(s+1)}{2}, t_1 = \frac{s(s-1)}{2}$
$\xi_n = 2(s^2+1)\xi_{n-2} - \xi_{n-4} + s^2, \xi_2 = \frac{s(s^2+s+2)}{2}, \xi_1 = \frac{(s-1)(s^2+2)}{2}$
$T_{t_n} = 2(2(s^2+1)^2-1)T_{t_{n-2}} - T_{t_{n-4}} + \frac{s^2(s^2+3)}{4}$
$T_{t_2} = \frac{s(s+1)(s(s+1)+2)}{8}, T_{t_1} = \frac{s(s-1)(s(s-1)+2)}{8}$
$T_{\xi_n} = 2(2(s^2+1)^2-1)T_{\xi_{n-2}} - T_{\xi_{n-4}} + \frac{s^2(s^2+2)(s^2+3)}{4}$
$T_{\xi_2} = \frac{s(s+1)(s^2+2)(s(s+1)+2)}{8}, T_{\xi_1} = \frac{s(s-1)(s^2+2)(s(s-1)+2)}{8}$

Table 8. Expressions of $t_n, \xi_n, T_{t_n}, T_{\xi_n}$ in function of s for $k = s^2 + 4$

$k = s^2 + 4$, s even, $r = 2$
$t_n = (s^2+2)t_{n-2} - t_{n-4} + \frac{s^2}{2}, t_2 = \frac{s(s+1)}{4}, t_1 = \frac{s(s-1)}{4}$
$\xi_n = (s^2+2)\xi_{n-2} - \xi_{n-4} + \frac{s^2}{2}, \xi_2 = \frac{s(s^2+s+4)}{4}, \xi_1 = \frac{(s-1)(s^2+4)}{4}$
$T_{t_n} = ((s^2+2)^2-2)T_{t_{n-2}} - T_{t_{n-4}} + \frac{s^2(s^2+5)}{16}$
$T_{t_2} = \frac{s(s+1)(s(s+1)+4)}{32}, T_{t_1} = \frac{s(s-1)(s(s-1)+4)}{32}$
$T_{\xi_n} = ((s^2+2)^2-2)T_{\xi_{n-2}} - T_{\xi_{n-4}} + \frac{s^2(s^2+4)(s^2+5)}{16}$
$T_{\xi_2} = \frac{s(s+1)(s^2+4)(s(s+1)+4)}{32}, T_{\xi_1} = \frac{s(s-1)(s^2+4)(s(s-1)+4)}{32}$
$k = s^2 + 4, s \equiv 1 (\bmod 4), r = 4$
$t_n = (s^2(s^2+3)^2+2)t_{n-4} - t_{n-8} + \frac{s^2(s^2+3)^2}{2}$,
$t_4 = \frac{s(s^2+3)(s(s^2+3)+(s^2+1))}{4}, t_3 = \frac{s(s^2+3)(s(s^2+3)-(s^2+1))}{4}$,
$t_2 = \frac{s(s^2(s+1)+2(2s+1))}{4}, t_1 = \frac{s(s-1)}{4}$
$\xi_n = (s^2(s^2+3)^2+2)\xi_{n-4} - \xi_{n-8} + \frac{s^2(s^2+3)^2}{2}$,
$\xi_4 = \frac{s(s^2+3)(s(s^2+3)+(s^2+1)(s^2+4))}{4}, \xi_3 = \frac{(s^2+1)(s^2+4)(s^3-s^2+3s-1)}{4}$
$\xi_2 = \frac{s(s^2+4)(s^2+s+2)}{4}, \xi_1 = \frac{(s-1)(s^2+4)}{4}$
$T_{t_n} = ((s^2+2)^2-2)(s^6(s^2+8)+4s^2(5s^2+4)+1)T_{t_{n-4}} - T_{t_{n-8}} + \frac{s^2(s^2+1)^2(s^2+3)^2(s^2+5)}{16}$
$T_{t_4} = \frac{s(s^2+1)(s^2+3)((s^2+1)+s(s^2+3))(s^2(s^2+5)+s(s^2+3)+4)}{32}$
$T_{t_3} = \frac{s(s^2+1)(s^2+3)(-(s^2+1)+s(s^2+3))(s^2(s^2+5)-s(s^2+3)+4)}{32}$
$T_{t_2} = \frac{s(s^2+2)(s^2+s+2)(s^2(s+1)+2(2s+1))}{32}, T_{t_1} = \frac{s(s-1)(s(s-1)+4)}{32}$
$T_{\xi_n} = ((s^2+2)^2-2)(s^6(s^2+8)+4s^2(5s^2+4)+1)T_{\xi_{n-4}} - T_{\xi_{n-8}} + \frac{s^2(s^2+1)^2(s^2+3)^2(s^2+4)(s^2+5)}{16}$
$T_{\xi_4} = \frac{s(s^2+1)(s^2+3)(s^2+4)(s(s^2+3)+s^2+1)(s^2(s^2+5)+s(s^2+3)+4)}{32}$
$T_{\xi_3} = \frac{s(s^2+1)(s^2+3)(s^2+4)(s(s^2+3)-(s^2+1))(s^2(s^2+5)-s(s^2+3)+4)}{32}$
$T_{\xi_2} = \frac{s(s^2+2)(s^2+4)(s^2+s+2)(s^2(s+1)+2(2s+1))}{32}, T_{\xi_1} = \frac{s(s-1)(s^2+4)(s(s-1)+4)}{32}$

In Table 9, for $k = s^2 + 4$ and $s \equiv 3 \,(\bmod 4)$, one has $(x_f, y_f) = \left(\frac{(s^2+1)^2(s^2+4)}{2} - 1, \frac{s(s^2+1)(s^2+3)}{2}\right), r = 4$, and $(X_{f_{3,4}}, Y_{f_{3,4}}) = \left(\pm(\frac{s(s^2+s+4)}{2} + 1), \frac{s(s+1)}{2} + 1\right)$ yields, by (28), $(\xi_1, t_1) = \left(\frac{s(s^2+s+4)}{4}), \frac{s(s+1)}{4}\right)$, and, by (29), $(\xi_2, t_2) = \left(\frac{(s^2+2)(s^2(s-1)+2(2s-1))}{4}, \frac{s(s^2(s-1)+2(2s-1))}{4}\right)$.

We can now calculate for each non-square integer k, the fundamental solutions x_f and y_f of the simple Pell equation and the four recurrent relations yielding $t_n, \xi_n, T_{t_n}, T_{\xi_n}$, and the first $2r$ values of each variable. These are given in [27].

5. Conclusions

We have shown that the problem of finding all triangular numbers that are multiples of other triangular numbers with non-square integer multiplier k can be solved using solutions of Pell equations with a simple change of variables. Only those r fundamental solutions (X_{fi}, Y_{fi}) of the generalized Pell equation with X_{fi} odd and Y_{fi} odd or even (if y_f is not even) provide solutions to the problem of finding triangular numbers that are multiple of other triangular numbers. General expressions of fundamental solutions of the Pell equations are given for some cases of the multiplier k in function of the closest natural square s^2. Infinitely many solutions are then found on r branches corresponding to each of the r generalized fundamental solutions (X_{f_i}, Y_{f_i}) and these solutions can be found either by a general relation involving $\sqrt{k}$, or by two sets of recurrent relations, or by Chebyshev polynomial solutions. Among these, the second set of recurrent relations are found to be the same as those found previously without using the Pell equation solving method.

Furthermore, the number r of generalized fundamental solutions (X_{f_i}, Y_{f_i}) with X_{f_i}, odd and Y_{f_i} odd or even (if y_f is not even) corresponds to the rank of these second set recurrent relations. The two constants $\kappa = t_r + t_{r-1}$ and $\delta = t_r - t_{r-1}$ are also related to, respectively, the fundamental solutions x_f and y_f of the simple Pell equation, as $\kappa = x_f - 1$ and $\delta = y_f$ or $\delta = 2x_f y_f$, except for some cases of $k \equiv 0 \pmod 8$.

Finally, we have found simplified expressions of recurrent relations of the four variables for k expressed in function of s, the closest integer smaller than $\sqrt{k}$. Recurrent relations yielding t, ξ, T_t and T_ξ can then be calculated for non-square k.

Table 9. Expressions of $t_n, \xi_n, T_{t_n}, T_{\xi_n}$ in function of s for $k = s^2 + 4$

$k = s^2 + 4, s \equiv 3(\bmod 4), r = 4$
$t_n = (s^2 + 2)(s^2(s^2 + 4) + 1)t_{n-4} - t_{n-8} + \frac{s^2(s^2+3)^2}{2},$
$t_4 = \frac{s(s^2+3)(s^2(s+1)+3s+1)}{4}, t_3 = \frac{s(s^2+3)(s^2(s-1)+3s-1)}{4},$
$t_2 = \frac{s(s^2(s-1)+2(2s-1))}{4}, t_1 = \frac{s(s+1)}{4}$
$\xi_n = (s^2 + 2)(s^2(s^2 + 4) + 1)\xi_{n-4} - \xi_{n-8} + \frac{s^2(s^2+3)^2}{2},$
$\xi_4 = \frac{s(s^2+3)(s^2(s+1)+5s^2+3s+4)}{4}, \xi_3 = \frac{(s^2+1)(s^2+4)(s^2(s-1)+3s-1)}{4}$
$\xi_2 = \frac{(s^2+2)(s^2(s-1)+2(2s-1))}{4}, \xi_1 = \frac{s(s^2+s+4)}{4}$
$T_{t_n} = ((s^2 + 2)^2 - 2)(s^6(s^2 + 8) + 4s^2(5s^2 + 4) + 1)T_{t_{n-4}} - T_{t_{n-8}} + \frac{s^2(s^2+1)^2(s^2+3)^2(s^2+5)}{16}$
$T_{t_4} = \frac{s(s^2+1)(s^2+3)(s^2(s+1)+3s+1)(s^2(s+1)+s(5s+3)+4)}{32}$
$T_{t_3} = \frac{s(s^2+1)(s^2+3)(s^2(s-1)+3s-1)(s^2(s-1)+s(5s-3)+4}{32}$
$T_{t_2} = \frac{s(s^2+2)(s(s-1)+2)(s^2(s-1)+2(2s-1))}{32}, T_{t_1} = \frac{s(s+1)(s(s+1)+4)}{32}$
$T_{\xi_n} = ((s^2 + 2)^2 - 2)(s^6(s^2 + 8) + 4s^2(5s^2 + 4) + 1)T_{\xi_{n-4}} - T_{\xi_{n-8}} + \frac{s^2(s^2+1)^2(s^2+3)^2(s^2+4)(s^2+5)}{16}$
$T_{\xi_4} = \frac{s(s^2+1)(s^2+3)(s^2+4)(s^2(s+1)+3s+1)(s^2(s+1)+s(5s+3)+4)}{32}$
$T_{\xi_3} = \frac{s(s^2+1)(s^2+3)(s^2+4)(s^2(s-1)+3s-1)(s^2(s-1)+s(5s-3)+4)}{32}$
$T_{\xi_2} = \frac{s(s^2+2)(s^2+4)(s(s-1)+2)(s^2(s-1)+2(2s-1))}{32}, T_{\xi_1} = \frac{s(s+1)(s^2+4)(s(s+1)+4)}{32}$

References

[1] Weisstein, E. W. (2021). *Triangular Number*. From MathWorld–A Wolfram Web Resource.

[2] Andrews, G. E. (1971). *Number Theory*. Dover.

[3] Cunningham, D. J. (1901). Address of the President of the Anthropological Section of the British Association, II. *Science, 14*(356), 640-647.

[4] de Joncourt, E. (1762). *The Nature and Notable Use of the Most Simple Trigonal Numbers*. The Hague: Husson.

[5] Roegel, D. (2013). A Reconstruction of Joncourt's Table of Triangular Numbers (1762).

[6] Chahal, J., & D'Souza, H. (2017). Some remarks on triangular numbers. In *Number Theory With an Emphasis on the Markoff Spectrum* (pp. 61-67). Routledge.

[7] Breiteig, T. (2015). Quotients of triangular numbers. *The Mathematical Gazette, 99*(545), 243-255.

[8] Chahal, J., Griffin, M., & Priddis, N. (2019). When are multiples of polygonal Numbers again polygonal numbers?. *Hardy-Ramanujan Journal*, 58-67.

[9] Dickson, L. E. (1920). *History of the Theory of Numbers*, vol. 2, Diophantine Analysis, Carnegie Inst.

[10] Pletser, V. (2022). Recurrent relations for triangular numbers multiples of other triangular numbers. *Indian Journal of Pure and Applied Mathematics, 53*(3), 782-791.

[11] Sloane, N. J. A., ed. (2021). The On-Line Encyclopedia of Integer Sequences. https://oeis.org.

[12] Well, A. (1984). *Number Theory, an Approach Through History*. Birkhausiur Boston, 40-49.

[13] Lenstra Jr, H. W. (2002). Solving the Pell equation. *Notices of the Ams, 49*(2), 182-192.

[14] Lemmermeyer, F. (2021). The Pell Equation. http://www.fen.bilkent.edu.tr/~franz/publ/pell.htm
[15] O'Connor, J. J. & Robertson, E. F. (2002). Pell's equation. https://mathshistory.st-andrews.ac.uk/HistTopics/Pell/
[16] Nagell, T. (2021). *Introduction to Number Theory* (Vol. 163). American Mathematical Society: 8-238.
[17] Weisstein, E. W. (2021). *Pell Equation*. from MathWorld–A Wolfram Web Resource.
[18] Jacobson, M. J., & Williams, H. C. (2009). *Solving the Pell Equation* (pp. 353-359). New York: Springer.
[19] Robertson, J. P. (2004). *Solving the Generalized Pell Equation* $X^2 - Dy^2 = N$. Unpublished manuscript.
[20] Lagrange, J. L. (1867). Solution d'un Probleme d'Arithmetique. In *Serret, J.-A., Ed., Oeuvres De Lagrange, Vol. 1, Gauthier-Villars*, Paris, 671–731.
[21] Chrystal, G. (1964). *Algebra: An Elementary Text-Book.*–. Black.
[22] Mollin, R. A. (1997). *Fundamental Number Theory With Applications*. Crc Press: 1-313.
[23] Matthews, K. (2000). The Diophantine Equation $x^2 - Dy^2 = N, D > 0$. *Expositiones Mathematicae, 18*(4), 323-332.
[24] Matthews, K. (2014). *Quadratic Diophantine Equations Bcmath Programs.*
[25] Pletser, V. (2014). On continued fraction development of quadratic irrationals having all periodic terms but last equal and associated general solutions of the Pell equation. *Journal of Number Theory, 136*, 339-353.
[26] Pletser, V. (2014). Finding all squared integers expressible as the sum of consecutive squared integers using generalized Pell equation solutions with Chebyshev polynomials. arXiv preprint arXiv:1409.7972.
[27] Pletser, V. (2022). Recurrent relations for triangular numbers multiples of other triangular numbers. *Indian Journal of Pure and Applied Mathematics, 53*(3), 782-791.