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Existence of solutions for implicit fractional differential inclusions involving Hilfer-Katugampola fractional derivatives

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Abstract: This paper investigates the existence of solutions for initial value problems (IVPs) involving implicit fractional differential inclusions defined via the Hilfer-Katugampola fractional derivative. The Hilfer-Katugampola operator, recently introduced as a generalization of Katugampola and Caputo-Katugampola derivatives, encompasses a wide class of fractional operators. We establish existence results for the multivalued fractional differential problem under convexity and compactness assumptions on the multivalued right-hand side, leveraging Bohnenblust-Karlin fixed point theorem and contraction principles for multivalued maps. An illustrative example is provided to demonstrate the applicability of the main theoretical results. Our work contributes to the emerging theory of fractional differential inclusions governed by fractional derivatives of generalized type.

Keywords: Hilfer-Katugampola fractional derivative, fractional differential inclusions, implicit fractional differential equations, multivalued maps, fixed point theorem

MSC: 26A33, 34A08.

1. Introduction

The theory of fractional calculus has recently undergone significant growth with the development of new fractional differential operators that generalize classical derivatives to better capture memory and hereditary properties of complex systems [1–7]. Among these operators, the Hilfer-Katugampola fractional derivative, introduced by Oliveira et al. [8], provides a unifying generalization that incorporates parameters controlling both the order and the type of the fractional derivative. This operator extends the Katugampola and Caputo-Katugampola derivatives, enabling a more flexible modeling framework adapted to various applications.

Motivated by these advances, this paper investigates the existence of solutions for initial value problems involving implicit fractional differential inclusions formulated with the Hilfer-Katugampola fractional derivative. The considered problem is of the form

$$ {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\iota) \in \aleph(\iota, z(\iota), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\iota)), \ \ \iota \in \mathcal{J} := [a,b], \tag{1} $$

$$ ({}^{\rho}\mathcal{I}_{a^+}^{1-\ell} z)(a) = \theta, \ \ \theta \in \mathbb{R}, \ \ell = \ell_1 + \ell_2(1-\ell_1), \tag{2} $$

where $\ell_1 \in (0,1)$, $\ell_2 \in [0,1]$, and ρ a positive real number. Let $\aleph$ be a set-valued mapping defined from $\mathcal{J} \times \mathbb{R} \times \mathbb{R}$ into $\mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ denotes the collection of all nonempty subsets of the real numbers $\mathbb{R}$, ${}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2}$ is the Hilfer Katugampola fractional derivative of order ℓ_1 and type ℓ_2 and ${}^{\rho}\mathcal{I}_{a^+}^{\ell_1}$, ${}^{\rho}\mathcal{I}_{a^+}^{1-\ell}$ are Katugampola fractional integral of order ℓ_1 and $1-\ell$, respectively with $a > 0$.

The multivalued mapping $\aleph$ takes values in convex and compact subsets of $\mathbb{R}$, allowing the formulation of fractional differential inclusions that can describe systems with uncertainty or multistate behaviors. Utilizing fixed point theorems for multivalued maps, specifically the Bohnenblust-Karlin theorem and contraction

principles, we establish existence results for solutions under natural assumptions such as upper semicontinuity and boundedness of $\aleph$.

The organization of this manuscript is outlined as follows. In §2, essential definitions and foundational concepts related to fractional calculus and multifunctions are reviewed. §3 contains the principal existence theorems concerning the initial value problem. An illustrative example demonstrating the practical use of these findings is provided in §4. This study adds to the expanding body of research on fractional differential inclusions and emphasizes the relevance of generalized fractional derivatives for modeling complex phenomena.

2. Preliminaries

In this part, we establish the notations, fundamental definitions, and essential preliminary results that will be utilized consistently throughout the rest of this work.
Define $C(\mathcal{J})$ as the set of all continuous functions $u : \mathcal{J} \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max_{\iota \in \mathcal{J}} |u(\iota)|.$$

The notation $\mathcal{L}^1(\mathcal{J})$ refers to the collection of Lebesgue integrable functions $u : \mathcal{J} \to \mathbb{R}$ with the associated norm

$$\|u\|_{\mathcal{L}_1} = \int_a^b |u(\iota)| d\iota.$$

Consider the weighted space,

$$C_{1-\ell,\rho}(\mathcal{J}) = \{w : \mathcal{J}^* := (a, b] \to \mathbb{R} : \left(\frac{\iota^\rho - a^\rho}{\rho}\right)^{1-\ell} w(\iota) \in C(\mathcal{J})\},\ 0 < \ell \leq 1,$$

obviously, $C_{1-\ell,\rho}(\mathcal{J})$ is Banach space equipped with the norm

$$\|w\|_{C_{1-\ell,\rho}} = \left\| \left(\frac{\iota^\rho - a^\rho}{\rho}\right)^{1-\ell} w(\iota) \right\|_C = \max_{\iota \in \mathcal{J}} \left| \left(\frac{\iota^\rho - a^\rho}{\rho}\right)^{1-\ell} w(\iota) \right|.$$

We proceed by presenting several fundamental results and characteristics related to fractional calculus.

Definition 1. [9] For $\iota > a$, the Katugampola fractional integral of left-sided type and order $\ell_1 \in \mathbb{C}$ with $(Re(\ell_1) > 0)$ is defined as follows:

$$^\rho\mathcal{I}_{a^+}^{\ell_1} h(\iota) = \frac{\rho^{1-\ell_1}}{\Gamma(\ell_1)} \int_a^\iota (\iota^\rho - s^\rho)^{\ell_1 - 1} s^{\rho-1} h(s) ds, \tag{3}$$

if the integral exists, where $\Gamma(.)$ is the Gamma function.

Definition 2. [9] The Katugampola left-sided fractional derivative, which is associated with the Katugampola fractional integral given by Eq. (3), is defined for $\iota > a$ as follows:

$$^\rho\mathcal{D}_{a^+}^{\ell_1} h(\iota) = \frac{\rho^{\ell_1 - n + 1}}{\Gamma(n - \ell_1)} \left(\iota^{1-\rho} \frac{d}{d\iota}\right) \int_a^\iota (\iota^\rho - s^\rho)^{n - \ell_1 - 1} s^{\rho-1} h(s) ds,$$

where $n = [\ell_1] + 1$, if the integral exists.

Lemma 1. [8] *For $\iota > a$, then the following hold:*

$$\left(^\rho\mathcal{I}_{a^+}^{\ell_1} \left(\frac{s^\rho - a^\rho}{\rho}\right)^\zeta\right)(\iota) = \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta + \ell_1 + 1)} \left(\frac{\iota^\rho - a^\rho}{\rho}\right)^{\zeta + \ell_1}, \ell_1 \geq 0,\ \zeta > 0.$$

Proposition 1. [8] *Let $\ell_1,\ \ell_2 > 0$, $\rho \in \mathbb{R}$, then for $h \in C_{1-\ell,\rho}(\mathcal{J})$ and $\iota \in \mathcal{J}^*$, we have*

(i) ${}^{\rho}J^{*\ell_1}_{a^+}\,{}^{\rho}\mathcal{I}^{\ell_2}_{a^+}h = {}^{\rho}\mathcal{I}^{\ell_2}_{a^+}\,{}^{\rho}\mathcal{I}^{\ell_1}_{a^+}h(\imath) = {}^{\rho}\mathcal{I}^{\ell_1+\ell_2}_{a^+}h.$
(ii) ${}^{\rho}\mathcal{D}^{\ell_1}_{a^+}\,{}^{\rho}\mathcal{I}^{\ell_1}_{a^+}h = h.$

In particular, if $h \in C(\mathcal{J})$ then these equalities hold at $\imath \in \mathcal{J}$.

Theorem 1. *(Linearity property). Let $0 < \ell_1 < 1$, then for $h_1, h_2 \in C_{1-\ell,\rho}(\mathcal{J})$*

(i) ${}^{\rho}\mathcal{I}^{\ell_1}_{a^+}(h_1+h_2) = {}^{\rho}\mathcal{I}^{\ell_1}_{a^+}h_1 + {}^{\rho}\mathcal{I}^{\ell_1}_{a^+}h_2.$
(ii) ${}^{\rho}\mathcal{D}^{\ell_1}_{a^+}(h_1+h_2) = {}^{\rho}\mathcal{D}^{\ell_1}_{a^+}h_1 + {}^{\rho}\mathcal{D}^{\ell_1}_{a^+}h_2.$

Lemma 2. [8] *Let $0 < \ell_1 < 1$, $0 \le \ell < 1$. Then ${}^{\rho}\mathcal{I}^{\ell_1}_{a^+}$ is bounded from $C_{1-\ell,\rho}(\mathcal{J})$, into $C_{1-\ell,\rho}(\mathcal{J})$.*

Definition 3. [8] The Hilfer-Katugampola derivative of fractional order ℓ_1 for a function $h \in C_{1-\ell,\rho}$, is given by

$$ {}^{\rho}\mathcal{D}^{\ell_1,\ell_2}_{a^+}h(\imath) := ({}^{\rho}\mathcal{I}^{\ell_2(1-\ell_1)}_{a^+}\delta^{\rho}_{\rho}\mathcal{I}^{(1-\ell_1)(1-\ell_2)}_{a^+}h)(\imath), $$

where $0 < \ell_1 < 1$, $0 \le \ell_2 \le 1$, $\ell = \ell_1 + \ell_2 - \ell_1\ell_2$.

Remark 1. [8] The Hilfer Katugampola operator ${}^{\rho}D^{\ell_1,\ell_2}_{a^+}$ can be written in terme of Katugampola functional derivative as

$$ {}^{\rho}\mathcal{D}^{\ell_1,\ell_2}_{a^+} = {}^{\rho}\mathcal{I}^{\ell_2(1-\ell_1)}_{a^+}\delta_{\rho}\,{}^{\rho}\mathcal{I}^{1-\ell}_{a^+} = {}^{\rho}\mathcal{I}^{\ell_2(1-\ell_1)}_{a^+}{}^{\rho}\mathcal{D}^{\ell}_{a^+}, \quad \ell = \ell_1+\ell_2-\ell_1\ell_2. $$

Lemma 3. [8] *Let $0 < \ell_1 < 1,\ 0 \le \ell_2 \le 1,\ \ell = \ell_1+\ell_2-\ell_1\ell_2$, if $h \in C^{\ell}_{1-\ell,\rho}(\mathcal{J})$, where*

$$ C^{\ell}_{1-\ell,\rho}(\mathcal{J}) := \{h \in C_{1-\ell,\rho}(\mathcal{J}), {}^{\rho}\mathcal{D}^{\ell}_{a^+}h \in C_{1-\ell,\rho}(\mathcal{J})\}. $$

Then

(i) ${}^{\rho}\mathcal{I}^{\ell_1}_{a^+}\,{}^{\rho}\mathcal{D}^{\ell_1,\ell_2}_{a^+}h = {}^{\rho}\mathcal{I}^{\ell}_{a^+}\,{}^{\rho}D^{\ell}_{a^+}h, .$
(ii) ${}^{\rho}\mathcal{D}^{\ell}_{a^+}\,{}^{\rho}\mathcal{I}^{\ell_1}_{a^+}h = {}^{\rho}D^{\ell_2(1-\ell_1)}_{a^+}h, .$
(iii) ${}^{\rho}\mathcal{D}^{\ell_1,\ell_2}_{a^+}\,{}^{\rho}\mathcal{I}^{\ell_1}_{a^+}h = {}^{\rho}\mathcal{I}^{\ell_2(1-\ell_1)}_{a^+}\,{}^{\rho}D^{\ell_2(1-\ell_1)}_{a^+}h.$

2.1. Set-valued mappings

Consider X and Y as Banach spaces. We define several families of subsets of X as follows:

$$\begin{aligned}
\mathcal{P}(X) &= \{K \subseteq X : K \neq \emptyset\},\\
\mathcal{P}_{cl}(X) &= \{K \in \mathcal{P}(X) : K \text{ is closed}\},\\
\mathcal{P}_{b}(X) &= \{K \in \mathcal{P}(X) : K \text{ is bounded}\},\\
\mathcal{P}_{cv}(X) &= \{K \in \mathcal{P}(X) : K \text{ is convex}\},\\
\mathcal{P}_{cp}(X) &= \{K \in \mathcal{P}(X) : K \text{ is compact}\},\\
\mathcal{P}_{cp,c}(X) &= \{K \in \mathcal{P}(X) : K \text{ is both compact and convex}\}.
\end{aligned}$$

A set-valued map $T : X \to \mathcal{P}(Y)$ is termed compact if the union $\bigcup_{x\in X} T(x)$ forms a compact subset of Y. The images $T(x)$ are said to have properties such as convexity, closedness, or compactness if these hold for each $x \in X$. Additionally, T is bounded on bounded subsets of X provided that for every bounded subset $B \subseteq X$, the set $\bigcup_{x\in B} T(x)$ remains bounded in Y.

The map T is upper semicontinuous (abbreviated u.s.c.) at a point $x_0 \in X$ if $T(x_0)$ is nonempty and closed, and for every open set N containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. Furthermore, when T takes bounded subsets of X into relatively compact subsets of Y, it is said to be completely continuous.

When T is completely continuous and its values are nonempty and compact, the concepts of upper semicontinuity and closedness of the graph coincide. More precisely, if sequences $x_n \to x_*$ in X and $y_n \to y_*$ in Y satisfy $y_n \in T(x_n)$, then necessarily $y_* \in T(x_*)$.

Let $\mathcal{P}_{b,cl,c}(X)$ denote the collection of all nonempty, bounded, closed, and convex subsets of X. A set-valued mapping $T : \mathcal{J} \to \mathcal{P}_{cl}(X)$ is measurable if for each $y \in X$, the function $\iota \mapsto \text{dist}(y, T(\iota))$ is measurable on $\mathcal{J}$.

If $X \subseteq Y$, a fixed point $y \in X$ of T is defined by the condition $y \in T(y)$. The set of all such fixed points is represented by $\text{Fix}(T)$.

The norm on $T(y)$ is given by $\|T(y)\|_{\mathcal{P}} = \sup\{|x| : x \in T(y)\}$. For a map $T : \mathcal{J} \to \mathcal{P}_{cl}(\mathbb{R})$, measurability means the function

$$\iota \mapsto d(x, T(\iota)) = \inf_{y \in T(\iota)} |x - y|,$$

is measurable for each $x \in \mathbb{R}$.

Given $y \in C_{\ell,\rho}(\mathcal{J}, \mathbb{R})$, define the set of measurable selections of $\aleph$ by

$$S_{\aleph,z} = \{g \in \mathcal{L}^1(\mathcal{J}) : g(\iota) \in \aleph(\iota, z(\iota), {}^{\rho}\mathcal{D}^{\ell_1,\ell_2}_{a^+} z(\iota)) \text{ for almost every } \iota \in \mathcal{J}\}.$$

Consider now a metric space (X, d) induced by the norm $|\cdot|$. The Hausdorff metric $\mathcal{H}_d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty]$ is defined by

$$\mathcal{H}_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\},$$

where

$$d(A, b) = \inf_{a \in A} d(a, b), \quad d(a, B) = \inf_{b \in B} d(a, b).$$

With this metric, $(\mathcal{P}_{b,cl}(X), \mathcal{H}_d)$ forms a metric space, whereas $(\mathcal{P}_{cl}(X), \mathcal{H}_d)$ is a generalized metric space (cf. [10]).

Definition 4. Consider a multivalued mapping $\mathcal{N} : X \to \mathcal{P}_{cl}(X)$. We say that

(a) $\mathcal{N}$ is ξ-Lipschitz continuous if there is a positive constant $\xi > 0$ so that for every pair $x, y \in X$, the Hausdorff distance between the images satisfies

$$\mathcal{H}_d(\mathcal{N}(x), \mathcal{N}(y)) \leq \xi d(x, y).$$

(b) Furthermore, $\mathcal{N}$ is termed a contraction if the above condition holds with some ξ strictly less than 1.

The subsequent lemmas serve as foundational tools for the developments that follow.

Lemma 4 (Bohnenblust-Karlin [11])**.** *Let X be a Banach space and $K \subseteq X$ a nonempty, closed, convex set. Consider a multivalued map $\mathcal{N} : K \to \mathcal{P}_{cl,c}(K)$ that is upper semicontinuous, with the property that $\mathcal{N}(K)$ is relatively compact in X. Then, $\mathcal{N}$ admits at least one fixed point in the set K.*

Lemma 5 (Dei)**.** *Assume (X, d) is a complete metric space. Suppose the multivalued operator $\mathcal{N} : X \to \mathcal{P}_{cl}(X)$ fulfills a contraction condition. Under these assumptions, the set of fixed points* $\text{Fix}(\mathcal{N})$ *is guaranteed to be nonempty.*

For an extensive treatment of the theory of multivalued mappings, readers may consult the works of Aubin and Cellina [12], as well as Deimling [13].

3. Main results

To begin, we clarify the notion of a solution for the problem described by (1) − (2).

Definition 5. A function z belonging to the space $C_{1-\ell,\rho}(\mathcal{J})$ is called a solution of the problem (1) − (2) if there exists a function $g \in \mathcal{L}^1(\mathcal{J})$ such that for almost every $\iota \in \mathcal{J}$,

$$g(\iota) \in \aleph\big(\iota, z(\iota), {}^{\rho}\mathcal{D}^{\ell_1,\ell_2}_{a^+} z(\iota)\big),$$

and z satisfies the fractional differential equation

$$({}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z)(\imath) = g(\imath) \quad on \quad \mathcal{J},$$

along with the initial condition

$$({}^{\rho}\mathcal{I}_{a^+}^{1-\ell} z)(a) = \theta.$$

Based on Theorem 3 in [8] (page 11), we can state the following supporting lemma.

Lemma 6 (See [8]). *Let the parameters satisfy $0 < \ell_1 < 1$, $0 \leq \ell_2 \leq 1$, and $\rho > 0$, and define $\ell = \ell_1 + \ell_2(1 - \ell_1)$. Suppose g belongs to the space $C_{1-\ell,\rho}(\mathcal{J})$. A function z solves the fractional integral equation*

$$z(\imath) = \frac{\theta}{\Gamma(\ell)} \left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{\ell-1} + \int_a^{\imath} s^{\rho-1} \left(\frac{\imath^{\rho} - s^{\rho}}{\rho} \right)^{\ell_1-1} \frac{g(s)}{\Gamma(\ell_1)} \, ds,$$

recisely when this condition holds z is a solution to the fractional initial value problem.

$${}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath) = g(\imath), \quad \imath \in \mathcal{J},$$

subject to the initial condition

$$({}^{\rho}\mathcal{I}^{1-\ell} z)(a) = \theta, \quad \theta \in \mathbb{R}, \quad \ell = \ell_1 + \ell_2(1 - \ell_1).$$

Remark 2. By Lemma 6 we deduce easily

$$g(\imath) \in \aleph(\imath, z(\imath), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z) \Longleftrightarrow g(\imath) \in \aleph \left(\imath, \frac{\theta}{\Gamma(\ell)} \left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{\ell-1} + {}^{\rho}\mathcal{I}_{a^+}^{\ell_1} z(\imath), z(\imath) \right) \ a.e. \ \imath \in \mathcal{J}.$$

Our focus shifts to establishing the existence of solutions for problem $(1) - (2)$ under the assumption that the multivalued operator on the right-hand side takes values that are convex sets. More specifically, we consider the case where $\aleph$ is a multivalued mapping whose values are both compact and convex.

We begin by stating the assumptions listed below:

(H1) The multivalued mapping $\aleph : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ defined by $(\imath, y, z) \mapsto \aleph(\imath, y, z)$ satisfies:

(i) For every fixed $y, z \in \mathbb{R}$, the map is measurable with respect to the variable $\imath$.
(ii) For almost every $\imath \in \mathcal{J}$, it is upper semicontinuous in the variables $(y, z) \in \mathbb{R} \times \mathbb{R}$.

(H2) A continuous function $\varphi : \mathcal{J} \to \mathbb{R}^+$ exists such that for every $\imath \in \mathcal{J}$ and all $y, z \in \mathbb{R}$, the following holds:

$$\|\aleph(\imath, y, z)\|_{\mathcal{P}} = \sup\{|g| : g \in \aleph(\imath, y, z)\} \leq \frac{\varphi(\imath)}{1 + |y| + |z|}.$$

(H3) There are functions $p, q \in C(\mathcal{J})$ with the property that for almost every $\imath \in \mathcal{J}$ and every $y, \overline{y}, z, \overline{z} \in \mathbb{R}$,

$$\mathcal{H}_d(\aleph(\imath, y, z), \aleph(\imath, \overline{y}, \overline{z})) \leq p(\imath)|y - \overline{y}| + q(\imath)|z - \overline{z}|.$$

The initial existence theorem follows directly from Lemma 4.

Theorem 2. *Suppose the conditions listed in (**H1**) through (**H3**) hold true. Under these hypotheses, the initial value problem given by $(1) - (2)$ admits at least one solution defined on the interval $\mathcal{J}$.*

Proof. To prove the result, we rewrite the problem $(1) - (2)$ as an equivalent fixed point formulation. Define the multivalued operator

$$\mathcal{T} : C_{1-\ell,\rho}(\mathcal{J}) \longrightarrow \mathcal{P}(C_{1-\ell,\rho}(\mathcal{J})),$$

by

$$(\mathcal{T}z)(\imath) = \{\hbar \in C_{1-\ell,\rho}(\mathcal{J}) : \hbar(\imath) = \frac{\theta}{\Gamma(\ell)} \left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{\ell-1} + \int_a^{\imath} s^{\rho-1} \left(\frac{\imath^{\rho} - s^{\rho}}{\rho} \right)^{\ell_1-1} \frac{g(s)}{\Gamma(\ell_1)} ds\}. \tag{4}$$

It is evident that any fixed point of the operator $\mathcal{T}$ corresponds directly to a solution of the problem (1) − (2).

Define now

$$B_\eta = \left\{ z \in C_{1-\ell,\rho}(\mathcal{J}) : \|z\|_{C_{1-\ell,\rho}} \leq \eta \right\},$$

where,

$$\eta = \frac{|\theta|}{\Gamma(\ell)} + \frac{\widetilde{\varphi}}{\Gamma(\ell_1+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\ell_1+1-\ell},$$

Let us denote $\widetilde{\varphi} = \max_{\iota \in \mathcal{J}} \varphi(\iota)$. It is straightforward to verify that the set B_η forms a closed and convex subset within the space $C_{1-\ell,\rho}(\mathcal{J})$.

Our goal is to demonstrate that the operator $\mathcal{T}$ meets all the criteria specified in Lemma 4.

The argument proceeds through several stages:

Step 1. For every $z \in B_\eta$, the set $\mathcal{T}(z)$ is convex.

To justify this, suppose $\hbar_1, \hbar_2 \in \mathcal{T}(z)$. Then there exist functions $g_1, g_2 \in S_{\aleph,z}$ such that for all $\iota \in \mathcal{J}$, the following holds:

$$\hbar_i(\iota) = \frac{\theta}{\Gamma(\ell)} \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{\ell-1} + \int_a^\iota s^{\rho-1} \left(\frac{\iota^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \frac{g_i(s)}{\Gamma(\ell_1)} ds, \ i = 1, 2.$$

For any scalar $0 \leq d \leq 1$, and for every $\iota \in \mathcal{J}$, we have

$$(d\hbar_1 + (1-d)\hbar_2)(\iota) = \int_a^\iota s^{\rho-1} \left(\frac{\iota^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \frac{[dg_1(s) + (1-d)g_2(s)]}{\Gamma(\ell_1)} ds.$$

Because the set $S_{\aleph,z}$ is convex—this follows from the fact that the values of the multivalued mapping $\aleph$ are convex we conclude that

$$d\hbar_1 + (1-d)\hbar_2 \in \mathcal{T}(z).$$

Step 2. $\mathcal{T}(B_\eta)$ is uniformly bounded.

For any $z \in B_\eta$, $\hbar \in \mathcal{T}(z)$ and each $\iota \in \mathcal{J}$, we have

$$\begin{aligned}
\left| \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{1-\ell} \hbar(\iota) \right| \leq & \frac{|\theta|}{\Gamma(\ell)} + \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{1-\ell} \int_a^\iota s^{\rho-1} \left(\frac{\iota^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \frac{|g(s)|}{\Gamma(\ell_1)} ds \\
\leq & \frac{|\theta|}{\Gamma(\ell)} + \frac{\left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{1-\ell}}{\Gamma(\ell_1)} \int_a^\iota s^{\rho-1} \left(\frac{\iota^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \varphi(s) ds \\
\leq & \frac{|\theta|}{\Gamma(\ell)} + \frac{\widetilde{\varphi}}{\Gamma(\ell_1)} \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{1-\ell} \int_a^\iota s^{\rho-1} \left(\frac{\iota^\rho - s^\rho}{\rho} \right)^{\ell_1-1} ds \\
\leq & \frac{|\theta|}{\Gamma(\ell)} + \frac{\widetilde{\varphi}}{\Gamma(\ell_1+1)} \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{1-\ell} \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{\ell_1} \\
\leq & \frac{|\theta|}{\Gamma(\ell)} + \frac{\widetilde{\varphi}}{\Gamma(\ell_1+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\ell_1+1-\ell}.
\end{aligned}$$

Therefore,

$$\|\hbar\|_{C_{1-\ell,\rho}} \leq \frac{|\theta|}{\Gamma(\ell)} + \frac{\widetilde{\varphi}}{\Gamma(\ell_1+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\ell_1+1-\ell} = \eta.$$

It follows that the operator satisfies $\mathcal{T}(B_\eta) \subseteq B_\eta$, and since B_η is a bounded set, the image $\mathcal{T}(B_\eta)$ is uniformly bounded as well.

Step 3. The set $\mathcal{T}(B_\eta)$ is equicontinuous.

Take any $\iota_1, \iota_2 \in \mathcal{J}$ with $\iota_1 < \iota_2$ and any $z \in B_\eta$. For each $\hbar \in \mathcal{T}(z)$ and for all $\iota \in \mathcal{J}$, applying assumption (H2) yields

$$\begin{aligned}
&\left| \left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1-\ell} \hbar(\iota_2) - \left(\frac{\iota_1^\rho - a^\rho}{\rho} \right)^{1-\ell} \hbar(\iota_1) \right| \\
&\leq \left| \frac{\left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1-\ell}}{\Gamma(\ell_1)} \int_a^{\iota_2} s^{\rho-1} \left(\frac{\iota_2^\rho - s^\rho}{\rho} \right)^{\ell_1-1} g(s)ds - \frac{\left(\frac{\iota_1^\rho - a^\rho}{\rho} \right)^{1-\ell}}{\Gamma(\ell_1)} \int_a^{\iota_1} s^{\rho-1} \left(\frac{\iota_1^\rho - s^\rho}{\rho} \right)^{\ell_1-1} g(s)ds \right| \\
&\leq \frac{\left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1-\ell}}{\Gamma(\ell_1)} \int_{\iota_1}^{\iota_2} s^{\rho-1} \left(\frac{\iota_2^\rho - s^\rho}{\rho} \right)^{\ell_1-1} |g(s)|ds + \int_a^{\iota_1} \left| \left[\left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1-\ell} s^{\rho-1} \left(\frac{\iota_2^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \right. \right. \\
&\left. \left. - \left(\frac{\iota_1^\rho - a^\rho}{\rho} \right)^{1-\ell} s^{\rho-1} \left(\frac{\iota_1^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \right] \right| \frac{|g(s)|}{\Gamma(\ell_1)} ds \\
&\leq \frac{\left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1-\ell}}{\Gamma(\ell_1)} \int_{\iota_1}^{\iota_2} s^{\rho-1} \left(\frac{\iota_2^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \varphi(s)ds + \int_a^{\iota_1} \left| \left[\left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1-\ell} s^{\rho-1} \left(\frac{\iota_2^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \right. \right. \\
&\left. \left. - \left(\frac{\iota_1^\rho - a^\rho}{\rho} \right)^{1-\ell} s^{\rho-1} \left(\frac{\iota_1^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \right] \right| \frac{\varphi(s)}{\Gamma(\ell_1)} ds.
\end{aligned}$$

By virtue of the continuity property of the function φ, it follows that

$$\begin{aligned}
&\left| \left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1-\ell} (\hbar z)(\iota_2) - \left(\frac{\iota_1^\rho - a^\rho}{\rho} \right)^{1-\ell} (\hbar z)(\iota_1) \right| \\
&\leq \frac{\widetilde{\varphi}}{\Gamma(\ell_1+1)} \left[2 \left(\frac{\iota_1^\rho - a^\rho}{\rho} \right)^{1-\ell} \left(\frac{\iota_2^\rho - \iota_1^\rho}{\rho} \right)^{1+\ell_1-\ell} + \left(\frac{\iota_2^\rho - a^\rho}{\rho} \right)^{1+\ell_1-\ell} - \left(\frac{\iota_1^\rho - a^\rho}{\rho} \right)^{1+\ell_1-\ell} \right].
\end{aligned}$$

As $\iota_1 \to \iota_2$, the right-hand side of the above inequality approaches zero.

Combining the results of Steps 1 through 3 with the Arzelà-Ascoli theorem, we deduce that the operator

$$\mathcal{T} : C_{1-\ell,\rho}(\mathcal{J}) \longrightarrow \mathcal{P}(C_{1-\ell,\rho}(\mathcal{J})),$$

is completely continuous.

Step 4: The operator $\mathcal{T}$ possesses a closed graph.

Suppose that $z_n \to z_*$, $\hbar_n \in \mathcal{T}(z_n)$, and $\hbar_n \to \hbar_*$. The goal is to verify that $\hbar_* \in \mathcal{T}(z_*)$. Since $\hbar_n \in \mathcal{T}(z_n)$, it follows that there exist $g_n \in S_{\aleph,z_n}$ such that for every $\iota \in \mathcal{J}$,

$$\hbar_n(\iota) = \frac{\theta}{\Gamma(\ell)} \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{\ell-1} + \int_a^\iota s^{\rho-1} \left(\frac{\iota^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \frac{g_n(s)}{\Gamma(\ell_1)} ds.$$

It remains to demonstrate the existence of a function $g_*(\iota) \in S_{\aleph,z_*}$ such that for every $\iota \in \mathcal{J}$,

$$\hbar_*(\iota) = \frac{\theta}{\Gamma(\ell)} \left(\frac{\iota^\rho - a^\rho}{\rho} \right)^{\ell-1} + \int_a^\iota s^{\rho-1} \left(\frac{\iota^\rho - s^\rho}{\rho} \right)^{\ell_1-1} \frac{g_*(s)}{\Gamma(\ell_1)} ds.$$

Owing to the upper semicontinuity of the mapping $\aleph(\iota, \cdot, \cdot)$, for any $\epsilon > 0$, there exists an integer $n_0(\epsilon) \geq 0$ such that for all $n \geq n_0$, the following inclusion holds:

$$g_n(\iota) \in \aleph(\iota, z(\iota), {}^\rho\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\iota)) \subset \aleph(\iota, z_*(\iota), {}^\rho\mathcal{D}_{a^+}^{\ell_1,\ell_2} z_*(\iota)) + \epsilon B(0,1) \; a.e. \; \iota \in \mathcal{J}.$$

Because the multivalued map $\aleph$ takes compact values, one can extract a subsequence $g_{n_m}(\cdot)$ for which

$$g_{n_m}(.) \to g_*(.) \; as \; m \to \infty,$$

$$g_*(\imath) \in \aleph(\imath, z_*(\imath),^{\rho} \mathcal{D}_{a^+}^{\ell_1,\ell_2} z_*(\imath)) \; a.e. \; \imath \in \mathcal{J}.$$

For each $\sigma \in \aleph(\imath, z_*(\imath),^{\rho} \mathcal{I}_{a^+}^{\ell_1} z_*(\imath))$, we obtain

$$|g_{n_m}(\imath) - g_*(\imath)| \leq |g_{n_m}(\imath) - \sigma| + |\sigma - g_*(\imath)|,$$

and so

$$|g_{n_m}(\imath) - g_*(\imath)| \leq d(g_{n_m}(\imath), \aleph(\imath, z_*(\imath)),^{\rho} \mathcal{I}_{a^+}^{\ell_1} z_*(\imath)).$$

By employing an analogous argument where the positions of g_{n_m} and g_* are swapped, it follows that

$$\begin{aligned}
&|g_{n_m}(\imath) - g_*(\imath)| \\
&\quad \leq \mathcal{H}_d \left(\aleph(\imath, z_{n_m}(\imath),^{\rho} \mathcal{D}_{a^+}^{\ell_1,\ell_2} z_{n_m}(\imath)), \aleph(\imath, z^*(\imath),^{\rho} \mathcal{D}_{a^+}^{\ell_1,\ell_2} z^*(\imath)) \right) \\
&\quad \leq \mathcal{H}_d \left(\aleph(\imath, \frac{\theta}{\Gamma(\ell)} \left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{\ell-1} +^{\rho} \mathcal{I}_{a^+}^{\ell_1} z_{n_m}(\imath), z_{n_m}(\imath)), \aleph(\imath, \frac{\theta}{\Gamma(\ell)} \left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{\ell-1} +^{\rho} \mathcal{I}_{a^+}^{\ell_1} z^*(\imath), z^*(\imath)) \right) \\
&\quad \leq p(\imath)^{\rho} \mathcal{I}_{a^+}^{\ell_1} |z_{n_m}(\imath) - z^*(\imath)| + q(\imath) |z_{n_m}(\imath) - z^*(\imath)|.
\end{aligned}$$

Then, for $\imath \in \mathcal{J}$, we get

$$\begin{aligned}
&\left| \left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{1-\ell} (\hbar_{n_m}(\imath) - \hbar_*(\imath)) \right| \\
&\quad \leq \frac{\left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{1-\ell}}{\Gamma(\ell_1)} \int_a^{\imath} s^{\rho-1} \left(\frac{\imath^{\rho} - s^{\rho}}{\rho} \right)^{\ell_1-1} |g_{n_m}(s) - g_*(s)| ds \\
&\quad \leq \frac{\left(\frac{\imath^{\rho} - a^{\rho}}{\rho} \right)^{1-\ell}}{\Gamma(\ell_1)} \int_a^{\imath} s^{\rho-1} \left(\frac{\imath^{\rho} - s^{\rho}}{\rho} \right)^{\ell_1-1} \left[p(\imath)^{\rho} \mathcal{I}_{a^+}^{\ell_1} |z_{n_m}(\imath) - z^*(\imath)| + q(\imath) |z_{n_m}(\imath) - z^*(\imath)| \right] ds
\end{aligned}$$

Therefore

$$\|\hbar_{n_m} - \hbar_*\|_{C_{1-\ell,\rho}} \leq \left[\frac{p^*}{\Gamma(2\ell_1 + 1)} \left(\frac{b^{\rho} - a^{\rho}}{\rho} \right)^{2\ell_1} + \frac{q^*}{\Gamma(\ell_1 + 1)} \left(\frac{b^{\rho} - a^{\rho}}{\rho} \right)^{\ell_1} \right] \|z_{n_m} - z_*\|_{C_{1-\ell,\rho}},$$

where $p^* = \max_{\imath \in \mathcal{J}} p(\imath) \; and \; q^* = \max_{\imath \in \mathcal{J}} q(\imath)$.

Hence

$$\|\hbar_{n_m} - \hbar_*\|_{C_{1-\ell,\rho}} \to 0 \; as \; m \; \to \infty.$$

Consequently, applying Lemma 4 allows us to conclude that the operator $\mathcal{T}$ admits a fixed point z within the set $B_{\eta} \subset C_{1-\ell,\rho}(\mathcal{J})$. This fixed point corresponds to a solution of the initial value problem (1) − (2). This concludes the proof. □

4. Example

To illustrate the applicability of our findings, we examine the following fractional initial value problem,

$$^{\rho}\mathcal{D}_{a^+}^{\frac{1}{2},\frac{1}{2}} z(\imath) \in \aleph(\imath, z(\imath),^{\rho} \mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath)), \;\; \imath \in \mathcal{J} := \left[\frac{\pi}{2}, \pi \right], \tag{5}$$

$$(^{\rho}\mathcal{I}^{\frac{1}{4}} z)(\frac{\pi}{2}) = \left(1 - \frac{\pi}{2} \right), \tag{6}$$

where $\rho > 0, \; \ell = \frac{3}{4}$.

Set

$$\aleph(\imath, z(\imath), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath)) = \{g \in \mathbb{R} : \phi_1(\imath, z(\imath), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath)) \leq g \leq \phi_2(\imath, z(\imath), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath))\},$$

where the functions $\phi_1, \phi_2 : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are measurable with respect to the variable $\imath$. We suppose that for each fixed $\imath \in \mathcal{J}$, the function $\phi_1(\imath, \cdot, \cdot)$ is lower semicontinuous, meaning that for every $\mu \in \mathbb{R}$, the set

$$\{z \in \mathbb{R} : \phi_1(\imath, z(\imath), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath)) > \mu\},$$

is open. Similarly, for each fixed $\imath \in \mathcal{J}$, the function $\phi_2(\imath, \cdot, \cdot)$ is assumed to be upper semicontinuous, that is, for every $\mu \in \mathbb{R}$, the set

$$\{z \in \mathbb{R} : \phi_2(\imath, z(\imath), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath)) < \mu\},$$

is open. Further, we assume that

$$\max(|\phi_1(\imath, z(\imath), {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath))|,\ |\phi_2(\imath, z, {}^{\rho}\mathcal{D}_{a^+}^{\ell_1,\ell_2} z(\imath))|) \leq \frac{(\imath - \frac{\pi}{2})^{-\frac{1}{4}} \sin(\imath - \frac{\pi}{2})}{\left(1 + \sqrt{(\imath - \frac{\pi}{2})}(1 + |y| + |z|)\right)},$$

for every $\imath \in \mathcal{J}$ and $z \in \mathbb{R}$.

The condition (**H2**) holds true with

$$\begin{cases} \varphi(\imath) = \frac{(\imath - \frac{\pi}{2})^{-\frac{1}{4}} |\sin(\imath - \frac{\pi}{2})|}{1 + \sqrt{\imath - \frac{\pi}{2}}}, \ \imath \in \mathcal{J}^*, \\ \varphi(\frac{\pi}{2}) = 0. \end{cases}$$

As a consequence, the multivalued map $\aleph$ possesses compact and convex values and is also upper semicontinuous (refer to [13]). Given that all the hypotheses of Theorem 2 are met, it follows that the initial value problem $(5) - (6)$ admits at least one solution.

5. Conclusion

The present work focuses on establishing the existence of solutions for implicit fractional differential inclusions involving the recently introduced Hilfer-Katugampola fractional derivative, which generalizes the Katugampola and Caputo-Katugampola operators. By formulating the problem as a fixed point problem for multivalued operators on appropriate weighted function spaces, we utilized the Bohnenblust-Karlin fixed point theorem and contraction principles for multivalued maps to establish existence results under natural assumptions such as convexity, compactness, upper semicontinuity, and boundedness of the multivalued right-hand side.

The illustrative example demonstrates how these theoretical results apply to a concrete fractional initial value problem, confirming the broad applicability of our approach. This work contributes to the growing literature on fractional differential inclusions by extending existence theory to operators of generalized fractional type, enriching the modeling tools available for complex dynamical systems exhibiting memory and hereditary properties.

Future work may consider uniqueness, stability, and numerical methods for such fractional inclusions as well as extensions to more general fractional operators and systems in higher dimensions.

Conflicts of Interest: "The authors declare no conflict of interest."

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