Open Journal of
Mathematical Sciences

PSR Press

Article

Window-weighted Wijsman convergence and pre-Cauchy subsequence criteria in idempotent bicomplex metric hyperspaces

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Received: 03 Jun 2026; Revised: 02 Jul 2026; Accepted: 12 Jul 2026; Published: 22 Jul 2026

Abstract: We study window-weighted Wijsman statistical convergence for sequences of non-empty closed sets in hyperspaces associated with idempotent bicomplex metric spaces. The ambient distance is assumed to have the form $\rho = d_1e_1 + d_2e_2$, where d_1 and d_2 are ordinary metrics on the same underlying set. Thus the present paper concerns this idempotent, componentwise class of bicomplex-valued metrics, rather than arbitrary bicomplex-valued metric structures. For a closed set A, the point-to-set distance is represented by the two scalar profiles $d_1(x, A)$ and $d_2(x, A)$. We define window-weighted Wijsman statistical convergence and the corresponding pre-Cauchy condition by requiring simultaneous control of these two profiles along weighted moving windows. The main results give exact componentwise characterizations, establish invariance under weighted-equivalent changes of the window scheme, and show that a window-weighted Wijsman pre-Cauchy sequence becomes statistically convergent whenever it has an ordinarily Wijsman convergent trace of positive lower window weight. We also prove stability under negligible perturbations and show that ordinary Wijsman convergence on a trace of full window density determines the statistical convergence of the whole sequence. Examples illustrate the dependence on the window scheme, the strict weakness of the pre-Cauchy condition, and the effect of imposing simultaneous closedness with respect to two non-equivalent component metrics.

Keywords: idempotent bicomplex metric space, Wijsman convergence, hyperspace of closed sets, window-weighted convergence, pre-Cauchy condition, statistical convergence, full-density subsequence

MSC: 40A35, 54B20, 54E35, 30G35.

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1. Introduction

The study of convergence for sequences of closed sets is sensitive to two different kinds of structure. The first is geometric: a closed set is not observed through its elements alone, but through the point-to-set distance functions generated by the ambient metric. The second is asymptotic: the way in which the indices are sampled may change the limiting behaviour that one is able to detect. Ordinary convergence tests the whole sequence along the natural order of the positive integers, and for many hyperspace problems this requirement is too rigid. A sequence may fail to converge in the ordinary Wijsman sense because of exceptional blocks or sparse disturbances, while still exhibiting stable behaviour when it is examined along suitable moving windows. This observation is the main motivation for the present paper.

Moving-window methods provide a natural framework for separating persistent asymptotic behaviour from index-dependent irregularities. In scalar sequence spaces, such methods refine ordinary statistical convergence by replacing initial segments with intervals that move with the index. In the setting of closed sets, however, the situation is more delicate. A Wijsman-type condition is not imposed directly on the sets themselves, but on all scalar distance profiles associated with points of the ambient space. Thus, when one passes from scalar sequences to hyperspace-valued sequences, the sampling procedure must be compatible with the family of point-to-set distances that defines the convergence. This leads to two basic questions. First, when do two window schemes determine the same statistical Wijsman behaviour? Second, when does a pairwise pre-Cauchy condition inside the windows contain enough information to recover an actual Wijsman statistical limit?

The idempotent bicomplex metric setting considered in this paper is the componentwise class $\rho = d_1 e_1 + d_2 e_2$, where d_1 and d_2 are ordinary metrics on the same underlying set. Thus the bicomplex terminology is used in this idempotent sense, and the results are not claimed to cover arbitrary bicomplex-valued metric structures. In this framework, Wijsman convergence of closed sets is governed by the two scalar point-to-set distance profiles $d_1(x, A_k)$ and $d_2(x, A_k)$, for each fixed $x \in X$. This leads to a specialized window-weighted Wijsman convergence theory in the idempotent bicomplex setting, based on the simultaneous control of two scalar point-to-set distance profiles. The purpose of the present work is to study how weighted window comparison, pre-Cauchy regularity, and convergence along traces of positive or full window density interact in this setting.

The algebraic background comes from bicomplex analysis, which goes back to Segre's work on hypercomplex systems and was later developed systematically in multicomplex function theory [1,2]. The idempotent representation has since become a standard tool in bicomplex analysis and related structures; see, for example, [3]. In recent years, convergence and summability methods have also been investigated in bicomplex-valued settings. Statistical convergence in bicomplex metric spaces was studied in [4], while statistical boundedness was considered in [5]. Cesàro-type convergence of bicomplex sequences was treated in [6]. Further results on double sequences and almost convergence of bicomplex numbers appear in [7,8]. Difference sequence spaces and related geometric properties were studied in [9,10]; see also [11] for additional developments in this direction.

The second source of motivation is the theory of deferred and window-based summability. Agnew's deferred Cesàro method introduced the idea of testing asymptotic behaviour over intervals depending on the index rather than only over initial segments [12]. This viewpoint later led to deferred statistical convergence and its variants, as developed in [13,14]. Related statistical and ideal-type methods have appeared in metric and operator-theoretic contexts [15,16]. Related Cauchy and convergence notions in nonclassical metric settings may be found in [17,18]. Similar ideas have also been used in normed and generalized normed settings [19,20]. Further operator-theoretic and function-space manifestations may be found in [21,22]. Extensions involving uncertainty, credibility, and neutrosophic structures were investigated in [23,24]. More recent developments in related nonclassical frameworks include [25]. The classical basis of statistical convergence itself is due to Fast and Fridy [26,27].

On the hyperspace side, Wijsman convergence is one of the most widely used modes of convergence for sequences of closed sets, precisely because it is defined through point-to-set distance functions. Its origins are in the work of Wijsman [28,29], and the general topology of closed and closed convex sets is treated in depth in Beer's monograph [30]. Statistical convergence for sequences of sets was introduced by Nuray and Rhoades [31]. More recent neighbourhood-based viewpoints were considered in [32]. These works show that Wijsman-type methods are flexible enough to connect hyperspace topology with summability and density methods. Nevertheless, the interaction between window-weighted statistical methods and idempotent bicomplex hyperspaces has not been systematically developed.

The precise contribution of the present paper is the following. We do not attempt to develop a general theory of arbitrary bicomplex-valued metrics. Instead, we work in the idempotent class $\rho = d_1 e_1 + d_2 e_2$, where d_1 and d_2 are ordinary metrics. In this framework, Wijsman-type convergence of closed sets is governed by the two scalar point-to-set distance profiles

$$k \mapsto d_1(x, A_k), \quad k \mapsto d_2(x, A_k),$$

for every fixed $x \in X$. The mathematical advance is to formulate and analyze weighted moving-window versions of Wijsman statistical convergence and Wijsman pre-Cauchy regularity in this two-profile setting, while keeping the common hyperspace

$$\mathrm{CL}(X) = \{A \subseteq X : A \neq \varnothing,\ A \text{ is closed in both } (X, d_1) \text{ and } (X, d_2)\}.$$

Thus the results are best understood as structural transfer and stability principles for two coupled scalar distance profiles associated with an idempotent bicomplex metric hyperspace.

More specifically, the paper addresses two main questions. First, under what condition do two weighted window schemes generate the same Wijsman statistical convergence for hyperspace-valued sequences? This question is answered at the scalar level by Proposition 1 and then transferred to the Wijsman setting in Theorem 3. Second, when does window-weighted Wijsman pre-Cauchy behaviour imply statistical convergence to a specific closed set? The answer is given by Theorem 4: a pre-Cauchy sequence becomes statistically convergent once it has an ordinarily Wijsman convergent trace whose lower window weight is positive. This positive trace-density assumption supplies enough mass inside the windows to convert pairwise stabilization into convergence to the trace limit.

The main results should therefore be read in a modest but useful sense. The componentwise characterizations show that the idempotent bicomplex Wijsman conditions are exactly equivalent to the corresponding scalar window conditions for the two distance profiles. The comparison theorem identifies the correct invariance level of the window method: the convergence depends on the asymptotic weighted content of the windows rather than on their particular endpoints. The pre-Cauchy trace theorem gives a recovery principle for limits from pairwise regularity, and the stability results show that negligible perturbations do not affect the resulting convergence.

The examples at the end of the paper show that the assumptions cannot be removed in a formal way. They demonstrate that non-equivalent component metrics may affect the admissible hyperspace, that the same sequence may have different statistical behaviour under different window schemes, that the pre-Cauchy condition does not imply statistical convergence without an additional trace hypothesis, and that convergence on subsets of window-weighted density one can determine the statistical convergence of the whole sequence even when ordinary Wijsman convergence fails. These examples also show why the window structure is part of the asymptotic structure being studied rather than an auxiliary convention.

The paper is organized as follows. §2 introduces the idempotent bicomplex hyperspace framework, the point-to-set distance notation, and the weighted window structure used throughout the paper. §3 establishes the scalar window estimates needed for the later Wijsman-type arguments. §4 defines window-weighted Wijsman statistical convergence and the corresponding pre-Cauchy condition, and proves the two-component characterizations, the window comparison theorem, and the pre-Cauchy trace upgrade result. §5 studies stability under negligible perturbations and proves a convergence theorem for subsequences of full window density. §6 presents examples and counterexamples showing the role of the window scheme, the strict weakness of the pre-Cauchy condition, and the sharpness of the trace assumptions.

2. Preliminaries: Bicomplex hyperspaces and weighted windows

In this section we fix the notation and the basic framework used throughout the paper. The construction has two layers. The first one is geometric: the ambient bicomplex metric is represented by two ordinary metrics. The second one is asymptotic: convergence will be tested along weighted moving windows. Keeping these two layers separate is useful, since all later Wijsman-type conditions will be reduced to scalar estimates for the two component distance functions.

We recall the elementary bicomplex notation needed in the sequel. For background on bicomplex numbers and their idempotent decomposition, we refer to [1,2]. A concise introduction to bicomplex numbers may also be found in [3]. Let $\mathbb{C}_2 = \{z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}\}$, where $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. Define $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$. Then $e_1 + e_2 = 1$, $e_1 e_2 = 0$, $e_1^2 = e_1$ and $e_2^2 = e_2$. Thus every $\xi \in \mathbb{C}_2$ has a unique idempotent representation $\xi = \xi_1 e_1 + \xi_2 e_2$, $\xi_1, \xi_2 \in \mathbb{C}$.

In the present paper, metric values are taken from the positive hyperbolic cone

$$\mathbb{D}^+ = \{\alpha e_1 + \beta e_2 : \alpha, \beta \geq 0\}.$$

We use the usual componentwise partial order on $\mathbb{D}^+$, namely $\alpha_1 e_1 + \beta_1 e_2 \preceq \alpha_2 e_1 + \beta_2 e_2$ if and only if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Convergence in $\mathbb{D}^+$ is always understood componentwise. Thus

$$\alpha_n e_1 + \beta_n e_2 \longrightarrow \alpha e_1 + \beta e_2,$$

means precisely that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in $\mathbb{R}$.

Definition 1. Let X be a non-empty set. A mapping $\rho : X \times X \to \mathbb{D}^+$ is called an *idempotent bicomplex metric* on X if there exist two ordinary metrics d_1 and d_2 on X such that

$$\rho(x,y) = d_1(x,y)e_1 + d_2(x,y)e_2 \quad (x,y \in X).$$

The pair (X,ρ) is then called an *idempotent bicomplex metric space*.

Remark 1. Definition 1 specifies the class of bicomplex-valued metrics used in this paper. It is not intended to cover every possible bicomplex-valued metric structure. We deliberately restrict attention to metrics whose values lie in the positive hyperbolic cone and admit the idempotent representation $\rho = d_1e_1 + d_2e_2$ with two ordinary component metrics d_1 and d_2. This restriction is natural for the present purpose because Wijsman convergence of closed sets can then be described through the two real-valued point-to-set distance functions $d_1(x,A)$ and $d_2(x,A)$. Accordingly, all results below should be interpreted within this idempotent componentwise framework.

Throughout the paper, whenever (X,ρ) is fixed, we write $\rho = d_1e_1 + d_2e_2$ and regard d_1 and d_2 as fixed component metrics. This convention is important because every Wijsman-type condition below is imposed simultaneously on the two point-to-set distance profiles generated by d_1 and d_2.

Let $A \subseteq X$ be non-empty and let $x \in X$. For $j = 1,2$, define

$$d_j(x,A) = \inf\{d_j(x,a) : a \in A\},$$

and set

$$\rho(x,A) = d_1(x,A)e_1 + d_2(x,A)e_2.$$

We denote by $\mathrm{CL}(X)$ the family of all non-empty subsets of X which are closed in both metric spaces (X,d_1) and (X,d_2).

The simultaneous closedness requirement may be stronger than closedness with respect to only one component metric when the two metric topologies are not equivalent. The following simple example illustrates this point.

Example 1. Let $X = \mathbb{R}$. Define $d_1(x,y) = |x-y|$ and let d_2 be the discrete metric,

$$d_2(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Then d_1 and d_2 do not induce the same topology. Indeed, every subset of $\mathbb{R}$ is closed in $(\mathbb{R},d_2)$, while only the usual closed subsets are closed in $(\mathbb{R},d_1)$. Hence

$$\mathrm{CL}(X) = \{A \subseteq \mathbb{R} : A \neq \varnothing,\ A \text{ is closed in the usual metric}\}.$$

For instance, $A = (0,1)$ is closed in the discrete metric d_2, but it is not closed in the usual metric d_1, and therefore $A \notin \mathrm{CL}(X)$. This shows that the common hyperspace is determined by the simultaneous closedness condition and not by either component metric alone.

The following elementary facts justify the use of $\rho(x,A)$ as a point-to-set distance in the present setting.

Lemma 1. *Let $A \in \mathrm{CL}(X)$. Then the following assertions hold.*

1. *For each $j \in \{1,2\}$, the function $x \mapsto d_j(x,A)$ is 1-Lipschitz on (X,d_j).*
2. *For $x \in X$, one has $\rho(x,A) = 0$ if and only if $x \in A$.*
3. *If $A, B \in \mathrm{CL}(X)$ and $\rho(x,A) = \rho(x,B)$ for every $x \in X$, then $A = B$.*

Proof. We first prove (1). Let $j \in \{1,2\}$ be fixed, and consider $x,y \in X$. For each $a \in A$, the triangle inequality in (X,d_j) asserts that

$$d_j(x,a) \leq d_j(x,y) + d_j(y,a).$$

By taking the infimum over $a \in A$, we obtain

$$d_j(x,A) \leq d_j(x,y) + d_j(y,A).$$

Reversing x and y yields

$$d_j(y,A) \leq d_j(x,y) + d_j(x,A).$$

The two inequalities imply that

$$|d_j(x,A) - d_j(y,A)| \leq d_j(x,y),$$

which proves that the mapping $x \mapsto d_j(x,A)$ is 1-Lipschitz with respect to d_j.

We now prove (2). If $x \in A$, then $d_j(x,A) = 0$ for $j = 1,2$, so $\rho(x,A) = 0$. On the other hand, assume that $\rho(x,A) = 0$. Since convergence and equality in $\mathbb{D}^+$ are determined componentwise, it follows that $d_1(x,A) = 0$ and $d_2(x,A) = 0$. Since A is closed in (X,d_1), the condition $d_1(x,A) = 0$ implies that $x \in A$. Consequently, $\rho(x,A) = 0$ implies that x belongs to A.

Assume that $A, B \in \mathrm{CL}(X)$ and that $\rho(x,A) = \rho(x,B)$ for all $x \in X$. Let $x \in A$. By (2), $\rho(x,A) = 0$. Hence $\rho(x,B) = 0$, and another application of (2) gives $x \in B$. Thus $A \subseteq B$. Interchanging A and B gives $B \subseteq A$. Therefore $A = B$. $\square$

Remark 2. The idempotent representation used in this paper separates the bicomplex metric into two ordinary metric components. Therefore, many arguments reduce to scalar estimates applied to the two distance profiles. The point of the present formulation is not to claim a phenomenon beyond this componentwise structure, but to keep track of the two profiles simultaneously on the common hyperspace

$$\mathrm{CL}(X) = \{A \subseteq X : A \neq \varnothing,\ A \text{ is closed in both } (X,d_1) \text{ and } (X,d_2)\}.$$

When d_1 and d_2 generate different topologies, this simultaneous closedness condition can have a visible effect, as shown in Example 1. Thus the hyperspace is attached to the pair (d_1,d_2), although the convergence tests themselves are expressed through scalar component estimates.

Definition 2. Let (A_k) be a sequence in $\mathrm{CL}(X)$ and let $A \in \mathrm{CL}(X)$. We say that (A_k) is *Wijsman convergent* to A in the idempotent bicomplex sense if $\rho(x,A_k) \to \rho(x,A)$ for every $x \in X$. Equivalently, (A_k) is Wijsman convergent to A if and only if, for every $x \in X$, $d_1(x,A_k) \to d_1(x,A)$ and $d_2(x,A_k) \to d_2(x,A)$.

The equivalence in Definition 2 follows directly from the componentwise interpretation of convergence in $\mathbb{D}^+$. Lemma 1 also shows that a Wijsman limit, when it exists in $\mathrm{CL}(X)$, is uniquely determined by the limiting point-to-set distance profile.

We now introduce the window notation. This part is independent of the bicomplex structure and will later be applied to the scalar distance profiles $d_1(x,A_k)$ and $d_2(x,A_k)$.

Definition 3. A *window scheme* is a sequence $\mathcal{W} = \{I_n\}$ of finite intervals of positive integers of the form

$$I_n = (\mu_n, \lambda_n] \cap \mathbb{N},$$

where $0 \leq \mu_n < \lambda_n$ and $h_n := \lambda_n - \mu_n \to \infty$. No monotonicity of (μ_n) or (λ_n) is assumed unless explicitly stated. In particular, the windows need not be nested, disjoint, or ordered by inclusion. The proofs below use only the interval form, the condition $h_n \to \infty$, and the weighted normalization $W_n \to \infty$.

Let $w = (w_k)$ be a sequence of positive real numbers. For each $n \in \mathbb{N}$, we write

$$W_n = \sum_{k \in I_n} w_k,$$

for the weighted mass of the n-th window. Throughout the paper we assume $W_n \to \infty$. Thus $\mathcal{W} = \{I_n\}$ denotes the window scheme, whereas W_n denotes only the corresponding weighted mass.

Definition 4. Let $E \subseteq \mathbb{N}$. If the following limit exists, we call it the *window-weighted density* of E with respect to $(\mathcal{W}, w)$:

$$\delta_{\mathcal{W}}^{w}(E) = \lim_{n\to\infty} \frac{1}{W_n} \sum_{k\in I_n\cap E} w_k.$$

The density in Definition 4 measures the relative weighted size of E inside the moving windows I_n. In particular, it is adapted to situations where the relevant asymptotic information is localized in blocks rather than distributed along the initial segments $\{1, \dots, n\}$.

The following observation will be used repeatedly when finite modifications are considered.

Lemma 2. *If $F \subseteq \mathbb{N}$ is finite, then* $\lim_{n\to\infty} \frac{1}{W_n} \sum\limits_{k\in I_n\cap F} w_k = 0$.

Proof. Since F is finite and $w_k > 0$ for every k, the quantity $C_F = \sum\limits_{k\in F} w_k$ is finite. For any $n \in \mathbb{N}$, it holds that

$$0 \le \sum_{k\in I_n\cap F} w_k \le \sum_{k\in F} w_k = C_F.$$

Dividing by W_n, we obtain

$$0 \le \frac{1}{W_n} \sum_{k\in I_n\cap F} w_k \le \frac{C_F}{W_n}.$$

As W_n approaches infinity, the right-hand side converges to 0. Therefore, the required limit is established. □

3. Scalar window estimates

The arguments in the subsequent sections are based on scalar estimates applied to the two component distance profiles $k \mapsto d_1(x, A_k)$ and $k \mapsto d_2(x, A_k)$. For this reason, we first record the corresponding scalar notions and comparison principles for weighted moving windows. These results will later be applied pointwise in $x \in X$.

Let $\mathcal{W} = \{I_n\}$ be a fixed window scheme and let $W_n = \sum\limits_{k\in I_n} w_k$. For a real sequence $u = (u_k)$, a number $\ell \in \mathbb{R}$, and $\varepsilon > 0$, define

$$A_n^{\mathcal{W}}(u, \ell; \varepsilon) = \frac{1}{W_n} \sum_{\substack{k\in I_n \\ |u_k-\ell|\ge\varepsilon}} w_k,$$

and

$$P_n^{\mathcal{W}}(u; \varepsilon) = \frac{1}{W_n^2} \sum_{\substack{k,m\in I_n \\ |u_k-u_m|\ge\varepsilon}} w_k w_m.$$

Definition 5. Let $\mathcal{W} = \{I_n\}$ be a window scheme. A real sequence (u_k) is said to be *window-weighted statistically convergent* to $\ell \in \mathbb{R}$ with respect to $\mathcal{W}$ if $A_n^{\mathcal{W}}(u, \ell; \varepsilon) \to 0$ $(n \to \infty)$ for every $\varepsilon > 0$.

Definition 6. Let $\mathcal{W} = \{I_n\}$ be a window scheme. A real sequence (u_k) is said to be *window-weighted pre-Cauchy* with respect to $\mathcal{W}$ if $P_n^{\mathcal{W}}(u; \varepsilon) \to 0$ $(n \to \infty)$ for every $\varepsilon > 0$.

We next specify when two window schemes are regarded as asymptotically the same for the present weighted method.

Definition 7. Let $\mathcal{W} = \{I_n\}$ and $\mathcal{V} = \{J_n\}$ be two window schemes. Put $W_n = \sum\limits_{k\in I_n} w_k$ and $V_n = \sum\limits_{k\in J_n} w_k$. We say that $\mathcal{W}$ and $\mathcal{V}$ are *equivalent with respect to w* if $\frac{1}{W_n} \sum\limits_{k\in I_n\triangle J_n} w_k \to 0$.

Remark 3. Although Definition 7 is normalized by W_n, it is symmetric in the asymptotic sense needed here. Indeed, Proposition 1 shows that

$$\frac{V_n}{W_n} \to 1,$$

and consequently

$$\frac{1}{V_n} \sum_{k \in I_n \triangle J_n} w_k \to 0.$$

Thus the same equivalence relation is obtained if the normalization is taken with respect to V_n instead of W_n.

The next proposition is the main scalar comparison tool. It shows that equivalent windows have the same asymptotic effect on both one-parameter weighted averages and two-parameter pre-Cauchy averages.

Proposition 1. *Let $\mathcal{W} = \{I_n\}$ and $\mathcal{V} = \{J_n\}$ be two window schemes which are equivalent with respect to w. Then the following assertions hold.*

(1) *One has $\frac{V_n}{W_n} \to 1$ and $\frac{1}{V_n} \sum\limits_{k \in I_n \triangle J_n} w_k \to 0$.*

(2) *If (a_k) is a bounded non-negative real sequence, then*

$$\frac{1}{W_n} \sum_{k \in I_n} w_k a_k - \frac{1}{V_n} \sum_{k \in J_n} w_k a_k \to 0.$$

(3) *A real sequence (u_k) is window-weighted statistically convergent to ℓ with respect to $\mathcal{W}$ if and only if it is window-weighted statistically convergent to ℓ with respect to $\mathcal{V}$.*

(4) *A real sequence (u_k) is window-weighted pre-Cauchy with respect to $\mathcal{W}$ if and only if it is window-weighted pre-Cauchy with respect to $\mathcal{V}$.*

Proof. Define $D_n = \sum\limits_{k \in I_n \triangle J_n} w_k$. By hypothesis, $\frac{D_n}{W_n} \to 0$. Since

$$|W_n - V_n| = \left| \sum_{k \in I_n} w_k - \sum_{k \in J_n} w_k \right| \le \sum_{k \in I_n \triangle J_n} w_k = D_n,$$

we obtain

$$\left| 1 - \frac{V_n}{W_n} \right| = \frac{|W_n - V_n|}{W_n} \le \frac{D_n}{W_n} \to 0.$$

Consequently, $V_n / W_n \to 1$. Thus, $\frac{D_n}{V_n} = \frac{D_n}{W_n} \cdot \frac{W_n}{V_n} \to 0$, thereby proving (1).

We now prove (2). Let $0 \le a_k \le M$ for all $k \in \mathbb{N}$, where $M > 0$. Put $H_n = I_n \cap J_n$. Then

$$\left| \frac{1}{W_n} \sum_{k \in I_n} w_k a_k - \frac{1}{V_n} \sum_{k \in J_n} w_k a_k \right| \le \left| \frac{1}{W_n} \sum_{k \in H_n} w_k a_k - \frac{1}{V_n} \sum_{k \in H_n} w_k a_k \right| + \frac{1}{W_n} \sum_{k \in I_n \setminus H_n} w_k a_k + \frac{1}{V_n} \sum_{k \in J_n \setminus H_n} w_k a_k.$$

For the first term, since

$$\sum_{k \in H_n} w_k a_k \le M \sum_{k \in H_n} w_k \le M W_n,$$

we have

$$\left| \frac{1}{W_n} \sum_{k \in H_n} w_k a_k - \frac{1}{V_n} \sum_{k \in H_n} w_k a_k \right| \le M \left| 1 - \frac{W_n}{V_n} \right|.$$

For the remaining two terms, we use $I_n \setminus H_n \subseteq I_n \triangle J_n$ and $J_n \setminus H_n \subseteq I_n \triangle J_n$. Hence

$$\frac{1}{W_n} \sum_{k \in I_n \setminus H_n} w_k a_k \le M \frac{D_n}{W_n},$$

and

$$\frac{1}{V_n}\sum_{k\in J_n\setminus H_n} w_k a_k \le M\frac{D_n}{V_n}.$$

Combining these estimates gives

$$\left|\frac{1}{W_n}\sum_{k\in I_n} w_k a_k - \frac{1}{V_n}\sum_{k\in J_n} w_k a_k\right| \le M\left|1-\frac{W_n}{V_n}\right| + M\frac{D_n}{W_n} + M\frac{D_n}{V_n}.$$

Each term on the right tends to 0 by (1), and (2) follows.

To prove (3), fix $\varepsilon > 0$ and apply (2) to the bounded non-negative sequence

$$a_k = \begin{cases} 1, & |u_k-\ell| \ge \varepsilon, \\ 0, & |u_k-\ell| < \varepsilon. \end{cases}$$

Then

$$\frac{1}{W_n}\sum_{\substack{k\in I_n\\ |u_k-\ell|\ge\varepsilon}} w_k - \frac{1}{V_n}\sum_{\substack{k\in J_n\\ |u_k-\ell|\ge\varepsilon}} w_k \to 0.$$

Therefore the first sequence of weighted exceptional averages tends to 0 if and only if the second one does. Since this holds for every $\varepsilon > 0$, the equivalence of the two statistical convergence statements follows.

It remains to prove (4). Fix $\varepsilon > 0$ and set

$$b_{k,m} = \begin{cases} 1, & |u_k-u_m| \ge \varepsilon, \\ 0, & |u_k-u_m| < \varepsilon. \end{cases}$$

We shall compare the double weighted averages over $I_n \times I_n$ and $J_n \times J_n$. Again put $H_n = I_n \cap J_n$. Since

$$\sum_{k\in H_n} w_k = W_n - \sum_{k\in I_n\setminus J_n} w_k \ge W_n - D_n,$$

we have

$$0 \le 1 - \left(\frac{\sum_{k\in H_n} w_k}{W_n}\right)^2 \le 1 - \left(1-\frac{D_n}{W_n}\right)^2.$$

The right-hand side is

$$2\frac{D_n}{W_n} - \left(\frac{D_n}{W_n}\right)^2 \le 2\frac{D_n}{W_n},$$

for all n. Hence

$$\frac{1}{W_n^2}\sum_{(k,m)\in I_n^2\setminus H_n^2} w_k w_m = 1 - \left(\frac{\sum_{k\in H_n} w_k}{W_n}\right)^2 \to 0.$$

Using $D_n/V_n \to 0$, the same argument gives

$$\frac{1}{V_n^2}\sum_{(k,m)\in J_n^2\setminus H_n^2} w_k w_m \to 0.$$

Since $0 \le b_{k,m} \le 1$, removing the pairs outside H_n^2 changes the normalized sums by at most the normalized total weight of the removed pairs. Therefore,

$$\left|\frac{1}{W_n^2}\sum_{k,m\in I_n} w_k w_m b_{k,m} - \frac{1}{W_n^2}\sum_{k,m\in H_n} w_k w_m b_{k,m}\right| \to 0,$$

and

$$\left| \frac{1}{V_n^2} \sum_{k,m \in J_n} w_k w_m b_{k,m} - \frac{1}{V_n^2} \sum_{k,m \in H_n} w_k w_m b_{k,m} \right| \to 0.$$

It remains only to compare the two normalizations on H_n^2. Since

$$0 \leq \sum_{k,m \in H_n} w_k w_m b_{k,m} \leq \left(\sum_{k \in H_n} w_k \right)^2 \leq W_n^2,$$

we get

$$\left| \frac{1}{W_n^2} \sum_{k,m \in H_n} w_k w_m b_{k,m} - \frac{1}{V_n^2} \sum_{k,m \in H_n} w_k w_m b_{k,m} \right| \leq \left| 1 - \frac{W_n^2}{V_n^2} \right|.$$

Since $V_n / W_n \to 1$, the last term tends to 0. Consequently,

$$\frac{1}{W_n^2} \sum_{k,m \in I_n} w_k w_m b_{k,m} - \frac{1}{V_n^2} \sum_{k,m \in J_n} w_k w_m b_{k,m} \to 0.$$

With the above choice of $b_{k,m}$, this proves that $P_n^{\mathcal{W}}(u;\varepsilon) \to 0$ if and only if $P_n^{\mathcal{V}}(u;\varepsilon) \to 0$. Since $\varepsilon > 0$ was arbitrary, the equivalence of the two pre-Cauchy properties follows. □

The next lemma connects ordinary convergence with the window-weighted statistical convergence. It will be used later in the trace arguments.

Lemma 3. *Let (u_k) be a real sequence and let $\ell \in \mathbb{R}$. If $u_k \to \ell$ in the ordinary sense, then (u_k) is window-weighted statistically convergent to ℓ with respect to $\mathcal{W}$.*

Proof. Fix $\varepsilon > 0$. Since $u_k \to \ell$, there exists $k_0 \in \mathbb{N}$ such that $|u_k - \ell| < \varepsilon$ $(k \geq k_0)$. Therefore,

$$\{k \in \mathbb{N} : |u_k - \ell| \geq \varepsilon\} \subseteq \{1, 2, \ldots, k_0 - 1\}.$$

The set on the right is finite. By Lemma 2,

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ |u_k - \ell| \geq \varepsilon}} w_k \to 0.$$

Since $\varepsilon > 0$ was arbitrary, the desired window-weighted statistical convergence follows. □

Finally, statistical convergence in the window-weighted sense implies the corresponding pre-Cauchy condition, in the spirit of the statistically pre-Cauchy viewpoint of Connor, Fridy and Kline [33].

Lemma 4. *If a real sequence (u_k) is window-weighted statistically convergent to $\ell \in \mathbb{R}$ with respect to $\mathcal{W}$, then (u_k) is window-weighted pre-Cauchy with respect to $\mathcal{W}$.*

Proof. Fix $\varepsilon > 0$ and define $E_n = \left\{k \in I_n : |u_k - \ell| \geq \frac{\varepsilon}{2}\right\}$. Suppose that $k, m \in I_n$ and $|u_k - u_m| \geq \varepsilon$. If both $k \notin E_n$ and $m \notin E_n$, then $|u_k - \ell| < \frac{\varepsilon}{2}$ and $|u_m - \ell| < \frac{\varepsilon}{2}$. The triangle inequality would then give

$$|u_k - u_m| \leq |u_k - \ell| + |u_m - \ell| < \varepsilon,$$

which contradicts $|u_k - u_m| \geq \varepsilon$. Hence at least one of k and m belongs to E_n. Therefore,

$$\{(k,m) \in I_n \times I_n : |u_k - u_m| \geq \varepsilon\} \subseteq (E_n \times I_n) \cup (I_n \times E_n).$$

Using this inclusion, we obtain

$$P_n^{\mathcal{W}}(u;\varepsilon) = \frac{1}{W_n^2} \sum_{\substack{k,m\in I_n \\ |u_k-u_m|\geq\varepsilon}} w_k w_m \leq \frac{1}{W_n^2} \sum_{k\in E_n} \sum_{m\in I_n} w_k w_m + \frac{1}{W_n^2} \sum_{k\in I_n} \sum_{m\in E_n} w_k w_m$$
$$= \frac{2}{W_n} \sum_{k\in E_n} w_k = 2A_n^{\mathcal{W}}(u,\ell;\varepsilon/2).$$

Since (u_k) is window-weighted statistically convergent to ℓ, the last term tends to 0. Thus $P_n^{\mathcal{W}}(u;\varepsilon) \to 0$. Since $\varepsilon > 0$ was arbitrary, (u_k) is window-weighted pre-Cauchy. $\square$

4. Window-weighted Wijsman convergence

We now pass from scalar sequences to sequences of closed sets. The purpose of this section is to introduce the window-weighted Wijsman statistical convergence and its associated pre-Cauchy condition, and then to prove the main comparison and recovery results. The scalar tools from §3 will be applied to the component distance profiles $k \mapsto d_1(x, A_k)$ and $k \mapsto d_2(x, A_k)$, with $x \in X$ fixed.

Throughout this section, (A_k) denotes a sequence in $\mathrm{CL}(X)$. For $A \in \mathrm{CL}(X)$, $x \in X$, and $k, \ell \in \mathbb{N}$, define

$$\Lambda_k(x; A) = |d_1(x, A_k) - d_1(x, A)| + |d_2(x, A_k) - d_2(x, A)|,$$

and

$$\Lambda_{k,\ell}(x) = |d_1(x, A_k) - d_1(x, A_\ell)| + |d_2(x, A_k) - d_2(x, A_\ell)|.$$

Thus $\Lambda_k(x; A)$ measures the total two-component deviation of A_k from the candidate limit set A at the point x, while $\Lambda_{k,\ell}(x)$ measures the corresponding pairwise deviation between A_k and A_ℓ.

Definition 8. Let $A \in \mathrm{CL}(X)$. We say that (A_k) is *window-weighted Wijsman statistically convergent* to A if, for every $x \in X$ and every $\varepsilon > 0$,

$$\frac{1}{W_n} \sum_{\substack{k\in I_n \\ \Lambda_k(x;A)\geq\varepsilon}} w_k \to 0.$$

Definition 9. We say that (A_k) is *window-weighted Wijsman pre-Cauchy* if, for every $x \in X$ and every $\varepsilon > 0$,

$$\frac{1}{W_n^2} \sum_{\substack{k,\ell\in I_n \\ \Lambda_{k,\ell}(x)\geq\varepsilon}} w_k w_\ell \to 0.$$

The first result shows that these two definitions are exactly equivalent to the corresponding scalar window conditions for the two component metrics.

The following theorem is a structural transfer result. It does not claim that a new scalar phenomenon appears in the bicomplex setting. Rather, it records that the idempotent bicomplex Wijsman definitions are exactly equivalent to the simultaneous scalar window conditions for the two point-to-set distance profiles. This equivalence is essential for applying the scalar comparison and pre-Cauchy estimates to hyperspace-valued sequences.

Theorem 1 (Two-component characterizations)**.** *Let (A_k) be a sequence in $\mathrm{CL}(X)$.*

(1) *Let $A \in \mathrm{CL}(X)$. Then (A_k) is window-weighted Wijsman statistically convergent to A if and only if, for each $j \in \{1,2\}$, every $x \in X$, and every $\varepsilon > 0$,*

$$\frac{1}{W_n} \sum_{\substack{k\in I_n \\ |d_j(x,A_k)-d_j(x,A)|\geq\varepsilon}} w_k \to 0.$$

(2) *The sequence (A_k) is window-weighted Wijsman pre-Cauchy if and only if, for each $j \in \{1,2\}$, every $x \in X$, and every $\varepsilon > 0$,*

$$\frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ |d_j(x,A_k) - d_j(x,A_\ell)| \geq \varepsilon}} w_k w_\ell \to 0.$$

Proof. We prove the two assertions separately.

For (1), assume first that (A_k) is window-weighted Wijsman statistically convergent to A. Fix $j \in \{1,2\}$, $x \in X$, and $\varepsilon > 0$. Since

$$|d_j(x, A_k) - d_j(x, A)| \leq \Lambda_k(x; A),$$

for every $k \in \mathbb{N}$, we have the set inclusion

$$\{k \in I_n : |d_j(x, A_k) - d_j(x, A)| \geq \varepsilon\} \subseteq \{k \in I_n : \Lambda_k(x; A) \geq \varepsilon\}.$$

Multiplying by w_k, summing over the corresponding indices, and dividing by W_n, we obtain

$$0 \leq \frac{1}{W_n} \sum_{\substack{k \in I_n \\ |d_j(x,A_k) - d_j(x,A)| \geq \varepsilon}} w_k \leq \frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k(x;A) \geq \varepsilon}} w_k.$$

The right-hand side tends to 0, and therefore the componentwise statistical condition follows.

Conversely, assume that the componentwise condition holds for both $j = 1$ and $j = 2$. Fix $x \in X$ and $\varepsilon > 0$. If $\Lambda_k(x; A) \geq \varepsilon$, then

$$|d_1(x, A_k) - d_1(x, A)| + |d_2(x, A_k) - d_2(x, A)| \geq \varepsilon.$$

Since both summands are non-negative, at least one of them is not smaller than $\varepsilon/2$. Hence

$$\{k \in I_n : \Lambda_k(x; A) \geq \varepsilon\} \subseteq E_{1,n}(x, \varepsilon) \cup E_{2,n}(x, \varepsilon),$$

where

$$E_{j,n}(x, \varepsilon) = \left\{k \in I_n : |d_j(x, A_k) - d_j(x, A)| \geq \frac{\varepsilon}{2}\right\}.$$

It follows that

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k(x;A) \geq \varepsilon}} w_k \leq \sum_{j=1}^{2} \frac{1}{W_n} \sum_{k \in E_{j,n}(x,\varepsilon)} w_k.$$

By the assumed componentwise condition, both terms on the right tend to 0. Therefore (A_k) is window-weighted Wijsman statistically convergent to A.

We now prove (2). Assume first that (A_k) is window-weighted Wijsman pre-Cauchy. Fix $j \in \{1,2\}$, $x \in X$, and $\varepsilon > 0$. Since

$$|d_j(x, A_k) - d_j(x, A_\ell)| \leq \Lambda_{k,\ell}(x),$$

for all $k, \ell \in \mathbb{N}$, we have

$$\{(k, \ell) \in I_n \times I_n : |d_j(x, A_k) - d_j(x, A_\ell)| \geq \varepsilon\} \subseteq \{(k, \ell) \in I_n \times I_n : \Lambda_{k,\ell}(x) \geq \varepsilon\}.$$

After multiplying by $w_k w_\ell$, summing, and dividing by W_n^2, the desired componentwise pre-Cauchy condition follows.

Conversely, assume that the componentwise pairwise condition holds for $j = 1, 2$. Fix $x \in X$ and $\varepsilon > 0$. If $\Lambda_{k,\ell}(x) \geq \varepsilon$, then

$$|d_1(x, A_k) - d_1(x, A_\ell)| + |d_2(x, A_k) - d_2(x, A_\ell)| \geq \varepsilon.$$

Thus at least one of the two component terms is not smaller than $\varepsilon/2$. Therefore

$$\{(k, \ell) \in I_n \times I_n : \Lambda_{k,\ell}(x) \geq \varepsilon\} \subseteq F_{1,n}(x, \varepsilon) \cup F_{2,n}(x, \varepsilon),$$

where

$$F_{j,n}(x,\varepsilon) = \left\{(k,\ell) \in I_n \times I_n : |d_j(x,A_k) - d_j(x,A_\ell)| \geq \frac{\varepsilon}{2}\right\}.$$

Consequently,

$$\frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda_{k,\ell}(x) \geq \varepsilon}} w_k w_\ell \leq \sum_{j=1}^{2} \frac{1}{W_n^2} \sum_{(k,\ell) \in F_{j,n}(x,\varepsilon)} w_k w_\ell.$$

The two terms on the right tend to 0 by the componentwise pairwise assumption. Hence (A_k) is window-weighted Wijsman pre-Cauchy. □

The next theorem is the Wijsman analogue of the scalar implication from statistical convergence to the pre-Cauchy property.

Theorem 2. *If (A_k) is window-weighted Wijsman statistically convergent to $A \in \mathrm{CL}(X)$, then (A_k) is window-weighted Wijsman pre-Cauchy.*

Proof. Fix $x \in X$ and $\varepsilon > 0$. Define $E_n(x,\varepsilon) = \left\{k \in I_n : \Lambda_k(x;A) \geq \frac{\varepsilon}{2}\right\}$. Suppose that $k,\ell \in I_n$ and $\Lambda_{k,\ell}(x) \geq \varepsilon$. We claim that at least one of k and ℓ belongs to $E_n(x,\varepsilon)$. Indeed, if $k \notin E_n(x,\varepsilon)$ and $\ell \notin E_n(x,\varepsilon)$, then $\Lambda_k(x;A) < \frac{\varepsilon}{2}$ and $\Lambda_\ell(x;A) < \frac{\varepsilon}{2}$. For each $j \in \{1,2\}$, the triangle inequality in $\mathbb{R}$ gives

$$|d_j(x,A_k) - d_j(x,A_\ell)| \leq |d_j(x,A_k) - d_j(x,A)| + |d_j(x,A_\ell) - d_j(x,A)|.$$

Adding these inequalities for $j = 1,2$, we obtain $\Lambda_{k,\ell}(x) \leq \Lambda_k(x;A) + \Lambda_\ell(x;A) < \varepsilon$, which contradicts $\Lambda_{k,\ell}(x) \geq \varepsilon$. Hence

$$\{(k,\ell) \in I_n \times I_n : \Lambda_{k,\ell}(x) \geq \varepsilon\} \subseteq (E_n(x,\varepsilon) \times I_n) \cup (I_n \times E_n(x,\varepsilon)).$$

Therefore,

$$\begin{aligned} \frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda_{k,\ell}(x) \geq \varepsilon}} w_k w_\ell &\leq \frac{1}{W_n^2} \sum_{k \in E_n(x,\varepsilon)} \sum_{\ell \in I_n} w_k w_\ell + \frac{1}{W_n^2} \sum_{k \in I_n} \sum_{\ell \in E_n(x,\varepsilon)} w_k w_\ell \\ &= \frac{2}{W_n} \sum_{k \in E_n(x,\varepsilon)} w_k. \end{aligned}$$

Since (A_k) is window-weighted Wijsman statistically convergent to A, the last term tends to 0. Thus the pre-Cauchy condition follows. □

We next prove that the statistical Wijsman mode is unchanged under weighted-equivalent modifications of the window scheme.

Theorem 3 (Comparison of window schemes)**.** *Let $\mathcal{V} = \{J_n\}$ be another window scheme and put $V_n = \sum_{k \in J_n} w_k$. Assume that $\mathcal{W} = \{I_n\}$ and $\mathcal{V} = \{J_n\}$ are equivalent with respect to w, that is,*

$$\frac{1}{W_n} \sum_{k \in I_n \triangle J_n} w_k \to 0.$$

Then, for every sequence (A_k) in $\mathrm{CL}(X)$ and every $A \in \mathrm{CL}(X)$, (A_k) is window-weighted Wijsman statistically convergent to A with respect to $\mathcal{W}$ if and only if it is window-weighted Wijsman statistically convergent to A with respect to $\mathcal{V}$.

Proof. By Theorem 1, window-weighted Wijsman statistical convergence to A is equivalent to the scalar window-weighted statistical convergence, for every $x \in X$, of the two real sequences $u_k^{(j,x)} = d_j(x,A_k)$, $j = 1,2$, to the limits $\ell_j(x) = d_j(x,A)$. Since $\mathcal{W}$ and $\mathcal{V}$ are equivalent with respect to w, Proposition 1(3) shows that the scalar statistical convergence of each sequence $u_k^{(j,x)}$ is the same with respect to $\mathcal{W}$ and $\mathcal{V}$. Applying Theorem 1 once more gives the desired equivalence of the two Wijsman statistical convergence statements. □

The next result is the main recovery principle of the paper. A pre-Cauchy condition contains only pairwise information inside the windows and does not by itself identify a limit. The additional trace hypothesis supplies such a limit. The positive lower window weight condition

$$\liminf_{n\to\infty} \frac{1}{W_n} \sum_{k\in I_n\cap M} w_k > 0,$$

means that the convergent trace M contributes a non-negligible amount of weighted mass inside the windows. This mass is sufficient to convert pairwise stabilization into convergence to the trace limit. The argument is pointwise in $x \in X$; no uniform convergence over X is required.

Theorem 4. *Let (A_k) be a window-weighted Wijsman pre-Cauchy sequence in $\mathrm{CL}(X)$. Assume that there exist $A \in \mathrm{CL}(X)$ and $M \subseteq \mathbb{N}$ such that the following two conditions hold:*

(1) *for every $x \in X$, $\rho(x, A_k) \to \rho(x, A)$ $(k \to \infty,\ k \in M)$, or equivalently, $\Lambda_k(x; A) \to 0$ $(k \to \infty,\ k \in M)$;*
(2) $\liminf_{n\to\infty} \frac{1}{W_n} \sum_{k\in I_n\cap M} w_k > 0$.

Then (A_k) is window-weighted Wijsman statistically convergent to A.

Proof. Fix $x \in X$ and $\varepsilon > 0$. Set

$$\gamma = \liminf_{n\to\infty} \frac{1}{W_n} \sum_{k\in I_n\cap M} w_k.$$

By assumption, $\gamma > 0$. Choose a number β such that $0 < \beta < \gamma$. Then there exists $n_1 \in \mathbb{N}$ such that, for every $n \geq n_1$,

$$\sum_{k\in I_n\cap M} w_k \geq \beta W_n.$$

Since $\Lambda_k(x; A) \to 0$ along M, there exists $N_0 \in \mathbb{N}$ such that $\Lambda_k(x; A) < \frac{\varepsilon}{2}$ $(k \in M,\ k \geq N_0)$. Let $F = \{1, 2, \ldots, N_0 - 1\}$. Then $\Lambda_k(x; A) < \frac{\varepsilon}{2}$ $(k \in M \setminus F)$. By Lemma 2,

$$\frac{1}{W_n} \sum_{k\in I_n\cap F} w_k \to 0.$$

Therefore there exists $n_2 \in \mathbb{N}$ such that, for every $n \geq n_2$,

$$\sum_{k\in I_n\cap F} w_k < \frac{\beta}{2} W_n.$$

For $n \geq \max\{n_1, n_2\}$, define $G_n = (I_n \cap M) \setminus F$. Then

$$\sum_{\ell\in G_n} w_\ell \geq \sum_{\ell\in I_n\cap M} w_\ell - \sum_{\ell\in I_n\cap F} w_\ell \geq \frac{\beta}{2} W_n.$$

Moreover, every $\ell \in G_n$ satisfies $\Lambda_\ell(x; A) < \frac{\varepsilon}{2}$.

Now put $B_n(x, \varepsilon) = \{k \in I_n : \Lambda_k(x; A) \geq \varepsilon\}$. Take $k \in B_n(x, \varepsilon)$ and $\ell \in G_n$. Then $\Lambda_k(x; A) \geq \varepsilon$ and $\Lambda_\ell(x; A) < \frac{\varepsilon}{2}$. We next compare this with the pairwise quantity $\Lambda_{k,\ell}(x)$. Put

$$a = (d_1(x, A_k), d_2(x, A_k)), \qquad b = (d_1(x, A_\ell), d_2(x, A_\ell)),$$

and $c = (d_1(x, A), d_2(x, A))$. Then

$$\Lambda_{k,\ell}(x) = \|a - b\|_1, \qquad \Lambda_k(x; A) = \|a - c\|_1, \qquad \Lambda_\ell(x; A) = \|b - c\|_1.$$

By the reverse triangle inequality in the normed space $(\mathbb{R}^2, \|\cdot\|_1)$,

$$\|a-b\|_1 \geq \big|\|a-c\|_1 - \|b-c\|_1\big|.$$

Hence

$$\Lambda_{k,\ell}(x) \geq |\Lambda_k(x;A) - \Lambda_\ell(x;A)|.$$

Since $\Lambda_k(x;A) \geq \varepsilon$ and $\Lambda_\ell(x;A) < \varepsilon/2$, it follows that $\Lambda_{k,\ell}(x) > \frac{\varepsilon}{2}$. Thus

$$B_n(x,\varepsilon) \times G_n \subseteq \left\{(k,\ell) \in I_n \times I_n : \Lambda_{k,\ell}(x) \geq \frac{\varepsilon}{2}\right\}.$$

Consequently,

$$\frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda_{k,\ell}(x) \geq \varepsilon/2}} w_k w_\ell \geq \frac{1}{W_n^2} \sum_{k \in B_n(x,\varepsilon)} \sum_{\ell \in G_n} w_k w_\ell = \left(\frac{1}{W_n} \sum_{k \in B_n(x,\varepsilon)} w_k\right)\left(\frac{1}{W_n} \sum_{\ell \in G_n} w_\ell\right).$$

For all sufficiently large n, the second factor on the right is at least $\beta/2$. Therefore,

$$\frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda_{k,\ell}(x) \geq \varepsilon/2}} w_k w_\ell \geq \frac{\beta}{2} \cdot \frac{1}{W_n} \sum_{k \in B_n(x,\varepsilon)} w_k.$$

Since (A_k) is window-weighted Wijsman pre-Cauchy, the left-hand side tends to 0. Hence

$$\frac{1}{W_n} \sum_{k \in B_n(x,\varepsilon)} w_k \to 0.$$

This is precisely the window-weighted Wijsman statistical convergence of (A_k) to A at the fixed point x. Since $x \in X$ and $\varepsilon > 0$ were arbitrary, (A_k) is window-weighted Wijsman statistically convergent to A. □

The following trace version is often the form in which the preceding theorem is used.

Corollary 1. *Let (A_k) be a window-weighted Wijsman pre-Cauchy sequence in $\mathrm{CL}(X)$. Suppose that there exist $A \in \mathrm{CL}(X)$ and $M \subseteq \mathbb{N}$ such that $\rho(x, A_k) \to \rho(x, A)$ $(k \to \infty,\ k \in M)$ for every $x \in X$, and*

$$\liminf_{n\to\infty} \frac{1}{W_n} \sum_{k \in I_n \cap M} w_k > 0.$$

Then (A_k) is window-weighted Wijsman statistically convergent to A.

Proof. Ordinary Wijsman convergence of $(A_k)_{k\in M}$ to A means that, for every $x \in X$,

$$\rho(x, A_k) \to \rho(x, A) \quad (k \to \infty,\ k \in M).$$

Equivalently,

$$d_j(x, A_k) \to d_j(x, A) \quad (k \to \infty,\ k \in M,\ j = 1,2).$$

Therefore $\Lambda_k(x;A) \to 0$ $(k \to \infty,\ k \in M)$ for every $x \in X$. The conclusion now follows directly from Theorem 4. □

5. Stability and subsequences of full window density

We now study the stability of window-weighted Wijsman convergence under small perturbations of the sequence. The guiding principle is that changes which are negligible with respect to the chosen weighted windows should not affect either statistical convergence or the pre-Cauchy property. We also prove that ordinary Wijsman convergence along a trace $M \subseteq \mathbb{N}$ of window-weighted density one is sufficient to determine the statistical convergence of the whole sequence.

Let $\mathcal{A} = (A_k)$ and $\mathcal{B} = (B_k)$ be two sequences in $\mathrm{CL}(X)$. For $x \in X$, define

$$\Gamma_k^{\mathcal{A},\mathcal{B}}(x) = |d_1(x, A_k) - d_1(x, B_k)| + |d_2(x, A_k) - d_2(x, B_k)|.$$

This quantity measures the two-component pointwise distance between the distance profiles of A_k and B_k at the point x.

Definition 10. Two sequences $\mathcal{A} = (A_k)$ and $\mathcal{B} = (B_k)$ in $\mathrm{CL}(X)$ are said to be *window-weighted Wijsman equivalent* if, for every $x \in X$ and every $\varepsilon > 0$,

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Gamma_k^{\mathcal{A},\mathcal{B}}(x) \geq \varepsilon}} w_k \to 0.$$

In this case we write $\mathcal{A} \sim_{ww} \mathcal{B}$.

The definition is symmetric in (A_k) and (B_k). It says that the two sequences have the same distance profiles outside a set of indices whose weighted size inside the windows tends to zero.

Theorem 5. *Let $\mathcal{A} = (A_k)$ and $\mathcal{B} = (B_k)$ be sequences in $\mathrm{CL}(X)$, and let $C \in \mathrm{CL}(X)$. Assume that $\mathcal{A} \sim_{ww} \mathcal{B}$. If $\mathcal{A}$ is window-weighted Wijsman statistically convergent to C, then $\mathcal{B}$ is window-weighted Wijsman statistically convergent to C.*

Proof. Fix $x \in X$ and $\varepsilon > 0$. Define

$$\Lambda_k^A(x; C) = |d_1(x, A_k) - d_1(x, C)| + |d_2(x, A_k) - d_2(x, C)|,$$

and

$$\Lambda_k^B(x; C) = |d_1(x, B_k) - d_1(x, C)| + |d_2(x, B_k) - d_2(x, C)|.$$

For each $k \in \mathbb{N}$, the triangle inequality in $\mathbb{R}$ gives

$$|d_j(x, B_k) - d_j(x, C)| \leq |d_j(x, B_k) - d_j(x, A_k)| + |d_j(x, A_k) - d_j(x, C)|,$$

for $j = 1, 2$. Adding the two inequalities, we obtain

$$\Lambda_k^B(x; C) \leq \Gamma_k^{\mathcal{A},\mathcal{B}}(x) + \Lambda_k^A(x; C).$$

Now suppose that $\Lambda_k^B(x; C) \geq \varepsilon$. If also $\Gamma_k^{\mathcal{A},\mathcal{B}}(x) < \frac{\varepsilon}{2}$, then the preceding inequality implies $\Lambda_k^A(x; C) \geq \frac{\varepsilon}{2}$. Hence, for every n,

$$\{k \in I_n : \Lambda_k^B(x; C) \geq \varepsilon\} \subseteq \left\{k \in I_n : \Gamma_k^{\mathcal{A},\mathcal{B}}(x) \geq \frac{\varepsilon}{2}\right\} \cup \left\{k \in I_n : \Lambda_k^A(x; C) \geq \frac{\varepsilon}{2}\right\}.$$

Therefore,

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k^B(x;C) \geq \varepsilon}} w_k \leq \frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Gamma_k^{\mathcal{A},\mathcal{B}}(x) \geq \varepsilon/2}} w_k + \frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k^A(x;C) \geq \varepsilon/2}} w_k.$$

The first term tends to 0 by $\mathcal{A} \sim_{ww} \mathcal{B}$, and the second tends to 0 because $\mathcal{A}$ is window-weighted Wijsman statistically convergent to C. Hence

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k^B(x;C) \geq \varepsilon}} w_k \to 0.$$

Since $x \in X$ and $\varepsilon > 0$ were arbitrary, $\mathcal{B}$ is window-weighted Wijsman statistically convergent to C. $\square$

The same perturbation principle holds for the pre-Cauchy mode. In this case both coordinates of a pair (k, ℓ) must be controlled, which explains the appearance of $\varepsilon/4$ in the proof.

Theorem 6. *Let $\mathcal{A} = (A_k)$ and $\mathcal{B} = (B_k)$ be sequences in $\mathrm{CL}(X)$. Assume that $\mathcal{A} \sim_{ww} \mathcal{B}$. If $\mathcal{A}$ is window-weighted Wijsman pre-Cauchy, then $\mathcal{B}$ is window-weighted Wijsman pre-Cauchy.*

Proof. Fix $x \in X$ and $\varepsilon > 0$. For $k, \ell \in \mathbb{N}$, set

$$\Lambda^{A}_{k,\ell}(x) = |d_1(x, A_k) - d_1(x, A_\ell)| + |d_2(x, A_k) - d_2(x, A_\ell)|,$$

and

$$\Lambda^{B}_{k,\ell}(x) = |d_1(x, B_k) - d_1(x, B_\ell)| + |d_2(x, B_k) - d_2(x, B_\ell)|.$$

Define

$$E_n(x, \varepsilon) = \left\{ k \in I_n : \Gamma^{\mathcal{A},\mathcal{B}}_k(x) \geq \frac{\varepsilon}{4} \right\}.$$

For $j = 1, 2$, the triangle inequality yields

$$|d_j(x, B_k) - d_j(x, B_\ell)| \leq |d_j(x, B_k) - d_j(x, A_k)| + |d_j(x, A_k) - d_j(x, A_\ell)| + |d_j(x, A_\ell) - d_j(x, B_\ell)|.$$

Adding these two inequalities gives

$$\Lambda^{B}_{k,\ell}(x) \leq \Gamma^{\mathcal{A},\mathcal{B}}_k(x) + \Lambda^{A}_{k,\ell}(x) + \Gamma^{\mathcal{A},\mathcal{B}}_\ell(x).$$

Suppose that $\Lambda^{B}_{k,\ell}(x) \geq \varepsilon$ and that $k, \ell \notin E_n(x, \varepsilon)$. Then

$$\Gamma^{\mathcal{A},\mathcal{B}}_k(x) < \frac{\varepsilon}{4}, \qquad \Gamma^{\mathcal{A},\mathcal{B}}_\ell(x) < \frac{\varepsilon}{4}.$$

The preceding inequality then implies $\Lambda^{A}_{k,\ell}(x) \geq \frac{\varepsilon}{2}$. Consequently,

$$\{(k, \ell) \in I_n \times I_n : \Lambda^{B}_{k,\ell}(x) \geq \varepsilon\} \subseteq \{(k, \ell) \in I_n \times I_n : \Lambda^{A}_{k,\ell}(x) \geq \varepsilon/2\} \cup (E_n(x, \varepsilon) \times I_n) \cup (I_n \times E_n(x, \varepsilon)).$$

Multiplying by $w_k w_\ell$, summing, and dividing by W_n^2, we obtain

$$\begin{aligned} \frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda^{B}_{k,\ell}(x) \geq \varepsilon}} w_k w_\ell &\leq \frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda^{A}_{k,\ell}(x) \geq \varepsilon/2}} w_k w_\ell + \frac{1}{W_n^2} \sum_{k \in E_n(x,\varepsilon)} \sum_{\ell \in I_n} w_k w_\ell + \frac{1}{W_n^2} \sum_{k \in I_n} \sum_{\ell \in E_n(x,\varepsilon)} w_k w_\ell \\ &= \frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda^{A}_{k,\ell}(x) \geq \varepsilon/2}} w_k w_\ell + \frac{2}{W_n} \sum_{k \in E_n(x,\varepsilon)} w_k. \end{aligned}$$

The first term tends to 0 because $\mathcal{A}$ is window-weighted Wijsman pre-Cauchy. The second term tends to 0 by the equivalence $\mathcal{A} \sim_{ww} \mathcal{B}$. Hence

$$\frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda^{B}_{k,\ell}(x) \geq \varepsilon}} w_k w_\ell \to 0.$$

Since $x \in X$ and $\varepsilon > 0$ were arbitrary, $\mathcal{B}$ is window-weighted Wijsman pre-Cauchy. $\square$

Finite changes are a basic example of window-weighted Wijsman equivalence.

Corollary 2. *Let $\mathcal{A} = (A_k)$ and $\mathcal{B} = (B_k)$ be sequences in $\mathrm{CL}(X)$. Suppose that there exists a finite set $F \subseteq \mathbb{N}$ such that $A_k = B_k$ whenever $k \notin F$. Then $\mathcal{A} \sim_{ww} \mathcal{B}$. Consequently, for every $C \in \mathrm{CL}(X)$, $\mathcal{A}$ is window-weighted Wijsman statistically convergent to C if and only if $\mathcal{B}$ is window-weighted Wijsman statistically convergent to C. Moreover, $\mathcal{A}$ is window-weighted Wijsman pre-Cauchy if and only if $\mathcal{B}$ is window-weighted Wijsman pre-Cauchy.*

Proof. Fix $x \in X$ and $\varepsilon > 0$. If $k \notin F$, then $A_k = B_k$, and hence $\Gamma_k^{\mathcal{A},\mathcal{B}}(x) = 0$. Therefore,

$$\{k \in \mathbb{N} : \Gamma_k^{\mathcal{A},\mathcal{B}}(x) \geq \varepsilon\} \subseteq F.$$

Since F is finite, Lemma 2 gives

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Gamma_k^{\mathcal{A},\mathcal{B}}(x) \geq \varepsilon}} w_k \to 0.$$

Thus $\mathcal{A} \sim_{ww} \mathcal{B}$.

The statistical part follows from Theorem 5, first in the direction from $\mathcal{A}$ to $\mathcal{B}$ and then in the reverse direction, since $\mathcal{B} \sim_{ww} \mathcal{A}$ as well. The pre-Cauchy part follows in the same way from Theorem 6. □

We next prove the full-density subsequence principle. It states that if ordinary Wijsman convergence holds along a subset which occupies asymptotically all of the weighted mass inside the windows, then the full sequence is window-weighted Wijsman statistically convergent.

Theorem 7. *Let (A_k) be a sequence in $\mathrm{CL}(X)$, and let $C \in \mathrm{CL}(X)$. Suppose that there exists $M \subseteq \mathbb{N}$ such that $\delta_{\mathcal{W}}^{w}(M) = 1$ and*

$$\rho(x, A_k) \to \rho(x, C) \quad (k \to \infty,\ k \in M),$$

for every $x \in X$. Then (A_k) is window-weighted Wijsman statistically convergent to C.

Proof. Fix $x \in X$ and $\varepsilon > 0$. The convergence $\rho(x, A_k) \to \rho(x, C)$ $(k \to \infty,\ k \in M)$ means, componentwise, that

$$d_j(x, A_k) \to d_j(x, C) \quad (k \to \infty,\ k \in M,\ j = 1, 2).$$

Hence there exists $N_0 \in \mathbb{N}$ such that

$$|d_j(x, A_k) - d_j(x, C)| < \frac{\varepsilon}{2},$$

for every $k \in M$ with $k \geq N_0$ and for $j = 1, 2$. Let $F = \{1, 2, \ldots, N_0 - 1\}$. Then, for every $k \in M \setminus F$,

$$\Lambda_k(x; C) = |d_1(x, A_k) - d_1(x, C)| + |d_2(x, A_k) - d_2(x, C)| < \varepsilon.$$

Therefore,

$$\{k \in I_n : \Lambda_k(x; C) \geq \varepsilon\} \subseteq (I_n \setminus M) \cup (I_n \cap F).$$

It follows that

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k(x;C) \geq \varepsilon}} w_k \leq \frac{1}{W_n} \sum_{k \in I_n \setminus M} w_k + \frac{1}{W_n} \sum_{k \in I_n \cap F} w_k.$$

For the first term, we use

$$\frac{1}{W_n} \sum_{k \in I_n \setminus M} w_k = 1 - \frac{1}{W_n} \sum_{k \in I_n \cap M} w_k \to 0.$$

For the second term, Lemma 2 gives

$$\frac{1}{W_n} \sum_{k \in I_n \cap F} w_k \to 0.$$

Thus

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k(x;C) \geq \varepsilon}} w_k \to 0.$$

Since $x \in X$ and $\varepsilon > 0$ were arbitrary, (A_k) is window-weighted Wijsman statistically convergent to C. □

The following formulation records the same result in terms of subsequences of window-weighted density one.

Corollary 3. *Let (A_k) be a sequence in $\mathrm{CL}(X)$, and let $C \in \mathrm{CL}(X)$. Suppose that there exists $M \subseteq \mathbb{N}$ such that $\delta^{w}_{\mathcal{W}}(M) = 1$ and*

$$\rho(x, A_k) \to \rho(x, C) \qquad (k \to \infty,\ k \in M),$$

for every $x \in X$. Then (A_k) is window-weighted Wijsman statistically convergent to C.

Proof. The assumption that M has window-weighted density one means that

$$\frac{1}{W_n} \sum_{k \in I_n \cap M} w_k \to 1.$$

Ordinary Wijsman convergence of the trace $(A_k)_{k \in M}$ to C means that

$$\rho(x, A_k) \to \rho(x, C) \qquad (k \to \infty,\ k \in M),$$

for every $x \in X$. The conclusion follows from Theorem 7. □

6. Examples and sharpness

We conclude with examples which illustrate the sharpness and the structural role of the preceding results. The examples are chosen to show five different phenomena. First, non-equivalent component metrics may affect the admissible hyperspace through the simultaneous closedness condition. Second, the same sequence of closed sets may have different window-weighted Wijsman statistical limits under two different window schemes. Third, the pre-Cauchy condition is strictly weaker than statistical convergence. Fourth, in the pre-Cauchy upgrade theorem, the assumption that the convergent trace has positive lower window weight cannot be replaced by the mere existence of an infinite convergent trace. Fifth, a subset of window-weighted density one may recover statistical convergence even when ordinary Wijsman convergence fails because of unbounded moving perturbations.

We first give an example in which the two component metrics are not topologically equivalent. In the remaining examples, we use the common setting introduced after Example 2.

Example 2 (Effect of non-equivalent component metrics)**.** Let $X = \mathbb{R}$. Define $d_1(x, y) = |x - y|$ and let d_2 be the discrete metric,

$$d_2(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Then d_1 and d_2 generate different topologies. Every subset of X is closed with respect to d_2, whereas the d_1-closed sets are precisely the usual closed subsets of $\mathbb{R}$. Therefore the hyperspace $\mathrm{CL}(X)$ consists of the non-empty usual closed subsets of $\mathbb{R}$.

Let $A = [0, 1]$ and $B = (0, 1)$. Then B is closed in (X, d_2) but not in (X, d_1), so $B \notin \mathrm{CL}(X)$, while $A \in \mathrm{CL}(X)$. This shows that admissibility in the hyperspace is not determined by one component alone.

Furthermore, for $x \notin A$, the two component distance profiles behave differently:

$$d_1(x, A) = \inf_{a \in A} |x - a|,$$

whereas $d_2(x, A) = 1$. Thus the Wijsman profile associated with $\rho = d_1 e_1 + d_2 e_2$ contains both the usual metric distance information and the discrete component information. This illustrates why the two component metrics must be tracked simultaneously in the idempotent bicomplex formulation.

In all remaining examples, let

$$X = \mathbb{R}, \quad d_1(x, y) = |x - y|, \quad d_2(x, y) = \min\{|x - y|, 1\}.$$

Then d_1 and d_2 are ordinary metrics on X, and

$$\rho(x,y) = d_1(x,y)e_1 + d_2(x,y)e_2,$$

defines an idempotent bicomplex metric. Since d_1 and d_2 induce the same usual topology on $\mathbb{R}$, every non-empty usual closed interval used below belongs to $\mathrm{CL}(X)$.

Example 3 (The same sequence has different limits under different windows)**.** Let $C_0 = [0,1]$ and $C_1 = [2,3]$. For each $n \in \mathbb{N}$, define two blocks

$$L_n = (n^2, n^2+n] \cap \mathbb{N}, \quad R_n = (n^2+n, (n+1)^2] \cap \mathbb{N}.$$

The lengths of these blocks satisfy

$$|L_n| = n, \quad |R_n| = n+1,$$

and hence both block families are admissible window schemes. Define a sequence (A_k) in $\mathrm{CL}(X)$ by setting $A_1 = C_0$ and, for $k \geq 2$,

$$A_k = \begin{cases} C_1, & k \in L_n \text{ for some } n \in \mathbb{N}, \\ C_0, & k \in R_n \text{ for some } n \in \mathbb{N}. \end{cases}$$

We take $w_k \equiv 1$.

First use the window scheme $\mathcal{L} = \{L_n\}$. If $k \in L_n$, then $A_k = C_1$. Therefore, for every $x \in \mathbb{R}$,

$$d_j(x, A_k) = d_j(x, C_1) \quad (j = 1, 2,\ k \in L_n).$$

Consequently, for every $\varepsilon > 0$,

$$\frac{1}{|L_n|} |\{k \in L_n : \Lambda_k(x; C_1) \geq \varepsilon\}| = 0 \quad (n \in \mathbb{N}).$$

Thus (A_k) is window-weighted Wijsman statistically convergent to C_1 with respect to $\mathcal{L}$.

Now use the window scheme $\mathcal{R} = \{R_n\}$. If $k \in R_n$, then $A_k = C_0$. Hence, for every $x \in \mathbb{R}$,

$$d_j(x, A_k) = d_j(x, C_0) \quad (j = 1, 2,\ k \in R_n),$$

and the corresponding exceptional averages for convergence to C_0 are identically zero. Therefore (A_k) is window-weighted Wijsman statistically convergent to C_0 with respect to $\mathcal{R}$.

The two limits C_0 and C_1 are distinct. Moreover, the sequence is not ordinarily Wijsman convergent. Indeed, at $x = 0$,

$$d_1(0, C_0) = 0, \quad d_1(0, C_1) = 2.$$

Since $A_k = C_0$ for infinitely many k and $A_k = C_1$ for infinitely many k, the scalar sequence $d_1(0, A_k)$ has subsequences with values 0 and 2. Hence it is not convergent. This example shows that the choice of window scheme may determine not only whether convergence occurs, but also which Wijsman statistical limit is obtained.

Example 4 (Pre-Cauchy behaviour without statistical convergence)**.** Let $C_0 = [0,1]$ and $C_1 = [2,3]$, and consider the windows

$$I_n = (n^2, (n+1)^2] \cap \mathbb{N}, \quad w_k \equiv 1.$$

Then $W_n = |I_n| = (n+1)^2 - n^2 = 2n+1 \to \infty$. Define a sequence (A_k) in $\mathrm{CL}(X)$ as follows. Set $A_1 = C_0$. For $k \geq 2$, define

$$A_k = \begin{cases} C_0, & k \in I_n \text{ for some even } n, \\ C_1, & k \in I_n \text{ for some odd } n. \end{cases}$$

The value assigned to the finite initial index $k = 1$ is immaterial for the window-weighted conclusions, by Lemma 2. If $k, \ell \in I_n$, then $A_k = A_\ell$. Therefore $\Lambda_{k,\ell}(x) = 0$ $(x \in X,\ k, \ell \in I_n)$, and hence

$$\frac{1}{W_n^2} \sum_{\substack{k,\ell \in I_n \\ \Lambda_{k,\ell}(x) \geq \varepsilon}} 1 = 0,$$

for every $x \in X$, every $\varepsilon > 0$, and every n. Thus (A_k) is window-weighted Wijsman pre-Cauchy.

We show that (A_k) is not window-weighted Wijsman statistically convergent to any member of $\mathrm{CL}(X)$. Let $H \in \mathrm{CL}(X)$ be arbitrary and put $c = d_1(0, H)$. Since $d_1(0, C_0) = 0$ and $d_1(0, C_1) = 2$, we have $\max\{|c|, |2 - c|\} \geq 1$. If $|c| \geq 1$, then on every even window I_n all terms satisfy $A_k = C_0$, and hence

$$\Lambda_k(0; H) \geq |d_1(0, A_k) - d_1(0, H)| = |0 - c| \geq 1.$$

Therefore, for every even n,

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k(0;H) \geq 1/2}} 1 = 1.$$

If $|2 - c| \geq 1$, then the same argument on every odd window gives

$$\frac{1}{W_n} \sum_{\substack{k \in I_n \\ \Lambda_k(0;H) \geq 1/2}} 1 = 1,$$

for every odd n. In either case, the exceptional ratios fail to tend to zero. Therefore (A_k) is not window-weighted Wijsman statistically convergent to H. Since $H \in \mathrm{CL}(X)$ was arbitrary, no Wijsman statistical limit exists. This proves that the window-weighted Wijsman pre-Cauchy condition is strictly weaker than window-weighted Wijsman statistical convergence.

Example 5 (An infinite convergent trace of zero window weight is not enough)**.** This example shows that the positive lower window weight assumption in Theorem 4 cannot be replaced by the mere existence of an infinite ordinary Wijsman convergent trace.

Use the sequence (A_k) and the windows (I_n) from Example 4. Thus (A_k) is window-weighted Wijsman pre-Cauchy, but it is not window-weighted Wijsman statistically convergent to any element of $\mathrm{CL}(X)$.

Let $M = \{n^2 + 1 : n \in \mathbb{N},\ n \text{ is even}\}$. For every $k \in M$, one has $A_k = C_0$. Hence the trace $(A_k)_{k \in M}$ is constant and therefore ordinarily Wijsman convergent to C_0.

However, this trace has zero lower window weight. Indeed, for an even n, the set $I_n \cap M$ contains exactly one point, while for an odd n it is empty. Since $W_n = 2n + 1$, we have

$$\frac{1}{W_n} \sum_{k \in I_n \cap M} 1 = \begin{cases} \dfrac{1}{2n+1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Thus

$$\liminf_{n \to \infty} \frac{1}{W_n} \sum_{k \in I_n \cap M} 1 = 0.$$

Although (A_k) is pre-Cauchy and has an infinite trace ordinarily Wijsman convergent to C_0, the full sequence is not statistically convergent. Therefore the positive lower window weight condition in Theorem 4 is essential.

Example 6 (A full-density subsequence with unbounded perturbations)**.** Let $C = [0, 1]$. For each $n \in \mathbb{N}$, define the window

$$I_n = (2^{2n}, 2^{2n} + n^3] \cap \mathbb{N}, \quad w_k \equiv 1, \quad W_n = n^3.$$

Inside I_n, let the exceptional block be $E_n = (2^{2n}, 2^{2n} + n] \cap \mathbb{N}$. The use of 2^{2n} makes the windows eventually disjoint and avoids any possible overlap ambiguity. Now define a sequence (A_k) in $\mathrm{CL}(X)$ by

$$A_k = \begin{cases} [n, n+1], & k \in E_n \text{ for some } n \in \mathbb{N}, \\ C, & \text{otherwise.} \end{cases}$$

The exceptional sets are not fixed perturbations: their values move farther away from C as $n \to \infty$.
Let

$$M = \mathbb{N} \setminus \bigcup_{n=1}^{\infty} E_n.$$

The windows I_n are pairwise disjoint for all sufficiently large n. Moreover, $|I_n| = n^3$ and $|E_n| = n$. Therefore, inside the n-th window,

$$\frac{1}{W_n} \sum_{k \in I_n \cap M} 1 = \frac{|I_n| - |E_n|}{|I_n|} = \frac{n^3 - n}{n^3} = 1 - \frac{1}{n^2} \to 1.$$

Thus M has window-weighted density one. Along M, the sequence is constant: $A_k = C$ $(k \in M)$. Hence the trace $(A_k)_{k \in M}$ is ordinarily Wijsman convergent to C. By Theorem 7, the full sequence (A_k) is window-weighted Wijsman statistically convergent to C.

On the other hand, ordinary Wijsman convergence of the full sequence fails in a strong way. Choose $k_n \in E_n$. Then $A_{k_n} = [n, n+1]$. At $x = 0$, we have $d_1(0, A_{k_n}) = n$ and $d_1(0, C) = 0$. Therefore

$$d_1(0, A_{k_n}) - d_1(0, C) = n \to \infty.$$

Thus the scalar distance profile $d_1(0, A_k)$ is not convergent to $d_1(0, C)$. This shows that convergence on a subset of window-weighted density one can yield window-weighted Wijsman statistical convergence even when ordinary Wijsman convergence is destroyed by unbounded moving perturbations.

7. Conclusion

We have studied window-weighted Wijsman statistical convergence for sequences of closed sets in hyperspaces associated with idempotent bicomplex metric spaces. The metric structure considered here is the componentwise idempotent one, $\rho = d_1 e_1 + d_2 e_2$, and the resulting convergence theory is governed by the two scalar point-to-set distance profiles $d_1(x, A_k)$ and $d_2(x, A_k)$. Within this framework, we obtained componentwise characterizations of window-weighted Wijsman statistical convergence and of the corresponding pre-Cauchy condition. We also proved that weighted-equivalent window schemes generate the same statistical Wijsman convergence.

The main recovery result shows that window-weighted Wijsman pre-Cauchy behaviour can be upgraded to statistical convergence when the sequence has an ordinarily Wijsman convergent trace with positive lower window weight. This identifies the role of the trace-density assumption: it supplies enough weighted mass inside the moving windows to determine the statistical limit. Stability under negligible perturbations and the full-density trace theorem further show that the convergence mode is robust with respect to changes on sets of vanishing window weight.

The examples demonstrate that the window scheme is part of the asymptotic structure, that the pre-Cauchy condition is strictly weaker than statistical convergence without an additional trace hypothesis, and that simultaneous closedness with respect to two component metrics can affect the admissible hyperspace. Thus the paper provides a specialized and coherent contribution to weighted-window Wijsman convergence in the idempotent bicomplex setting, without claiming a general theory for arbitrary bicomplex-valued metrics.

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