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Wijsman statistical and strong Cesàro convergence of closed sets via power-type Musielak–Orlicz Modulars

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Abstract: We study convergence of sequences of nonempty closed sets in a metric space through the power-type Musielak–Orlicz modular $\Phi_n(t) = t^{p_n}$. We introduce Wijsman (p_n)-statistical convergence, Wijsman (p_n)-strong Cesàro convergence, and a lacunary block-mean statistical version associated with a lacunary sequence $\theta = (k_r)$. In the bounded-exponent case $1 \leq p_n \leq p_+ < \infty$, we prove that, for Wijsman bounded sequences, these three modes of convergence are equivalent whenever the lacunary moderate growth condition holds. In the power-type setting this condition is equivalent to $h_r/k_r \to 0$. We also give comparison results and examples showing why boundedness of the exponent sequence is essential. For random closed sets, we prove a pathwise Cesàro transfer result: at a fixed base point, almost sure modular strong Cesàro control, together with almost sure boundedness of the distance evaluations and bounded exponents, yields almost sure convergence of the Cesàro averages of the distance functionals to the Wijsman limit.

Keywords: Wijsman convergence, Musielak–Orlicz modular, variable exponent, lacunary sequence, statistical convergence, random closed set

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1. Introduction

Let X be a metric space with a metric d, and let $\mathcal{C}(X)$ be the set of all nonempty closed subsets of X. For a point $x \in X$ and a set $A \in \mathcal{C}(X)$, the distance from x to A is defined as $d(x, A) = \inf_{a \in A} d(x, a)$. A fundamental mode of convergence for sequences in $\mathcal{C}(X)$ is Wijsman convergence, which requires pointwise convergence of the distance functionals $d(x, A_n)$ toward $d(x, A)$ for every $x \in X$. Since it is formulated through real-valued functions, Wijsman convergence provides a natural and flexible approach for set convergence and interacts effectively with several topologies on families of closed sets [1,2].

Statistical convergence offers a different perspective on convergence by relaxing the requirement of convergence at all indices. In the classical form introduced by Fast and Steinhaus, the indices where convergence fails are required to form a set of natural density zero [3,4]. Later developments clarified the structure of statistical convergence and its relation to summability and Cesàro type averaging [5,6]. This circle of ideas has been influential in modern summability theory and sequence space methods [7,8]. When statistical ideas are combined with Wijsman convergence, one obtains Wijsman statistical convergence and its variants for sequences of closed sets, which have been investigated in different directions, including strong Cesàro type approaches [9,10]. Links between statistical convergence and strong summability methods were clarified early on through modulus based formulations and matrix summability schemes, which also helped to organize inclusion relations among these modes of convergence [11,12].

A further refinement is obtained by introducing modular weights through Orlicz type functions. In this setting one replaces the ordinary density and classical averaging procedures by expressions controlled by convex modulars, which allows one to capture finer growth and regularity effects. The Musielak–Orlicz approach , in which the modular is allowed to vary with the index, provides a particularly broad and adaptable setting [13,14]. Related work on Musielak–Orlicz sequence spaces and the associated convergence notions shows that allowing index-dependent modulars yields behaviors that do not arise in the classical Orlicz setting [13–15]. Modulus and Orlicz type weights also appear naturally in statistical convergence schemes beyond the classical density setting [16]. Within this approach, Φ-statistical convergence, Φ-strong Cesàro convergence and lacunary Φ-statistical convergence of sequences of closed sets can be formulated in a unified way. A systematic account of these notions, together with an equivalence theory under a moderate growth condition, is given in [17].

The present paper focuses on a power-type Musielak–Orlicz choice in which $\Phi_n(t) = t^{p_n}$ holds for $t \geq 0$ and for an exponent sequence (p_n) satisfying $p_n \geq 1$. This special case retains the modular character of the Musielak–Orlicz setting while introducing a variable growth rate through the exponents. It can be viewed as a discrete variable exponent modular that parallels variable growth methodologies in the Musielak–Orlicz literature [13,14]. In this context we define Wijsman p_n-statistical convergence and Wijsman p_n-strong Cesàro convergence, and we also introduce the corresponding lacunary version associated with a lacunary sequence. Our main deterministic equivalence result is proved for Wijsman bounded sequences under the lacunary moderate growth condition and under the bounded-exponent assumption $1 \leq p_n \leq p_+ < \infty$, in a form consistent with the general Musielak–Orlicz theory developed in [17]. The role of the exponent sequence is made explicit, and the bounded exponent case is treated in a way that yields a direct comparison with ordinary Cesàro type controls, which connects the modular scheme to standard summability arguments [7,8]. Related convergence and summability ideas also appear in operator-theoretic and probabilistic settings. For statistical convergence phenomena in operator-theoretic settings and in Fock-type spaces we refer to [18,19], while credibility-based models and further operator-oriented contributions along these lines can be found in [20–22].

The contribution of the paper is therefore not only the restatement of the general Musielak–Orlicz framework for the special choice $\Phi_n(t) = t^{p_n}$. Although the results are motivated by the general Musielak–Orlicz theory, the power-type case is treated here with explicit estimates and a self-contained proof of the main equivalence theorem. In the power-type case the moderate growth condition can be written explicitly as $h_r/k_r \to 0$, which clarifies the precise lacunary regime in which block control and ordinary Cesàro control are compatible. We also isolate the bounded-exponent comparison

$$\frac{1}{n}\sum_{k=1}^{n} t_k^{p_k} \to 0 \quad \Longrightarrow \quad \frac{1}{n}\sum_{k=1}^{n} t_k \to 0,$$

for bounded nonnegative sequences (t_k) when $1 \leq p_k \leq p_+ < \infty$. This comparison explains why the boundedness of (p_n) is needed in the main equivalence theorem. The examples given below show that unbounded exponents may lead to different behaviour, and the random closed set result is obtained by applying the deterministic Cesàro estimate along almost every sample path.

The power-type Orlicz viewpoint is also motivated by a broader line of recent work in which Orlicz based statistical, lacunary and ideal convergence notions are developed for multi-indexed sequences in generalized settings. Such developments include ideal convergence with Orlicz functions in neutrosophic n-normed spaces and generalized statistical convergence for uncertain sequences of fuzzy numbers [23,24]. Further directions along the same line involve λ-weak convergence defined by Orlicz functions and lacunary A-statistical schemes via Orlicz control [25–27]. Related Orlicz based Wijsman type lacunary and invariant statistical convergence phenomena for higher order index structures also support the relevance of modular set convergence contexts [28]. Ideal and statistical variants of convergence in generalized normed and random settings have also been investigated. For example, lacunary ideal convergence in random n-normed spaces and I-statistical convergence in 2-normed spaces were studied in [29,30], while completion issues in linear n-normed spaces were considered in [31].

Recent work also indicates that summability ideas for set-valued objects can be pursued in directions that are close to the present theme. In particular, I-statistical pre-Cauchy and frequent Cauchy criteria have been investigated for sequences of compact sets in the Hausdorff hyperspace of a metric space, providing a complementary set-convergence viewpoint beyond the pointwise distance-function approach [32]. Probabilistic variants of weighted and deferred summability, including rough asymptotically deferred weighted statistical equivalence of order α in probability, indicate that Cesàro-type averaging may respond differently to random perturbations under different summability procedures [33]. These developments reinforce the relevance of modular and weighted approaches when set convergence is analyzed through index-dependent growth.

We also include a probabilistic component motivated by random closed sets. Distance functionals of random closed sets are real-valued random variables, and Cesàro averaging of such functionals naturally leads to stabilization questions for distance evaluations. In the power-type modular setting, we prove a pathwise transfer result: under suitable boundedness and modular hypotheses at a fixed base point, the Cesàro averages of the distance functionals converge almost surely to the Wijsman limit. This pathwise formulation is related to, but weaker in probabilistic content than, the strong-law-type analogue discussed in [17].

The paper is organized as follows. §2 collects the necessary preliminaries on Wijsman convergence, Musielak–Orlicz modulars, and the relevant statistical and lacunary notions. §3 develops the power-type approach and proves the equivalence theorem under the bounded-exponent assumption and the lacunary moderate growth condition. §4 presents the probabilistic application to random closed sets. §5 provides illustrative examples and a comparative summary. The final section contains concluding remarks and possible directions for further work.

2. Preliminaries

This section collects the basic notions and notations that will be used throughout the paper. The setting is a metric space X equipped with a metric d. The symbol $\mathcal{C}(X)$ denotes the collection of all nonempty closed subsets of X.

For a point $x \in X$ and a set $A \in \mathcal{C}(X)$, the distance from x to A is defined by $d(x, A) = \inf_{a \in A} d(x, a)$. Given a sequence (A_n) in $\mathcal{C}(X)$ and a candidate limit set A in $\mathcal{C}(X)$, the deviation at x is measured by

$$\gamma_n(x) = \big|d(x, A_n) - d(x, A)\big|, \ x \in X.$$

This scalar quantity allows one to transfer questions on set convergence to questions on convergence of real-valued sequences.

Definition 1. A sequence (A_n) in $\mathcal{C}(X)$ is said to converge to A in $\mathcal{C}(X)$ in the sense of Wijsman, and this is denoted by $A_n \xrightarrow{W} A$, if $d(x, A_n) \to d(x, A)$ for every $x \in X$.

The definition implies that $A_n \xrightarrow{W} A$ holds precisely when $\gamma_n(x) \to 0$ for every $x \in X$. Since later arguments require bounds on the scalar sequence $\big(\gamma_n(x)\big)$ for each fixed point $x \in X$, a boundedness assumption at the level of distance functionals is often imposed.

Definition 2. A sequence (A_n) in $\mathcal{C}(X)$ is called Wijsman bounded if for every $x \in X$ the numerical sequence $\big(d(x, A_n)\big)$ is bounded. Equivalently, for every $x \in X$ one has $\sup_{n \in \mathbb{N}} d(x, A_n) < \infty$.

When a sequence is Wijsman bounded, the deviations $\gamma_n(x)$ are bounded for each fixed x, since $\gamma_n(x) \le d(x, A_n) + d(x, A)$ holds for all n. This elementary observation will be used repeatedly when comparing different modular expressions.

The next group of definitions introduces modular weights. These weights are encoded by Orlicz type functions and their indexed generalization, which provides a flexible tool to measure the size of exceptional index sets.

Definition 3. A function Φ from $[0, \infty)$ into $[0, \infty)$ is called an Orlicz function if it is nondecreasing, continuous and convex, satisfies $\Phi(0) = 0$, satisfies $\Phi(t) > 0$ for all $t > 0$, and satisfies $\Phi(t) \to \infty$ as $t \to \infty$.

Definition 4. A mapping Φ from $\mathbb{N} \times [0, \infty)$ into $[0, \infty)$ is called a Musielak–Orlicz function if for each $n \in \mathbb{N}$ the function Φ_n defined by $\Phi_n(t) = \Phi(n, t)$ is an Orlicz function.

For later estimates it is convenient to recall a standard consequence of convexity.

Lemma 1. *Let Φ be an Orlicz function. For every $t \geq 0$ and every λ satisfying $0 \leq \lambda \leq 1$, one has $\Phi(\lambda t) \leq \lambda\Phi(t)$.*

Proof. Convexity yields $\Phi(\lambda t + (1-\lambda)0) \leq \lambda\Phi(t) + (1-\lambda)\Phi(0)$. Since $\Phi(0) = 0$, the desired inequality follows. □

The modular weights enter the statistical approach through a density concept. The following definition is taken in the Musielak–Orlicz setting and reduces to the classical density-zero requirement when the weight is linear.

Definition 5. Let $\Phi = \{\Phi_n\}$ be a Musielak–Orlicz function. Let K be a subset of $\mathbb{N}$ and let $\varepsilon > 0$ be fixed. The Φ-density of K at level ε is defined by

$$\delta_\Phi(K,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_K(k)\,\Phi_k\left(\frac{1}{\varepsilon}\right),$$

where χ_K denotes the indicator function of K.

When $\Phi_k(t) = t$ holds for every k, the expression $\delta_\Phi(K,\varepsilon)$ equals ε^{-1} times the upper natural density of K. Therefore the condition $\delta_\Phi(K,\varepsilon) = 0$ is equivalent to the requirement that K has natural density zero.

We now record the convergence notions that will be used in the sequel. The first one controls the indices at which the deviation stays above a prescribed level, while the second one controls the average modular deviation.

Definition 6. Let $\Phi = \{\Phi_n\}$ be a Musielak–Orlicz function. A sequence (A_n) in $\mathcal{C}(X)$ is said to be Wijsman Φ-statistically convergent to A in $\mathcal{C}(X)$, and this is denoted by $A_n \xrightarrow{W_\Phi\text{-}st} A$, if for every $x \in X$ and every $\varepsilon > 0$,

$$\delta_\Phi\Big(\{n\in\mathbb{N} : \gamma_n(x) \geq \varepsilon\}, \varepsilon\Big) = 0.$$

Definition 7. Let $\Phi = \{\Phi_n\}$ be a Musielak–Orlicz function. A sequence (A_n) in $\mathcal{C}(X)$ is said to be Wijsman Φ-strong Cesàro convergent to A in $\mathcal{C}(X)$, and this is denoted by $A_n \xrightarrow{W_\Phi\text{-}sc} A$, if for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \Phi_k\big(\gamma_k(x)\big) = 0.$$

The next definitions introduce lacunary decompositions. These decompositions separate the index set into blocks whose lengths tend to infinity and allow a refined control of modular block means.

A sequence $\theta = (k_r)$ of positive integers is called lacunary in this paper if $k_0 = 0$, $k_r < k_{r+1}$ for all r, and the gaps

$$h_r = k_r - k_{r-1},$$

satisfy $h_r \to \infty$ as $r \to \infty$. For each $r \in \mathbb{N}$, the lacunary interval I_r is defined by $I_r = (k_{r-1}, k_r]$. For $r \geq 2$, we write

$$q_r = \frac{k_r}{k_{r-1}}.$$

The ratio q_r is useful for describing the relative growth of the lacunary sequence, but no standing assumption such as $\liminf_{r\to\infty} q_r > 1$ is imposed in this paper.

Definition 8. Let $\Phi = \{\Phi_n\}$ be a Musielak–Orlicz function and let $\theta = (k_r)$ be lacunary. Let $K \subset \mathbb{N}$ be a set of block indices and let $\varepsilon > 0$. The lacunary Φ-density of K with respect to θ at level ε is defined by

$$\delta_\Phi^\theta(K,\varepsilon) = \limsup_{R\to\infty} \frac{\sum\limits_{r\in K\cap\{1,\ldots,R\}} h_r\,\Phi_{k_r}\left(\frac{1}{\varepsilon}\right)}{\sum\limits_{r=1}^{R} h_r\,\Phi_{k_r}\left(\frac{1}{\varepsilon}\right)}.$$

Remark 1. The weight $\Phi_{k_r}(1/\varepsilon)$ is taken at the right endpoint of the block I_r in order to assign one modular scale to the whole block. Thus the lacunary density above is a density of block indices, weighted by the block lengths and by the modular scale attached to the endpoint of each block. This convention is compatible with the block-mean formulation used below and allows the lacunary condition to be compared directly with ordinary Cesàro modular means.

Definition 9. Let $\Phi = \{\Phi_n\}$ be a Musielak–Orlicz function and let $\theta = (k_r)$ be lacunary with intervals $I_r = (k_{r-1}, k_r]$. A sequence (A_n) in $\mathcal{C}(X)$ is said to be lacunary Wijsman Φ-statistically convergent to A in $\mathcal{C}(X)$ with respect to θ, and this is denoted by $A_n \xrightarrow{W_\Phi^\theta\text{-}st} A$, if for every $x \in X$ and every $\varepsilon > 0$,

$$\delta_\Phi^\theta\left(\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}\Phi_k(\gamma_k(x)) \geq \varepsilon\right\}, \varepsilon\right) = 0.$$

Remark 2. The lacunary notion used here is a block-mean statistical condition. It does not count, inside each block I_r, the indices for which $\gamma_k(x)$ is large. Instead, it first forms the modular block mean

$$\frac{1}{h_r}\sum_{k\in I_r}\Phi_k(\gamma_k(x)),$$

and then regards the block I_r as exceptional when this mean is at least ε. Thus the definition is closer to lacunary strong summability than to the classical lacunary statistical condition based on the relative number of exceptional indices inside each block. In the usual lacunary Wijsman statistical setting, one often counts the indices $k \in I_r$ for which $\gamma_k(x) \geq \varepsilon$ and then normalizes by h_r. By contrast, the present definition first averages the modular deviations over the whole block and then treats the block itself as exceptional when this modular mean is large. This is the intended form in the present paper, since the main comparison is between block modular means and ordinary Cesàro modular means.

Remark 3. All convergence notions in this paper are understood pointwise in the base point $x \in X$, in the sense that for each fixed x the corresponding scalar sequence $\big(d(x, A_n)\big)$ is required to satisfy the stated condition. No uniformity with respect to x is assumed unless explicitly stated.

The equivalence results in the next section require a compatibility condition between the lacunary gaps and the Musielak–Orlicz growth. For later reference, we record this condition here.

Definition 10. Let $\Phi = \{\Phi_n\}$ be a Musielak–Orlicz function and let $\theta = (k_r)$ be lacunary. The family Φ is said to satisfy the moderate growth condition with respect to θ if

$$\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\frac{\Phi_n(h_r)}{\Phi_n(k_r)} = 0.$$

Remark 4. The term lacunary is used here only in the sense that the block lengths satisfy $h_r \to \infty$. In many parts of the lacunary convergence literature one also assumes $\liminf_{r\to\infty} q_r > 1$. Such an assumption is not imposed in this paper. In fact, for the power-type family $\Phi_n(t) = t^{p_n}$, the moderate growth condition will be shown to be equivalent to $\frac{h_r}{k_r} \to 0$. Since

$$\frac{k_r}{k_{r-1}} = \frac{1}{1 - h_r/k_r} \quad (r \geq 2),$$

this condition implies $q_r \to 1$. Hence the relevant lacunary sequences in the main equivalence theorem have gaps tending to infinity but have small relative gaps.

At this point the basic convergence notions have been fixed in a form that is suitable for modular refinements. In the next section the Musielak–Orlicz family will be specialized to the power-type choice $\Phi_n(t) = t^{p_n}$, and the corresponding Wijsman p_n-statistical and Wijsman p_n-strong Cesàro convergences will be formulated explicitly in terms of the deviations $\gamma_n(x)$. The lacunary context introduced through the sequence θ and the intervals I_r will be used to compare block averages with ordinary Cesàro means. The moderate growth condition recorded above will play the central role in passing from lacunary modular control to ordinary modular control, and it will provide the key step in proving equivalence among the three convergence modes. Finally, the Wijsman boundedness assumption will be invoked whenever bounds on the sequence $\big(\gamma_n(x)\big)$ are needed for a fixed point $x \in X$ in order to justify comparisons between different modular expressions.

3. Power-type Musielak–Orlicz modulars and variable exponents

This section specializes the Musielak–Orlicz family to the power-type choice and develops the corresponding convergence notions and comparison principles. The purpose is twofold. The first aim is to make the growth condition in the equivalence theorem explicit in terms of the lacunary gaps and the exponent sequence. The second aim is to clarify how boundedness of the exponent sequence influences the relation between modular control and ordinary Cesàro control of the deviations.

Throughout this section the Musielak–Orlicz family $\Phi = \{\Phi_n\}$ is given by $\Phi_n(t) = t^{p_n}$ for $t \geq 0$, where $p_n \geq 1$ holds for every $n \in \mathbb{N}$. For convenience the quantities p_- and p_+ are defined by

$$p_- = \inf_{n\in\mathbb{N}} p_n, \text{ and } p_+ = \sup_{n\in\mathbb{N}} p_n.$$

When $p_+ < \infty$, the family has a uniform power control which provides a convenient connection between the modular expression t^{p_n} and the linear expression t under pointwise boundedness of the deviations.

3.1. Wijsman (p_n)-convergences

The definitions from §2 take a particularly concrete form in the power-type setting. The next two notions will be the basic objects in the equivalence results and in the probabilistic application.

Definition 11. A sequence $(A_n) \subset \mathcal{C}(X)$ is Wijsman (p_n)-statistically convergent to $A \in \mathcal{C}(X)$ if for every $x \in X$ and every $\varepsilon > 0$, letting $E(x,\varepsilon) = \{k \in \mathbb{N} : \gamma_k(x) \geq \varepsilon\}$, one has

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{E(x,\varepsilon)}(k) \left(\frac{1}{\varepsilon}\right)^{p_k} = 0.$$

Definition 12. A sequence $(A_n) \subset \mathcal{C}(X)$ is Wijsman (p_n)-strong Cesàro convergent to $A \in \mathcal{C}(X)$ if for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \gamma_k(x)^{p_k} = 0.$$

The lacunary version is obtained by replacing the Cesàro means with lacunary block means over the intervals I_r introduced in §2 and by using the lacunary Φ-density δ_Φ^θ.

The next elementary lemma records two monotonicity facts that will be used repeatedly.

Lemma 2. *Let t be a real number satisfying $0 \leq t \leq 1$ and let $1 \leq a \leq b$. Then $t^b \leq t^a$. If $t \geq 1$ holds, then $t^a \leq t^b$.*

Proof. If $t = 0$ or $t = 1$, the assertions are immediate. Assume $t \in (0,1)$ and consider the function $f(s) = t^s$ for $s \geq 1$. Writing $f(s) = e^{s\ln t}$, we obtain $f'(s) = t^s \ln t$. Since $\ln t < 0$ for $t \in (0,1)$ and $t^s > 0$, it follows that $f'(s) < 0$ on $[1,\infty)$, hence f is strictly decreasing there. Therefore, whenever $1 \leq a \leq b$, one has $t^b \leq t^a$.

If $t > 1$, the same computation yields $\ln t > 0$, hence $f'(s) > 0$ and f is strictly increasing on $[1,\infty)$. Consequently, $t^a \leq t^b$ for $1 \leq a \leq b$. The case $t = 1$ was already covered, so the proof is complete. □

A first consequence is that Wijsman convergence implies modular strong Cesàro convergence in the power-type setting, without any extra hypothesis on (p_n).

Proposition 1. *If $A_n \xrightarrow{W} A$, then A_n is Wijsman (p_n)-strong Cesàro convergent to A.*

Proof. Fix $x \in X$ and set $\gamma_n = \gamma_n(x) = |d(x, A_n) - d(x, A)|$. The assumption $A_n \xrightarrow{W} A$ yields $\gamma_n \to 0$. Since $p_n \geq 1$ for all n, there exists $N \in \mathbb{N}$ such that $0 \leq \gamma_n < 1$ for $n \geq N$, and hence $0 \leq \gamma_n^{p_n} \leq \gamma_n$ $(n \geq N)$. Therefore $\gamma_n^{p_n} \to 0$ as $n \to \infty$. To conclude, we use the classical Cesàro principle: if a real sequence (a_n) satisfies $a_n \to 0$, then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \longrightarrow 0.$$

Indeed, given $\varepsilon > 0$, choose N such that $|a_k| < \varepsilon$ for all $k \geq N$. Then for $n \geq N$,

$$\frac{1}{n}\sum_{k=1}^{n} |a_k| \leq \frac{1}{n}\sum_{k=1}^{N-1} |a_k| \; + \; \varepsilon\frac{n-(N-1)}{n} \leq \frac{1}{n}\sum_{k=1}^{N-1} |a_k| \; + \; \varepsilon,$$

and letting $n \to \infty$ gives $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |a_k| \leq \varepsilon$. Since ε is arbitrary, the Cesàro means converge to 0.

Applying this to $a_k = \gamma_k^{p_k}$ yields

$$\frac{1}{n}\sum_{k=1}^{n} \gamma_k(x)^{p_k} \to 0,$$

which is exactly the Wijsman (p_n)-strong Cesàro convergence of A_n to A. □

The same reasoning gives statistical convergence, and the statement is included for completeness since it clarifies the logical hierarchy.

Proposition 2. *If $A_n \xrightarrow{W} A$, then A_n is Wijsman (p_n)-statistically convergent to A.*

Proof. Fix $x \in X$ and $\varepsilon > 0$, and set $E(x,\varepsilon) = \{k \in \mathbb{N} : \gamma_k(x) \geq \varepsilon\}$. Since $\gamma_k(x) \to 0$, there exists $N \in \mathbb{N}$ such that $\gamma_k(x) < \varepsilon$ for all $k \geq N$. Hence $E(x,\varepsilon) \subset \{1, 2, \ldots, N-1\}$ is finite. Therefore, for every $n \geq N$,

$$\sum_{k=1}^{n} \chi_{E(x,\varepsilon)}(k)\, \varepsilon^{-p_k} = \sum_{k\in E(x,\varepsilon)} \varepsilon^{-p_k} =: C(x,\varepsilon) < \infty,$$

and consequently

$$0 \leq \frac{1}{n}\sum_{k=1}^{n} \chi_{E(x,\varepsilon)}(k)\, \varepsilon^{-p_k} \leq \frac{C(x,\varepsilon)}{n} \xrightarrow[n\to\infty]{} 0.$$

This proves the asserted Wijsman (p_n)-statistical convergence. □

Remark 5. The reverse implications are generally false. Statistical convergence allows infinitely many exceptional indices, and the strength of the modular Cesàro condition depends sensitively on the growth of (p_n). In particular, bounded and unbounded exponent regimes lead to different separation phenomena; see the examples given later in this section.

3.2. Moderate growth in the power-type setting

The equivalence results rely on a compatibility condition between the lacunary gaps and the growth of the modular. In the power-type case this condition can be rewritten in a transparent form.

We recall the moderate growth condition from §2 and specialize it to the power-type choice $\Phi_n(t) = t^{p_n}$.

Lemma 3. *Let $\theta = (k_r)$ be lacunary and let $\Phi_n(t) = t^{p_n}$ with $p_n \geq 1$. For $a_r := h_r/k_r \in (0,1)$ one has*

$$\sup_{n\in\mathbb{N}} \frac{\Phi_n(h_r)}{\Phi_n(k_r)} = \sup_{n\in\mathbb{N}} \left(\frac{h_r}{k_r}\right)^{p_n} = \left(\frac{h_r}{k_r}\right)^{p_-}.$$

Consequently, the moderate growth condition holds if and only if $h_r/k_r \to 0$.

Proof. Fix $r \in \mathbb{N}$ and set $a_r = h_r/k_r$. Since $0 < h_r < k_r$, we have $a_r \in (0,1)$. Then

$$\frac{\Phi_n(h_r)}{\Phi_n(k_r)} = \frac{h_r^{p_n}}{k_r^{p_n}} = a_r^{p_n}.$$

Because the map $p \mapsto a_r^p$ is strictly decreasing on $(0,\infty)$, the inequality $p_n \geq p_-$ yields $a_r^{p_n} \leq a_r^{p_-}$ for every n, hence $\sup_n a_r^{p_n} \leq a_r^{p_-}$.

To obtain the reverse inequality, let $\eta > 0$. By the definition of $p_- = \inf_{n\in\mathbb{N}} p_n$ there exists $n_\eta \in \mathbb{N}$ such that $p_{n_\eta} < p_- + \eta$. Since $a_r \in (0,1)$, monotonicity gives $a_r^{p_{n_\eta}} > a_r^{p_-+\eta}$, and therefore $\sup_{n\in\mathbb{N}} a_r^{p_n} \geq a_r^{p_-+\eta}$. Letting $\eta \downarrow 0$ and using continuity of $p \mapsto a_r^p$ yields $\sup_{n\in\mathbb{N}} a_r^{p_n} \geq a_r^{p_-}$. Hence $\sup_{n\in\mathbb{N}} a_r^{p_n} = a_r^{p_-}$.

Finally, since $p_- > 0$, for a sequence $(a_r) \subset (0,1)$ one has $a_r^{p_-} \to 0$ if and only if $a_r \to 0$. Applying this to $a_r = h_r/k_r$ proves the last assertion. □

Remark 6. Lemma 3 shows that, in the power-type case, the moderate growth requirement is equivalent to $h_r/k_r \to 0$. For $r \geq 2$ this also gives

$$q_r = \frac{k_r}{k_{r-1}} = \frac{1}{1 - h_r/k_r} \to 1.$$

Thus the lacunary sequences relevant to Theorem 1 do not have a fixed relative gap. This is why the present paper does not assume $\liminf_{r\to\infty} q_r > 1$. For example, $k_r = 2^r$ gives $h_r/k_r \to 1/2$ and therefore fails the moderate growth condition, whereas $k_r = r^2$ gives $h_r/k_r \to 0$ and satisfies it.

The main equivalence theorem in this section is the power-type specialization of the Musielak–Orlicz equivalence principle.

Theorem 1. *Assume $(A_n) \subset \mathcal{C}(X)$ is Wijsman bounded and that the exponent sequence satisfies $1 \leq p_n \leq p_+ < \infty$. Let $\Phi_n(t) = t^{p_n}$ and let $\theta = (k_r)$ be lacunary with gaps $h_r = k_r - k_{r-1}$. If Φ satisfies the moderate growth condition with respect to θ, then the following are equivalent:*

(i) *A_n is Wijsman (p_n)-statistically convergent to A,*
(ii) *A_n is lacunary Wijsman (p_n)-statistically convergent to A with respect to θ,*
(iii) *A_n is Wijsman (p_n)-strong Cesàro convergent to A.*

Proof. Fix $x \in X$ and set

$$\gamma_k = \gamma_k(x) = |d(x, A_k) - d(x, A)|.$$

Since (A_n) is Wijsman bounded, the numerical sequence $\big(d(x, A_k)\big)$ is bounded. Hence the deviation sequence (γ_k) is bounded for this fixed x. Choose

$$M_x := \max\left\{1, \sup_{k\in\mathbb{N}} \gamma_k(x)\right\}.$$

Then $1 \leq M_x < \infty$ and $0 \leq \gamma_k \leq M_x$ for all $k \in \mathbb{N}$. Put $u_k = \gamma_k^{p_k}$. Since $1 \leq p_k \leq p_+ < \infty$ and $M_x \geq 1$, we have

$$0 \leq u_k = \gamma_k^{p_k} \leq M_x^{p_k} \leq M_x^{p_+} \quad (k \in \mathbb{N}).$$

Equivalence of (i) and (iii). Assume first that (iii) holds, that is,

$$\frac{1}{n}\sum_{k=1}^{n} u_k \longrightarrow 0 \quad (n \to \infty).$$

Fix $\varepsilon > 0$ and define

$$E_\varepsilon = \{k \in \mathbb{N} : \gamma_k \geq \varepsilon\}, \qquad E_\varepsilon(n) = E_\varepsilon \cap \{1, \dots, n\}.$$

For $k \in E_\varepsilon$ one has $u_k = \gamma_k^{p_k} \geq \varepsilon^{p_k}$. Hence, with

$$c_\varepsilon = \begin{cases} \varepsilon^{p_+}, & 0 < \varepsilon \leq 1, \\ \varepsilon, & \varepsilon > 1, \end{cases}$$

one gets $u_k \geq c_\varepsilon$ for all $k \in E_\varepsilon$, and therefore

$$c_\varepsilon\, |E_\varepsilon(n)| \leq \sum_{k \in E_\varepsilon(n)} u_k \leq \sum_{k=1}^{n} u_k.$$

Dividing by n yields $|E_\varepsilon(n)|/n \to 0$. Moreover, for all k one has $\varepsilon^{-p_k} \leq C_\varepsilon$, where

$$C_\varepsilon = \begin{cases} \varepsilon^{-p_+}, & 0 < \varepsilon \leq 1, \\ \varepsilon^{-1}, & \varepsilon > 1, \end{cases}$$

and thus

$$\frac{1}{n} \sum_{k=1}^{n} \chi_{E_\varepsilon}(k)\, \varepsilon^{-p_k} = \frac{1}{n} \sum_{k \in E_\varepsilon(n)} \varepsilon^{-p_k} \leq C_\varepsilon \frac{|E_\varepsilon(n)|}{n} \longrightarrow 0.$$

This is exactly (i).

Conversely, assume (i). Fix $0 < \varepsilon \leq 1$ and keep the notation above. Since $p_k \geq 1$, one has $\varepsilon^{-p_k} \geq \varepsilon^{-1}$ for every k, hence

$$\frac{|E_\varepsilon(n)|}{n} = \frac{1}{n} \sum_{k \in E_\varepsilon(n)} 1 \leq \varepsilon \cdot \frac{1}{n} \sum_{k \in E_\varepsilon(n)} \varepsilon^{-p_k} = \varepsilon \cdot \frac{1}{n} \sum_{k=1}^{n} \chi_{E_\varepsilon}(k)\, \varepsilon^{-p_k} \longrightarrow 0.$$

Split the Cesàro means of (u_k) into two parts:

$$\frac{1}{n} \sum_{k=1}^{n} u_k = \frac{1}{n} \sum_{\gamma_k < \varepsilon} u_k + \frac{1}{n} \sum_{\gamma_k \geq \varepsilon} u_k.$$

If $\gamma_k < \varepsilon \leq 1$, then $u_k = \gamma_k^{p_k} \leq \gamma_k \leq \varepsilon$, so

$$\frac{1}{n} \sum_{\gamma_k < \varepsilon} u_k \leq \varepsilon.$$

On the other hand, the choice of M_x gives $u_k \leq M_x^{p_+}$ for all k, hence

$$\frac{1}{n} \sum_{\gamma_k \geq \varepsilon} u_k = \frac{1}{n} \sum_{k \in E_\varepsilon(n)} u_k \leq M_x^{p_+} \frac{|E_\varepsilon(n)|}{n} \longrightarrow 0.$$

Therefore $\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k \leq \varepsilon$ for every $0 < \varepsilon \leq 1$, and letting $\varepsilon \downarrow 0$ gives $\frac{1}{n} \sum_{k=1}^{n} u_k \to 0$, that is, (iii).

Equivalence of (ii) and (iii). For each $r \in \mathbb{N}$ set

$$I_r = (k_{r-1}, k_r], \qquad A_r = \frac{1}{h_r} \sum_{k \in I_r} u_k, \qquad B(\varepsilon) = \{r \in \mathbb{N} : A_r \geq \varepsilon\}.$$

Note that $h_r = |I_r|$ and $\sum_{r=1}^{R} h_r = k_R$.

Assume (ii). By definition, $\delta_\Phi^\theta\big(B(\varepsilon), \varepsilon\big) = 0$ for every $\varepsilon > 0$, with $\Phi_{k_r}(1/\varepsilon) = (1/\varepsilon)^{p_{k_r}}$. Since $1 \leq p_{k_r} \leq p_+$, there exist constants $m_\varepsilon, M_\varepsilon > 0$ such that

$$m_\varepsilon \leq \Phi_{k_r}\left(\frac{1}{\varepsilon}\right) \leq M_\varepsilon,$$

for all $r \in \mathbb{N}$. Hence

$$0 = \delta_\Phi^\theta\big(B(\varepsilon), \varepsilon\big) = \limsup_{R\to\infty} \frac{\sum\limits_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r\,\Phi_{k_r}(1/\varepsilon)}{\sum\limits_{r=1}^{R} h_r\,\Phi_{k_r}(1/\varepsilon)} \geq \frac{m_\varepsilon}{M_\varepsilon} \limsup_{R\to\infty} \frac{\sum\limits_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r}{\sum\limits_{r=1}^{R} h_r}.$$

Therefore

$$\frac{1}{k_R} \sum_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r \longrightarrow 0.$$

Now write

$$\frac{1}{k_R}\sum_{k=1}^{k_R} u_k = \frac{1}{k_R}\sum_{r=1}^{R}\sum_{k\in I_r} u_k = \frac{1}{k_R}\sum_{r=1}^{R} h_r A_r \leq \varepsilon + M_x^{p_+}\cdot\frac{1}{k_R}\sum_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r,$$

where the uniform bound $u_k \leq M_x^{p_+}$ is valid because $M_x \geq 1$. Thus

$$\limsup_{R\to\infty} \frac{1}{k_R}\sum_{k=1}^{k_R} u_k \leq \varepsilon.$$

Letting $\varepsilon \downarrow 0$ yields

$$\frac{1}{k_R}\sum_{k=1}^{k_R} u_k \longrightarrow 0.$$

To pass from the subsequence (k_R) to all n, choose R with $k_{R-1} < n \leq k_R$. Using $u_k \geq 0$ one has

$$\frac{1}{n}\sum_{k=1}^{n} u_k \leq \frac{1}{k_{R-1}}\sum_{k=1}^{k_R} u_k = \frac{k_R}{k_{R-1}}\cdot\frac{1}{k_R}\sum_{k=1}^{k_R} u_k.$$

By Lemma 3, the moderate growth assumption implies $h_R/k_R \to 0$, hence

$$\frac{k_R}{k_{R-1}} = 1 + \frac{h_R}{k_{R-1}} \longrightarrow 1.$$

Consequently $\frac{1}{n}\sum\limits_{k=1}^{n} u_k \to 0$, which is (iii).

Conversely, assume (iii). If $r \in B(\varepsilon)$, then $\sum\limits_{k\in I_r} u_k = h_r A_r \geq \varepsilon h_r$. Summing over $1 \leq r \leq R$ gives

$$\varepsilon \sum_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r \leq \sum_{r=1}^{R}\sum_{k\in I_r} u_k = \sum_{k=1}^{k_R} u_k.$$

Dividing by k_R yields

$$\frac{1}{k_R}\sum_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r \leq \frac{1}{\varepsilon}\cdot\frac{1}{k_R}\sum_{k=1}^{k_R} u_k \longrightarrow 0.$$

Using again $m_\varepsilon \leq \Phi_{k_r}(1/\varepsilon) \leq M_\varepsilon$ and $\sum\limits_{r=1}^{R} h_r = k_R$, one obtains

$$\frac{\sum\limits_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r\,\Phi_{k_r}(1/\varepsilon)}{\sum\limits_{r=1}^{R} h_r\,\Phi_{k_r}(1/\varepsilon)} \leq \frac{M_\varepsilon}{m_\varepsilon}\cdot\frac{\sum\limits_{r\in B(\varepsilon)\cap\{1,\dots,R\}} h_r}{\sum\limits_{r=1}^{R} h_r} \longrightarrow 0,$$

hence $\delta_\Phi^\theta\big(B(\varepsilon), \varepsilon\big) = 0$, which is (ii).

The equivalences (i)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iii) complete the proof. $\square$

Corollary 1. *Let $(A_n) \subset \mathcal{C}(X)$ be Wijsman bounded and assume that $1 \le p_n \le p_+ < \infty$. Let $\theta = (k_r)$ be a lacunary sequence with gaps $h_r = k_r - k_{r-1}$ and suppose that $\frac{h_r}{k_r} \longrightarrow 0$ $(r \to \infty)$. Then, for the power-type family $\Phi_n(t) = t^{p_n}$, the following are equivalent:*

(i) *A_n is Wijsman (p_n)-statistically convergent to A,*
(ii) *A_n is lacunary Wijsman (p_n)-statistically convergent to A with respect to θ,*
(iii) *A_n is Wijsman (p_n)-strong Cesàro convergent to A.*

Proof. For $\Phi_n(t) = t^{p_n}$, Lemma 3 shows that the moderate growth condition with respect to θ is equivalent to $h_r/k_r \to 0$. Hence the hypothesis guarantees that Φ satisfies the moderate growth condition. The conclusion follows by applying Theorem 1. $\square$

3.3. Bounded exponent comparisons and consequences

The next results explain why boundedness of (p_n) is structurally important. Under bounded exponents, modular Cesàro control of $\gamma_n(x)^{p_n}$ forces ordinary Cesàro control of $\gamma_n(x)$. This observation will be used in §4 to control Cesàro means of the distance functionals.

Lemma 4. *Assume that $1 \le p_n \le p_+ < \infty$ and let (t_n) be a bounded sequence of nonnegative real numbers. If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} t_k^{p_k} = 0,$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} t_k = 0.$$

Proof. Fix $\delta \in (0,1)$. For each $n \in \mathbb{N}$ consider the index sets

$$E_1(n) = \{1 \le k \le n : t_k < \delta\}, \quad E_2(n) = \{1 \le k \le n : \delta \le t_k < 1\}, \quad E_3(n) = \{1 \le k \le n : t_k \ge 1\}.$$

Then

$$\frac{1}{n}\sum_{k=1}^{n} t_k = \frac{1}{n}\sum_{k\in E_1(n)} t_k + \frac{1}{n}\sum_{k\in E_2(n)} t_k + \frac{1}{n}\sum_{k\in E_3(n)} t_k \le \delta + \frac{|E_2(n)|}{n} + \frac{1}{n}\sum_{k\in E_3(n)} t_k.$$

For $k \in E_2(n)$ we have $0 < \delta \le t_k < 1$, hence $t_k^{p_k} \ge \delta^{p_k} \ge \delta^{p_+}$ (since $\delta < 1$ and $p_k \le p_+$). Therefore

$$\delta^{p_+} |E_2(n)| \le \sum_{k\in E_2(n)} t_k^{p_k} \le \sum_{k=1}^{n} t_k^{p_k}, \qquad \text{so} \qquad \frac{|E_2(n)|}{n} \le \delta^{-p_+} \cdot \frac{1}{n}\sum_{k=1}^{n} t_k^{p_k}.$$

For $k \in E_3(n)$ we have $t_k \ge 1$ and $p_k \ge 1$, hence $t_k \le t_k^{p_k}$, and thus

$$\frac{1}{n}\sum_{k\in E_3(n)} t_k \le \frac{1}{n}\sum_{k=1}^{n} t_k^{p_k}.$$

Combining the above estimates yields, for every n,

$$\frac{1}{n}\sum_{k=1}^{n} t_k \le \delta + \left(1 + \delta^{-p_+}\right)\frac{1}{n}\sum_{k=1}^{n} t_k^{p_k}.$$

Taking $\limsup_{n\to\infty}$ and using the hypothesis gives

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_k \le \delta.$$

Since $\delta \in (0,1)$ was arbitrary, it follows that $\frac{1}{n}\sum_{k=1}^{n} t_k \to 0$. $\square$

Proposition 3. *Assume $1 \leq p_n \leq p_+ < \infty$ and (A_n) is Wijsman bounded. If A_n is Wijsman (p_n)-strong Cesàro convergent to A, then for every $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \gamma_k(x) = 0.$$

Proof. Fix $x \in X$ and set

$$t_k := \gamma_k(x) = \left|d(x, A_k) - d(x, A)\right|, \; k \in \mathbb{N}.$$

Since (A_n) is Wijsman bounded, the real sequence $\big(d(x, A_k)\big)$ is bounded; let $M_x := \sup_{k\in\mathbb{N}} d(x, A_k) < \infty$. Moreover $d(x, A) < \infty$, hence

$$0 \leq t_k \leq d(x, A_k) + d(x, A) \leq M_x + d(x, A) \; (k \in \mathbb{N}),$$

so (t_k) is bounded and nonnegative. The hypothesis that A_n is Wijsman (p_n)-strong Cesàro convergent to A yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} t_k^{p_k} = 0.$$

Because $1 \leq p_k \leq p_+ < \infty$, Lemma 4 applies to (t_k) and gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} t_k = 0.$$

That is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \gamma_k(x) = 0,$$

as claimed. □

The following statement is a pointwise consequence of the argument used in the proof of Theorem 1. We record it separately because it will be useful in interpreting the bounded-exponent regime and in contrasting it with Example 1.

Proposition 4. *Assume that $1 \leq p_n \leq p_+ < \infty$ and that (A_n) is Wijsman bounded. If A_n is Wijsman (p_n)-strong Cesàro convergent to A, then A_n is Wijsman (p_n)-statistically convergent to A.*

Proof. This follows from the implication (iii)⇒(i) proved in Theorem 1. Indeed, fixing $x \in X$ and $\varepsilon > 0$, set

$$E(\varepsilon) = \{k \in \mathbb{N} : \gamma_k(x) \geq \varepsilon\}.$$

The proof of Theorem 1 shows that the condition

$$\frac{1}{n} \sum_{k=1}^{n} \gamma_k(x)^{p_k} \to 0,$$

implies

$$\frac{1}{n} \sum_{k=1}^{n} \chi_{E(\varepsilon)}(k) \varepsilon^{-p_k} \to 0,$$

provided that $1 \leq p_n \leq p_+ < \infty$ and the deviations are bounded at the fixed point x. The latter boundedness follows from Wijsman boundedness. Hence the required Wijsman (p_n)-statistical convergence follows. □

Remark 7. The separate formulation of Proposition 4 is useful only in the bounded-exponent regime. When (p_n) is allowed to grow without bound, the modular averages

$$\frac{1}{n} \sum_{k=1}^{n} \gamma_k(x)^{p_k},$$

may converge to zero even in the presence of persistent deviations with $\gamma_k(x) < 1$. In that case one cannot recover ordinary Cesàro control of $(\gamma_k(x))$, ordinary Wijsman convergence, or Wijsman (p_n)-statistical convergence from the modular strong Cesàro condition alone; see Example 1.

3.4. Examples and remarks

The next two examples underline that the behavior of the exponent sequence (p_n) governs the strength of the power-type modular averaging, even for very simple set sequences in $\mathbb{R}$. In particular, rapidly increasing exponents may suppress bounded deviations below one, while statistical convergence may ignore deviations occurring along sufficiently sparse subsequences.

Example 1. Let $X = \mathbb{R}$ with the usual metric, let $A = \{0\}$, and set

$$A_n = \{1/2\} \quad (n \in \mathbb{N}).$$

For each $x \in \mathbb{R}$, the reverse triangle inequality gives

$$\gamma_n(x) = \left|d(x, A_n) - d(x, A)\right| = \left||x - 1/2| - |x|\right| \le |(x - 1/2) - x| = \frac{1}{2}.$$

Moreover, at the point $x = 0$ one has

$$\gamma_n(0) = \frac{1}{2} \quad (n \in \mathbb{N}).$$

Now choose the unbounded exponent sequence $p_n = n$. Since $0 \le \gamma_n(x) \le 1/2 < 1$, we obtain

$$\gamma_n(x)^{p_n} \le \left(\frac{1}{2}\right)^n \quad (n \in \mathbb{N}).$$

Therefore, for every fixed $x \in \mathbb{R}$,

$$\frac{1}{n}\sum_{k=1}^{n} \gamma_k(x)^{p_k} \le \frac{1}{n}\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \longrightarrow 0.$$

Thus (A_n) is Wijsman (p_n)-strong Cesàro convergent to A.
However, ordinary Wijsman convergence fails. Indeed, at $x = 0$,

$$d(0, A_n) = \frac{1}{2} \quad \text{and} \quad d(0, A) = 0,$$

for every n, so $\gamma_n(0) = 1/2$ does not tend to 0.

The same example also shows that Wijsman (p_n)-strong Cesàro convergence need not imply Wijsman (p_n)-statistical convergence when the exponents are unbounded. Fix $0 < \varepsilon < 1/2$ and take $x = 0$. Then

$$E(0, \varepsilon) = \{k \in \mathbb{N} : \gamma_k(0) \ge \varepsilon\} = \mathbb{N}.$$

Consequently,

$$\frac{1}{n}\sum_{k=1}^{n} \chi_{E(0,\varepsilon)}(k)\varepsilon^{-p_k} = \frac{1}{n}\sum_{k=1}^{n} \varepsilon^{-k}.$$

Since $\varepsilon^{-1} > 2$, the last sequence does not converge to 0. Hence (A_n) is not Wijsman (p_n)-statistically convergent to A.

This example shows that the bounded-exponent assumption $1 \le p_n \le p_+ < \infty$ in Theorem 1 is essential. Without this assumption, modular strong Cesàro convergence may hold while both ordinary Wijsman convergence and Wijsman (p_n)-statistical convergence fail.

Example 2. Let $X = \mathbb{R}$ with the usual metric and $A = \{0\}$. Define

$$A_n = \begin{cases} \{1/2\}, & \text{if } n \text{ is a perfect square,} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Assume $1 \leq p_n \leq p_+ < \infty$. Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. The deviation $\gamma_n(x) = |d(x, A_n) - d(x, A)|$ vanishes whenever n is not a square, hence the set

$$E(x, \varepsilon) = \{k \in \mathbb{N} : \gamma_k(x) \geq \varepsilon\},$$

is contained in the set of square indices. Therefore, for every n,

$$\frac{1}{n} \sum_{k=1}^{n} \chi_{E(x,\varepsilon)}(k)\, \varepsilon^{-p_k} \leq \varepsilon^{-p_+} \cdot \frac{\lfloor \sqrt{n} \rfloor}{n} \longrightarrow 0.$$

This shows that (A_n) is Wijsman (p_n)-statistically convergent to A. However, Wijsman convergence does not hold, since along the square subsequence one has $d(0, A_{m^2}) = 1/2$ while $d(0, A) = 0$.

Remark 8. Examples 1 and 2 highlight two complementary phenomena. Example 1 shows that unbounded exponents can make the modular terms $\gamma_n(x)^{p_n}$ negligible even when the deviations do not tend to zero. In that example, Wijsman (p_n)-strong Cesàro convergence holds, but ordinary Wijsman convergence fails, and Wijsman (p_n)-statistical convergence also fails for every $0 < \varepsilon < 1/2$ at the point $x = 0$. Thus the bounded-exponent hypothesis in Theorem 1 is not merely technical.

By contrast, Example 2 shows that statistical convergence may hold despite persistent deviations along a subsequence of asymptotic density zero. Together, these examples explain why the deterministic comparison results are formulated under the bounded-exponent assumption $1 \leq p_n \leq p_+ < \infty$, and why the probabilistic transfer result also requires bounded exponents.

To prepare for the probabilistic section, it is convenient to record that when $1 \leq p_n \leq p_+ < \infty$, modular strong Cesàro control of $\gamma_k(x)^{p_k}$ yields ordinary Cesàro control of $\gamma_k(x)$ for each fixed x; see Proposition 3. This observation will be used in §4 to obtain convergence of Cesàro averages of the distance functionals $d(x, A_n)$ from almost sure modular strong Cesàro assumptions at a fixed point.

3.5. A deterministic estimate for the random-set application

We record one short deterministic estimate that will be applied pathwise in §4. The estimate is stated at a fixed base point $x \in X$ and does not assert any uniformity over X.

Lemma 5. *Fix $x \in X$ and let $(A_n) \subset \mathcal{C}(X)$. Let $A \in \mathcal{C}(X)$ and define $\gamma_n(x) = |d(x, A_n) - d(x, A)|$. Then, for every $n \in \mathbb{N}$,*

$$\left| \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) - d(x, A) \right| \leq \frac{1}{n} \sum_{k=1}^{n} \gamma_k(x).$$

Proof. For each $k \in \mathbb{N}$ set $a_k = d(x, A_k) - d(x, A)$. Then

$$\left| \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) - d(x, A) \right| = \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right| \leq \frac{1}{n} \sum_{k=1}^{n} |a_k| = \frac{1}{n} \sum_{k=1}^{n} \gamma_k(x),$$

where the inequality is the triangle inequality for the absolute value on $\mathbb{R}$. □

Corollary 2. *Assume $1 \leq p_n \leq p_+ < \infty$. Fix $x \in X$ and let $(A_n) \subset \mathcal{C}(X)$ be such that $\big(d(x, A_n)\big)$ is bounded. If A_n is Wijsman (p_n)-strong Cesàro convergent to A, then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A).$$

Proof. Fix $x \in X$ and set $t_k = \gamma_k(x)$. By the boundedness of $\big(d(x, A_n)\big)$ and the finiteness of $d(x, A)$, the sequence (t_k) is bounded and nonnegative. The hypothesis gives

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_k^{p_k} = 0.$$

Lemma 4 yields $\frac{1}{n}\sum_{k=1}^{n} t_k \to 0$. Applying Lemma 5 completes the argument. □

3.6. Transition to the probabilistic section

§4 applies the preceding deterministic estimate to random closed sets on sample paths. The probabilistic result below does not derive the almost sure modular Cesàro condition from independence or from moment assumptions. Rather, it assumes this condition almost surely and then transfers it, point by point in x, to convergence of Cesàro averages of the corresponding distance functionals.

4. A pathwise Cesàro transfer result for random closed sets

Random closed sets constitute a standard model for set-valued uncertainty, where measurability is encoded through hit events and, equivalently, through the induced distance functionals. This viewpoint originates in the foundational work on random sets and stochastic geometry and has since been developed systematically; we refer to [34,35] for comprehensive accounts. In particular, for every fixed base point $x \in X$, the map $\omega \longmapsto d\big(x, A(\omega)\big)$ associated with a random closed set A is measurable, so that pointwise distance evaluations may be treated as ordinary real-valued random variables [35].

The aim of this section is to apply the deterministic Cesàro estimates from §3 to random closed sets along sample paths. The result below should not be read as a new law of large numbers. It assumes the almost sure modular strong Cesàro condition and the almost sure boundedness of the distance evaluations at a fixed base point, and then transfers these assumptions to almost sure convergence of the Cesàro averages of the corresponding distance functionals.

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and X is a complete and separable metric space. The separability assumption ensures that measurability of distance functionals can be verified via countable sublevel descriptions, and it is compatible with the standard hypotheses in the theory of random closed sets [34,35].

4.1. Random closed sets and measurability of distance functionals

We recall the basic definition of a random closed set and record the measurability of the induced distance functionals, which will be used repeatedly when formulating pointwise almost sure convergence statements.

Definition 13. A mapping $A : \Omega \to \mathcal{C}(X)$ is called a random closed set if, for every open set $U \subset X$, the hit event

$$\{\omega \in \Omega : \ A(\omega) \cap U \neq \varnothing\},$$

belongs to $\mathcal{F}$.

This hit-or-miss measurability is standard and yields a well-behaved set-valued random element; see [34,35]. The next lemma explains why the associated distance functionals are measurable.

Lemma 6. *Let A be a random closed set. Then for every $x \in X$, the mapping $\omega \mapsto d\big(x, A(\omega)\big)$ is $\mathcal{F}$-measurable.*

Proof. Fix $x \in X$. For each $q \in \mathbb{Q}$ with $q > 0$, one has $d\big(x, A(\omega)\big) < q$ if and only if $A(\omega) \cap B(x, q) \neq \varnothing$, where $B(x, q)$ denotes the open ball centered at x with radius q. By the definition of a random closed set, the event on the right-hand side belongs to $\mathcal{F}$, hence so does the sublevel set $\{\omega : \ d(x, A(\omega)) < q\}$. Since the family of rational radii is countable and these strict sublevel sets generate the Borel σ-algebra on $\mathbb{R}$, the mapping $\omega \mapsto d(x, A(\omega))$ is $\mathcal{F}$-measurable. □

Consequently, given a sequence (A_n) of random closed sets and a random closed set A, for each fixed $x \in X$ the mappings

$$a_n(x,\omega) = d\big(x, A_n(\omega)\big), \qquad \Delta_n(x,\omega) = \big|d\big(x, A_n(\omega)\big) - d\big(x, A(\omega)\big)\big|,$$

define real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

4.2. Almost sure modular Cesàro control at a fixed point

We now pass from the deterministic modular scheme to random closed sets by fixing a base point $x \in X$ and working with the induced real-valued distance functionals. This pointwise formulation is consistent with the Wijsman nature of the problem and avoids null-set selection issues that may arise when quantifying almost sure statements simultaneously over an uncountable family of points.

Definition 14. Let (A_n) be a sequence of random closed sets and let A be a random closed set. Fix $x \in X$. We say that (A_n) converges almost surely to A in the Wijsman (p_n)-strong Cesàro sense at x if

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \big|d(x, A_k(\omega)) - d(x, A(\omega))\big|^{p_k} = 0\right\}\right) = 1.$$

Definition 15. Fix $x \in X$. A sequence (A_n) of random closed sets is said to have almost surely bounded distance functionals at x if

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sup_{n\in\mathbb{N}} d(x, A_n(\omega)) < \infty\right\}\right) = 1.$$

The boundedness assumption is pointwise in the fixed base point x. It is expected to hold, for example, when the random sets remain almost surely in a bounded region relative to x, or when the distance evaluations $d(x, A_n(\omega))$ are controlled by an almost surely finite bound independent of n. Under the bounded-exponent regime, this assumption supplies the deterministic ingredient needed to pass from modular Cesàro control of the deviations to ordinary Cesàro control.

Theorem 2 (Pathwise Cesàro transfer)**.** *Assume that $1 \le p_n \le p_+ < \infty$. Let (A_n) be a sequence of random closed sets and let A be a random closed set. Fix $x \in X$. Assume that (A_n) has almost surely bounded distance functionals at x and that (A_n) converges almost surely to A in the Wijsman (p_n)-strong Cesàro sense at x. Then*

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k(\omega)) = d(x, A(\omega))\right\}\right) = 1.$$

Proof. Fix $x \in X$. Let Ω_1 be the event on which $\sup_{n\in\mathbb{N}} d(x, A_n(\omega)) < \infty$ holds, and let Ω_2 be the event on which

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \big|d(x, A_k(\omega)) - d(x, A(\omega))\big|^{p_k} = 0,$$

holds. By assumption, $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = 1$, hence $\mathbb{P}(\Omega_1 \cap \Omega_2) = 1$.

Fix $\omega \in \Omega_1 \cap \Omega_2$ and set

$$Y_k(\omega) = \big|d(x, A_k(\omega)) - d(x, A(\omega))\big|, \; k \in \mathbb{N}.$$

Since $\omega \in \Omega_1$, the sequence $\big(d(x, A_k(\omega))\big)$ is bounded, and therefore $\big(Y_k(\omega)\big)$ is bounded as well. Moreover, $\omega \in \Omega_2$ yields

$$\frac{1}{n} \sum_{k=1}^{n} Y_k(\omega)^{p_k} \longrightarrow 0.$$

Applying Lemma 4 to the bounded nonnegative sequence $t_k = Y_k(\omega)$ gives

$$\frac{1}{n}\sum_{k=1}^{n} Y_k(\omega) \longrightarrow 0.$$

Finally, by the triangle inequality,

$$\left|\frac{1}{n}\sum_{k=1}^{n} d(x, A_k(\omega)) - d(x, A(\omega))\right| = \left|\frac{1}{n}\sum_{k=1}^{n} \big(d(x, A_k(\omega)) - d(x, A(\omega))\big)\right| \le \frac{1}{n}\sum_{k=1}^{n} Y_k(\omega),$$

and the right-hand side converges to 0. This proves the claim on $\Omega_1 \cap \Omega_2$, hence almost surely. □

Remark 9. Theorem 2 is a pathwise consequence of deterministic Cesàro estimates. It does not prove the almost sure modular strong Cesàro condition from independence, identical distribution, or moment assumptions. Instead, that condition is assumed almost surely. The role of the theorem is to show that, once this modular condition and the pointwise boundedness of the distance evaluations hold outside a null set, the deterministic comparison argument applies on each such sample path.

The boundedness of (p_n) is used to pass from the modular averages $\frac{1}{n}\sum_{k=1}^{n} Y_k(\omega)^{p_k}$ to the ordinary Cesàro averages $\frac{1}{n}\sum_{k=1}^{n} Y_k(\omega)$. If (p_n) is unbounded, deviations that stay below 1 may be damped by large exponents, so the modular Cesàro condition can hold without implying ordinary Cesàro control of the deviations. Example 1 illustrates this effect in the deterministic setting.

4.3. Identification under an additional strong law assumption

Theorem 2 gives the almost sure limit of the Cesàro averages

$$\frac{1}{n}\sum_{k=1}^{n} d(x, A_k(\omega)),$$

once an almost sure Wijsman (p_n)-strong Cesàro limit at a fixed point is assumed. If, in addition, the distance functionals satisfy a classical strong law of large numbers, then this limit can be identified explicitly. Thus the strong law is an extra input in the following corollary, not a consequence of Theorem 2 alone.

Corollary 3. *Let $x \in X$ be fixed and assume $1 \le p_n \le p_+ < \infty$. Assume that $\big(d(x, A_n)\big)$ is an independent and identically distributed sequence of integrable random variables, and that (A_n) satisfies the assumptions of Theorem 2 at the same point x. Then $d(x, A(\omega)) = \mathbb{E}\big[d(x, A_1)\big]$ for almost every $\omega \in \Omega$.*

Proof. Fix the point $x \in X$. By integrability and independence, the strong law of large numbers yields a measurable set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x, A_k(\omega)) = \mathbb{E}\big[d(x, A_1)\big],\ \omega \in \Omega_0.$$

On the other hand, Theorem 2 provides a measurable set $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x, A_k(\omega)) = d(x, A(\omega)),\ \omega \in \Omega_1.$$

Therefore, for every $\omega \in \Omega_0 \cap \Omega_1$, the two limits coincide, and the desired identity follows. □

Remark 10. Corollary 3 is formulated at a fixed base point x. Suppose that its assumptions hold for every point of a dense countable set $\{x_j\}_{j\ge 1} \subset X$, and suppose also that the functions

$$x \mapsto \mathbb{E}\big[d(x, A_1)\big],$$

are finite on this set and extend to a finite continuous function on X. Then one may choose a single null set outside which

$$d(x_j, A(\omega)) = \mathbb{E}\big[d(x_j, A_1)\big],$$

holds simultaneously for all $j \geq 1$. Since $x \mapsto d(x, A(\omega))$ is 1-Lipschitz for each fixed ω, the equality extends from the dense set to all of X. Thus, under these additional assumptions, the distance functional of $A(\omega)$ is almost surely deterministic. We do not use this observation in the sequel.

4.4. A verified interval-valued example

We conclude the probabilistic part with an interval-valued example in which the assumptions of Theorem 2 can be checked directly.

Example 3. Let $X = \mathbb{R}$ with the usual metric, and let $Z : \Omega \to \mathbb{R}$ be a bounded real-valued random variable. Thus there exists $M > 0$ such that

$$|Z(\omega)| \leq M \quad \text{for almost every } \omega \in \Omega.$$

Let $r > 0$ and let (r_n) be a sequence of positive real numbers such that

$$r_n \to r \qquad \text{and} \qquad R := \sup_{n\in\mathbb{N}} r_n < \infty.$$

Define random closed intervals

$$A_n(\omega) = [Z(\omega) - r_n,\ Z(\omega) + r_n], \qquad A(\omega) = [Z(\omega) - r,\ Z(\omega) + r].$$

Each A_n and A is a random closed set. For every fixed $x \in \mathbb{R}$,

$$d(x, A_n(\omega)) = \big(|x - Z(\omega)| - r_n\big)_+, \qquad d(x, A(\omega)) = \big(|x - Z(\omega)| - r\big)_+,$$

where $t_+ = \max\{t, 0\}$.

We first verify the almost sure boundedness of the distance evaluations at the fixed point x. Outside a null set on which $|Z| \leq M$, one has

$$0 \leq d(x, A_n(\omega)) \leq |x - Z(\omega)| + r_n \leq |x| + M + R \qquad (n \in \mathbb{N}).$$

Hence

$$\sup_{n\in\mathbb{N}} d(x, A_n(\omega)) < \infty,$$

for almost every $\omega \in \Omega$.

Next, the map $s \mapsto (|x - Z(\omega)| - s)_+$ is 1-Lipschitz in s on $[0, \infty)$. Therefore

$$\big|d(x, A_n(\omega)) - d(x, A(\omega))\big| \leq |r_n - r| \qquad (n \in \mathbb{N}),$$

for almost every $\omega \in \Omega$. Assume also that $1 \leq p_n \leq p_+ < \infty$. Since $r_n \to r$, the sequence $|r_n - r|$ tends to 0. Consequently,

$$\frac{1}{n}\sum_{k=1}^{n}\big|d(x, A_k(\omega)) - d(x, A(\omega))\big|^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}|r_k - r|^{p_k} \longrightarrow 0.$$

Indeed, for all sufficiently large k one has $|r_k - r| < 1$, and then

$$|r_k - r|^{p_k} \leq |r_k - r|.$$

Thus (A_n) converges almost surely to A in the Wijsman (p_n)-strong Cesàro sense at the fixed point x.

All assumptions of Theorem 2 are therefore verified, and we obtain

$$\frac{1}{n}\sum_{k=1}^{n} d(x, A_k(\omega)) \longrightarrow d(x, A(\omega)) \quad \text{for almost every } \omega \in \Omega.$$

This example shows explicitly how the theorem applies when the distance evaluations are almost surely bounded and the modular Cesàro condition is checked directly.

The next section turns back to the deterministic setting and provides concrete examples that separate the convergence notions depending on the exponent regime and on the presence or failure of lacunary moderate growth.

5. Examples and a comparative summary

This section presents representative constructions that bring out the two main phenomena governing the power-type approach developed above. First, we make explicit how the lacunary moderate growth condition reduces to a simple asymptotic requirement on the ratio h_r/k_r in the case $\Phi_n(t) = t^{p_n}$. Second, we present a single Wijsman bounded family of closed sets in $\mathbb{R}$ that separates the convergence notions depending on the behavior of the exponent sequence (p_n), and we summarize the outcomes in a comparison table.

5.1. Moderate growth in the power-type case

The equivalence theorem in §3 relies on a compatibility requirement between the lacunary gaps and the modular growth. In the power-type setting this compatibility is entirely determined by the asymptotic behavior of the ratio h_r/k_r.

As already shown in Lemma 3, for the power-type family

$$\Phi_n(t) = t^{p_n},$$

the moderate growth condition with respect to $\theta = (k_r)$ is equivalent to

$$\frac{h_r}{k_r} \to 0.$$

Therefore, in the examples below, it is enough to check the simple ratio h_r/k_r instead of repeating the proof of the moderate growth criterion.

Example 4. Let $X = \mathbb{R}$ with the usual metric, let $A = \{0\}$, and define

$$A_n = \{1/n\} \qquad (n \in \mathbb{N}).$$

Choose the bounded exponent sequence

$$p_n = 1 + \frac{1}{n+1}.$$

Then

$$1 \le p_n \le \frac{3}{2} < \infty \qquad (n \in \mathbb{N}).$$

For every fixed $x \in \mathbb{R}$, the reverse triangle inequality gives

$$\gamma_n(x) = \big|d(x, A_n) - d(x, A)\big| = \big||x - 1/n| - |x|\big| \le \frac{1}{n}.$$

Hence $\gamma_n(x) \to 0$ for every $x \in \mathbb{R}$, and therefore $A_n \xrightarrow{W} A$.

Moreover, (A_n) is Wijsman bounded. Indeed, for each fixed $x \in \mathbb{R}$,

$$d(x, A_n) = |x - 1/n| \le |x| + 1 \qquad (n \in \mathbb{N}).$$

Now take the lacunary sequence

$$k_r = r^2 \qquad (r \in \mathbb{N}).$$

Then

$$h_r = k_r - k_{r-1} = r^2 - (r-1)^2 = 2r - 1,$$

and consequently

$$\frac{h_r}{k_r} = \frac{2r-1}{r^2} \longrightarrow 0.$$

Thus the moderate growth condition holds for the power-type family $\Phi_n(t) = t^{p_n}$. All hypotheses of Theorem 1 are satisfied. Therefore the following three assertions are equivalent and, in this example, all of them hold:

(i) A_n is Wijsman (p_n)-statistically convergent to A;
(ii) A_n is lacunary Wijsman (p_n)-statistically convergent to A with respect to $\theta = (r^2)$;
(iii) A_n is Wijsman (p_n)-strong Cesàro convergent to A.

This gives a concrete example of the equivalence regime described in Theorem 1.

Example 5. Let the lacunary sequence be

$$k_r = 2^r \qquad (r \in \mathbb{N}).$$

Then

$$h_r = k_r - k_{r-1} = 2^r - 2^{r-1} = 2^{r-1},$$

and hence

$$\frac{h_r}{k_r} = \frac{2^{r-1}}{2^r} = \frac{1}{2} \qquad (r \in \mathbb{N}).$$

Therefore

$$\frac{h_r}{k_r} \not\to 0.$$

By Lemma 3, the moderate growth condition fails for the power-type family $\Phi_n(t) = t^{p_n}$. Indeed, if $p_- = \inf_n p_n \geq 1$, then

$$\sup_{n\in\mathbb{N}} \frac{\Phi_n(h_r)}{\Phi_n(k_r)} = \left(\frac{h_r}{k_r}\right)^{p_-} = \left(\frac{1}{2}\right)^{p_-} \not\to 0.$$

Thus the geometric lacunary sequence $k_r = 2^r$ lies outside the scope of Theorem 1. This example does not assert that the three convergence notions must fail to be equivalent in every such case; it shows only that the moderate growth hypothesis required by the theorem is not satisfied.

5.2. A single family that separates the convergence notions

To separate the convergence notions within a single construction, it is useful to perturb a constant set sequence along blocks whose asymptotic frequency can be computed explicitly. The following example produces a Wijsman bounded sequence with a persistent deviation along a set of indices of positive natural density.

Example 6. Let $X = \mathbb{R}$ with the usual metric and let $A = \{0\}$. Fix $\alpha \in (0,1)$. For each $m \in \mathbb{N}$ define the block $J_m = \{m^2+1, \dots, m^2+m\}$. Define a sequence $(A_n) \subset \mathcal{C}(\mathbb{R})$ by

$$A_n = \begin{cases} \{\alpha\}, & n \in \bigcup_{m=1}^{\infty} J_m, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then (A_n) is Wijsman bounded, since for each $x \in \mathbb{R}$ one has $d(x, A_n) \in \{|x|, |x-\alpha|\}$ and hence $\sup_n d(x, A_n) \leq \max\{|x|, |x-\alpha|\}$. Moreover, Wijsman convergence to A fails because for $x = 0$ one has $\gamma_n(0) = |d(0, A_n) - d(0, A)| = \alpha$ for all $n \in \bigcup_{m\geq 1} J_m$.

Proposition 5. *Let (A_n) be defined as in Example 6. Then:*

(i) *If $1 \le p_n \le p_+ < \infty$, then A_n is not Wijsman (p_n)-statistically convergent to A.*
(ii) *If $p_n \equiv 1$, then A_n is not Wijsman (p_n)-strong Cesàro convergent to A.*
(iii) *If $p_n = n$, then A_n is Wijsman (p_n)-strong Cesàro convergent to A.*

Proof. Set $E = \bigcup_{m=1}^{\infty} J_m$. For $x = 0$ we have $\gamma_n(0) = \alpha$ for $n \in E$ and $\gamma_n(0) = 0$ for $n \notin E$.

We first compute the natural density of E. For $M \in \mathbb{N}$ define $N_M = M^2 + M$. The blocks $J_1, \dots, J_M$ are disjoint and contained in $\{1, \dots, N_M\}$, hence

$$|E \cap \{1, \dots, N_M\}| = \sum_{m=1}^{M} |J_m| = \sum_{m=1}^{M} m = \frac{M(M+1)}{2}.$$

Therefore

$$\frac{|E \cap \{1, \dots, N_M\}|}{N_M} = \frac{\frac{M(M+1)}{2}}{M^2 + M} = \frac{1}{2}.$$

If $N_M \le N \le N_{M+1}$, then

$$\frac{|E \cap \{1, \dots, N_M\}|}{N_{M+1}} \le \frac{|E \cap \{1, \dots, N\}|}{N} \le \frac{|E \cap \{1, \dots, N_{M+1}\}|}{N_M},$$

and the outer terms converge to $1/2$ as $M \to \infty$. Hence

$$\lim_{N \to \infty} \frac{|E \cap \{1, \dots, N\}|}{N} = \frac{1}{2}.$$

(i) Fix ε with $0 < \varepsilon \le \alpha$ and take $x = 0$. Then $E = \{k \in \mathbb{N} : \gamma_k(0) \ge \varepsilon\}$. Assume $1 \le p_n \le p_+ < \infty$. Since $p_k \ge 1$ and $0 < \varepsilon < 1$, one has $\varepsilon^{-p_k} \ge \varepsilon^{-1}$ for every $k \in \mathbb{N}$. Consequently, for each $N \in \mathbb{N}$,

$$\frac{1}{N} \sum_{k=1}^{N} \chi_{\{\gamma_k(0) \ge \varepsilon\}}(k)\, \varepsilon^{-p_k} = \frac{1}{N} \sum_{k=1}^{N} \chi_E(k)\, \varepsilon^{-p_k} \ge \varepsilon^{-1} \cdot \frac{|E \cap \{1, \dots, N\}|}{N}.$$

Passing to $\limsup$ and using $\lim_{N \to \infty} \frac{|E \cap \{1, \dots, N\}|}{N} = \frac{1}{2}$ yields

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{\{\gamma_k(0) \ge \varepsilon\}}(k)\, \varepsilon^{-p_k} \ge \frac{1}{2\varepsilon} > 0,$$

so Wijsman (p_n)-statistical convergence fails.

(ii) Let $p_n \equiv 1$ and take $x = 0$. Then for each N,

$$\frac{1}{N} \sum_{k=1}^{N} \gamma_k(0) = \alpha \cdot \frac{|E \cap \{1, \dots, N\}|}{N} \longrightarrow \frac{\alpha}{2} > 0.$$

Hence the Cesàro means of $\gamma_k(0)^{p_k} = \gamma_k(0)$ do not converge to 0, and therefore A_n is not Wijsman (p_n)-strong Cesàro convergent to A.

(iii) Let $p_n = n$. For any $x \in \mathbb{R}$ and $n \in E$, the reverse triangle inequality gives

$$\gamma_n(x) = \big| |x - \alpha| - |x| \big| \le |(x - \alpha) - x| = \alpha,$$

and $\gamma_n(x) = 0$ when $n \notin E$. Hence, for every N,

$$\frac{1}{N} \sum_{k=1}^{N} \gamma_k(x)^{p_k} = \frac{1}{N} \sum_{k=1}^{N} \chi_E(k)\, \gamma_k(x)^k \le \frac{1}{N} \sum_{k=1}^{N} \alpha^k \longrightarrow 0.$$

This proves that A_n is Wijsman (p_n)-strong Cesàro convergent to A. $\square$

Remark 11. If the exceptional indices are placed only at perfect squares, namely $A_{m^2} = \{\alpha\}$ and $A_n = \{0\}$ otherwise, then the exceptional set has natural density 0. In that case Wijsman (p_n)-statistical convergence may hold for bounded (p_n), and for $p_n \equiv 1$ the corresponding strong Cesàro condition also holds, since the Cesàro means of $\gamma_n(0)$ then tend to 0. In contrast, the block construction in Example 6 has exceptional density 1/2, which forces the failures in Proposition 5(i)–(ii) while still allowing (iii) when the exponents grow rapidly.

5.3. A comparison table

For ease of reference, we summarize the main regimes discussed above. The first row records the precise equivalence range covered by Theorem 1: Wijsman boundedness, bounded exponents, and the lacunary moderate growth condition are all part of the hypothesis. The second row indicates that, when the moderate growth condition fails, Theorem 1 gives no equivalence conclusion. The last two rows emphasize that bounded and unbounded exponent regimes lead to different behavior even before any lacunary comparison is considered.

Table 1. Comparison of the main convergence regimes in the power-type Musielak–Orlicz setting. The equivalence result requires Wijsman boundedness, bounded exponents, and the lacunary moderate growth condition

Lacunary condition	Exponent and boundedness assumptions	Conclusion or role in the paper
$h_r/k_r \to 0$; equivalently, the moderate growth condition holds for $\Phi_n(t) = t^{p_n}$	(A_n) is Wijsman bounded and $1 \le p_n \le p_+ < \infty$	The three notions are equivalent: Wijsman (p_n)-statistical convergence, lacunary Wijsman (p_n)-statistical convergence, and Wijsman (p_n)-strong Cesàro convergence; see Theorem 1, Corollary 1, and Example 4.
$h_r/k_r \not\to 0$; the moderate growth condition fails in the power-type case	No equivalence statement follows from Theorem 1	This regime is outside the scope of the main equivalence theorem. The geometric choice $k_r = 2^r$ illustrates this failure of moderate growth; see Example 5.
No lacunary comparison is used	(A_n) is Wijsman bounded and $1 \le p_n \le p_+ < \infty$	Modular strong Cesàro control implies ordinary Cesàro control of the deviations; see Lemma 4 and Proposition 3. Statistical convergence may still hold without ordinary Wijsman convergence; see Example 2.
No lacunary comparison is used	The exponent sequence is unbounded, for example $p_n = n$	Boundedness of (p_n) is essential. Modular strong Cesàro convergence may hold even when ordinary Wijsman convergence fails; see Example 1. In the same example, Wijsman (p_n)-statistical convergence also fails for suitable ε.

Table 1 emphasizes two points. First, the equivalence of the three convergence modes is asserted only in the bounded-exponent regime $1 \le p_n \le p_+ < \infty$, under Wijsman boundedness and the lacunary condition $h_r/k_r \to 0$. Second, the exponent regime controls how strongly the modular averages penalize deviations: bounded exponents preserve a link with ordinary Cesàro estimates, whereas rapidly growing exponents may suppress deviations below one and may therefore produce modular strong Cesàro convergence without ordinary Wijsman convergence.

6. Conclusion

This work developed a power-type Musielak–Orlicz setting for studying convergence of sequences of nonempty closed sets in a metric space. For the modular family $\Phi_n(t) = t^{p_n}$, we introduced Wijsman (p_n)-statistical convergence, Wijsman (p_n)-strong Cesàro convergence, and a lacunary block-mean statistical version determined by a lacunary sequence $\theta = (k_r)$.

The main deterministic result was proved under the bounded-exponent assumption $1 \le p_n \le p_+ < \infty$. More precisely, for Wijsman bounded sequences, the three convergence modes are equivalent when the lacunary moderate growth condition holds. In the power-type case this condition is exactly $\frac{h_r}{k_r} \to 0$. Thus the result gives a concrete power-type form of the Musielak–Orlicz equivalence principle, while also making explicit the lacunary and exponent restrictions under which the equivalence is valid.

The paper also clarified why the boundedness of the exponent sequence is essential. The comparison between modular Cesàro means and ordinary Cesàro means holds for bounded nonnegative sequences only under the restriction $1 \le p_n \le p_+ < \infty$. The examples show that, when the exponents are unbounded, modular strong Cesàro convergence may persist even though ordinary Wijsman convergence, and in some cases Wijsman (p_n)-statistical convergence, fails.

Finally, for random closed sets we proved a pathwise Cesàro transfer result at a fixed base point. The probabilistic conclusion should be read in this sense: once the modular strong Cesàro condition and the boundedness of the distance evaluations hold almost surely, the deterministic comparison argument can be applied outside a null set to obtain almost sure convergence of the Cesàro averages of the distance functionals to the Wijsman limit.

7. Open problems and further directions

The present results suggest several natural directions.

(i) *b-metric extensions.* To what extent do the equivalence theorem and the almost sure Cesàro stabilization remain valid in b-metric spaces, where the triangle inequality is weakened? It would be particularly interesting to identify hypotheses that simultaneously ensure a workable Wijsman-type set convergence and preserve the modular comparison arguments that drive the power-type theory; see, for instance, [36,37] for fixed point methods in related metric-type settings.

(ii) *Soft metric spaces.* Can one develop analogues of Wijsman (p_n)-statistical and Wijsman (p_n)-strong Cesàro convergence for closed sets in soft metric spaces, and can one formulate a corresponding moderate growth condition that yields an equivalence principle comparable to Theorem 1?

(iii) *Soft b-metric spaces.* Is it possible to obtain parallel deterministic and stochastic conclusions in soft b-metric spaces, combining the soft approach with the relaxed geometry of b-metrics? In this direction, a central problem is to identify boundedness hypotheses and growth conditions under which modular control still implies ordinary Cesàro control, even in the absence of a genuine metric structure.

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