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k-fractional Hermite-Mercer type inequalities for uniformly convex functions

Saad Ihsan Butt[1,*], Yamin Sayyari[2] and Muhammad Umar[1]

[1] Department of Mathematics, Comsats University Islamabad Lahore Campus Pakistan
[2] Department of Mathematics, Sirjan University Of Technology, Sirjan, Iran
[*] Correspondence: saadihsanbutt@gmail.com

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Abstract: In this paper, we give extensions of Jensen-Mercer inequality for functions whose derivatives in the absolute values are uniformly convex considering the class of $k-$fractional integral operators.

Keywords: Hermite-Mercer inequality, Uniformly convex functions, Integral inequalities, $k-$Fractional integrals

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1. Introduction

Fractional calculus may be describe as addition of the concept of derivative operator from integer order n to arbitrary order. In applied mathematics the fractional integrals are powerful tools to solve many problems from different fields of science and engineering. In the current decade, many mathematicians always merge and put an effort and new ideas into fractional analysis to bring a new dimension with different features in the field of mathematical analysis and applied mathematics.

Convex functions show a vital role in several zones of Mathematics. This theory provides us amazing framework to initiate and develop numerical tools to tackle and study complicated problems in mathematics. Due to number of expedient properties, they are magical especially in optimization theory. There is delightful connection between the theory of mathematical inequalities and convex functions. Convexity arises in several related topics of a basically obtimization, namely, in information theory, the theory of inequalities. Interested readers can refer to [1–7].

The generalized k-fractional conformable integrals are used throughout this paper (see [8]) are defined by

$$ {}_{k}^{\beta}J_{\mu_1^+}^{\alpha}\Theta(y_1)=\frac{1}{k\Gamma_k(\beta)}\int_{\mu_1}^{y_1}\left(\frac{(y_1-\mu_1)^{\alpha}-(\gamma-\mu_1)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{\Theta(\gamma)}{(\gamma-\mu_1)^{1-\alpha}}d\gamma, \tag{1}$$

and

$$ {}_{k}^{\beta}J_{\mu_1^-}^{\alpha}\Theta(y_1)=\frac{1}{k\Gamma_k(\beta)}\int_{y_1}^{\mu_1}\left(\frac{(\mu_1-y_1)^{\alpha}-(\mu_1-\gamma)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{\Theta(\gamma)}{(\mu_1-\gamma)^{1-\alpha}}d\gamma. \tag{2}$$

If we set $k=1$ in (1) and (2) then it reduces to generalized fractional conformable integrals (see [9,10]) are defined by

$$ {}^{\beta}J_{\mu_1^+}^{\alpha}\Theta(y_1)=\frac{1}{\Gamma(\beta)}\int_{\mu_1}^{y_1}\left(\frac{(y_1-\mu_1)^{\alpha}-(\gamma-\mu_1)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\Theta(\gamma)}{(\gamma-\mu_1)^{1-\alpha}}d\gamma, \tag{3}$$

and

$$ {}^{\beta}J_{\mu_1^-}^{\alpha}\Theta(y_1)=\frac{1}{\Gamma(\beta)}\int_{y_1}^{\mu_1}\left(\frac{(\mu_1-y_1)^{\alpha}-(\mu_1-\gamma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\Theta(\gamma)}{(\mu_1-\gamma)^{1-\alpha}}d\gamma. \tag{4}$$

Note that if we choose $\alpha = 1$ in (3) and (4) then it reduces to classical Riemann-Liouville fractional integral operator.

Integral inequalities have an important role in the expansion of all branches of mathematics. One of the most powerful of these integral inequalities is the Hermite-Hadamard inequality obtained for convex function.

The Jensen-Mercer inequality is an important mathematical result that extends Jensen's inequality to weighted sums of convex functions, illustrating how the average of the function values relates to the function evaluated at the average point, thereby highlighting the fundamental properties of convexity in inequality analysis.

Theorem 1. [11] *Let $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a convex mapping. Then the inequality*

$$\Theta\left(\mu_1 + \nu_1 - \sum_{i=1}^{n} p_i x_i\right) \leq \Theta(\mu_1) + \Theta(\nu_1) - \sum_{i=1}^{n} p_i \Theta(x_i),$$

holds for all $x_i \in [\mu_1, \nu_1]$ and $p_i \in [0,1]$ with $\sum_{i=1}^{n} p_i = 1$.

Theorem 2. [12] *Let $\alpha, \beta, k > 0$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a convex mapping. Then*

$$\begin{aligned}\Theta\left(\mu_1 + \nu_1 - \frac{x_1 + y_1}{2}\right) \leq & \frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma(\beta+k)}{(y_1 - x_1)^{\alpha\frac{\beta}{k}}} \\ & \times \left\{ {}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^{+}}\Theta(\mu_1+\nu_1-x_1) + {}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^{-}}\Theta(\mu_1+\nu_1-y_1) \right\} \\ \leq & \Theta(\mu_1) + \Theta(\nu_1) - \left(\frac{\Theta(x_1) + \Theta(y_1)}{2}\right),\end{aligned}$$

for all $x_1, y_1 \in [\mu_1, \nu_1]$.

Theorem 3. [13] *Let $\Theta : I \longrightarrow \mathbb{R}$ be an uniformly convex function with modulus $\varphi : \mathbb{R}_+ \longrightarrow [0, +\infty]$ on I, $\{x_k\}_{k=1}^{n} \subseteq [a,b]$ be a sequence and let π be a permutation on $\{1, ..., n\}$ such that $x_{\pi(1)} \leq x_{\pi(2)} \leq ... \leq x_{\pi(n)}$. Then the inequality*

$$\Theta\left(\sum_{k=1}^{n} p_k x_k\right) \leq \sum_{k=1}^{n} p_k \Theta(x_k) - \sum_{k=1}^{n-1} p_{\pi(k)} p_{\pi(k+1)} \varphi\left(x_{\pi(k+1)} - x_{\pi(k)}\right), \tag{5}$$

holds for every convex combination $\sum_{k=0}^{n} p_k x_k$ of points $x_k \in I$.

Lemma 1. *([12], Lemma 2.5) Let $\alpha, \beta \in \mathbb{R}$ and $\Theta : [\mu_1, \nu_1] \longrightarrow \mathbb{R}$ be a differentiable mapping such that $\Theta' \in L[\mu_1, \nu_1]$. Then the inequality*

$$\begin{aligned}& \frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma(\beta+k)}{(y_1 - x_1)^{\alpha\frac{\beta}{k}}} \left\{ {}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^{+}}\Theta(\mu_1+\nu_1-x_1) + {}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^{-}}\Theta(\mu_1+\nu_1-y_1) \right\} \\ & - \Theta\left(\mu_1 + \nu_1 - \frac{x_1 + y_1}{2}\right) \\ & \quad = \frac{y_1 - x_1}{4}\alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1 - (1-\gamma)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} \\ & \quad \times \left[\Theta'\left(\mu_1 + \nu_1 - \left(\frac{2-\gamma}{2}x_1 + \frac{\gamma}{2}y_1\right)\right) - \Theta'\left(\mu_1 + \nu_1 - \left(\frac{\gamma}{2}x_1 + \frac{2-\gamma}{2}y_1\right)\right)\right] d\gamma,\end{aligned}$$

holds for all $x_1, y_1 \in [\mu_1, \nu_1]$.

Let $\Theta : [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $\Theta' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|\Theta'(x_1)| \leq M$, then the following inequality:

$$\left| \Theta(x_1) - \frac{1}{b-a} \int_a^b \Theta(y_1)\, dy_1 \right| \leq \frac{M}{b-a} \left[\frac{(x_1 - a)^2 + (b - x_1)^2}{2} \right], \tag{6}$$

holds. This result is known in the literature as the Ostrowski inequality. For recent results and generalizations concerning Ostrowski-inequality see [14–16] and the references therein.

In recent years, several significant contributions have been made by researcher in the development of Hermite–Jensen–Mercer and Hermite–Mercer type inequalities within fractional and k-fractional frameworks. In [12], authors established foundational results on Hermite–Jensen–Mercer-type inequalities via k-fractional integrals, providing an essential basis for subsequent generalizations. Further advancements were presented in Butt et al. [17], where the authors investigated Hermite–Jensen–Mercer inequalities in the setting of Ψ-Riemann–Liouville k-fractional operators, enriching the structure of fractional Mercer-type inequalities. Additionally, Butt et al. [18] extended these ideas by developing fractional Hermite–Jensen–Mercer inequalities with respect to another function, offering a broader analytical framework for applications. These works collectively demonstrate the relevance and growing importance of k-fractional approaches in the study of Mercer-type inequalities and form a strong motivation for the present investigation.

2. Main results

In this section we obtain some Hermite-Jensen-Mercer type inequalities with the help of fractional integrals.

Definition 1. [19] Let $\Theta : [\mu_1, \nu_1] \longrightarrow \mathbb{R}$ be a function. Then Θ is uniformly convex with modulus $\varphi : R_{\geq 0} \longrightarrow [0, +\infty)$ if is increasing, vanishes only at 0, and

$$\Theta(\gamma x_1 + (1-\gamma) y_1) + \gamma (1-\gamma) \varphi(|x_1 - y_1|) \leq \gamma \Theta(x_1) + (1-\gamma) \Theta(y_1),$$

for every $|x_1 - y_1| \in [0, 1]$ and $x_1, y_1 \in [\mu_1, \nu_1]$.

Uniformly convex function is stronger than a convex function. Almost all convex functions on the finite interval $[a, b]$ can be considered as a uniformly convex function. The algebraic properties of uniform convex functions are given in the following references (see Bauschke [19] page 144 and Zalinescu [20] section 4). We have given a few examples below:

i) ${x_1}^2 : \mathbb{R} \to \mathbb{R}$ is uniformly with modulus $\varphi(x_1) = {x_1}^2$, because $(\gamma a + (1-\gamma) b)^2 + \gamma(1-\gamma)(b-a)^2 = \gamma a^2 + (1-\gamma) b^2$,

ii) $e^{x_1} : (0, \infty) \to \mathbb{R}$ is uniformly with modulus $\varphi(x_1) = \frac{1}{2}{x_1}^2$,

iii) $1/x_1 : (a, b) \to \mathbb{R}$ is uniformly with modulus $\varphi(x_1) = \frac{1}{b^3}{x_1}^2$, $a > 0$.

iv) ${x_1}^4 : (a, b) \to \mathbb{R}$ is uniformly with modulus $\varphi(x_1) = 6a^2{x_1}^2$, $a > 0$.

Definition 2. [21] Let $\Theta : [\mu_1, \nu_1] \longrightarrow \mathbb{R}$ be a function. Then Θ is strongly convex with modulus $c > 0$ if

$$\Theta(\gamma x_1 + (1-\gamma) y_1) + c\gamma(1-\gamma)(x_1 - y_1)^2 \leq \gamma \Theta(x_1) + (1-\gamma) \Theta(y_1),$$

for every $\gamma \in [0, 1]$ and $x_1, y_1 \in [\mu_1, \nu_1]$.

Theorem 4. [22] *Let $\alpha, \beta > 0$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be an uniformly convex mapping with modulus φ. Then the inequality*

$$\begin{aligned} \Theta(\mu_1 + \nu_1 - (\gamma x_1 + (1-\gamma) y_1)) &\leq \Theta(\mu_1) + \Theta(\nu_1) - \gamma \Theta(x_1) - (1-\gamma) \Theta(y_1) - \gamma(1-\gamma) \varphi(|x_1 - y_1|) \\ &\quad - \frac{2\varphi(\nu_1 - \mu_1)}{(\nu_1 - \mu_1)^2} (\gamma(\nu_1 - x_1)(x_1 - \mu_1) + (1-\gamma)(\nu_1 - y_1)(y_1 - \mu_1)), \end{aligned} \tag{7}$$

holds for all $x_1, y_1 \in [\mu_1, \nu_1]$.

Throughout the paper we will use:

$$\mathbb{S} := \Bigg| \frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}} k\Gamma(\beta+k)}{(y_1-x_1)^{\alpha\frac{\beta}{k}}} \left\{ {}_k^{\beta}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^+}\Theta(\mu_1+\nu_1-x_1) + {}_k^{\beta}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^-}\Theta(\mu_1+\nu_1-y_1) \right\}$$
$$- \Theta\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right) \Bigg|,$$

and $H_n = B\left(\frac{}{k}+1, \frac{n}{}\right)$ for $n = \{1,2,3\}$, where B is beta functions.

Theorem 5. *Let* $\alpha, \beta, k > 0$ *and* $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ *be an uniformly convex mapping with modulus* φ. *Then the inequality*

$$\begin{aligned}
&\Theta\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right) + \frac{k}{8\beta}\int_0^1 u^{\frac{\beta}{k}-1}\varphi\left((1-u)^{\frac{1}{\alpha}}|x_1-y_1|\right)du \\
&\leq \frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma(\beta+k)}{(y_1-x_1)^{\alpha\frac{\beta}{k}}} \\
&\quad \times \left\{ {}_k^{\beta}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^+}\Theta(\mu_1+\nu_1-x_1) + {}_k^{\beta}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^-}\Theta(\mu_1+\nu_1-y_1) \right\} \\
&\leq \Theta(\mu_1) + \Theta(\nu_1) - \left(\frac{\Theta(x_1)+\Theta(y_1)}{2}\right) \\
&\quad - \frac{2\alpha^{-\frac{\beta}{k}}\varphi(\nu_1-\mu_1)}{\frac{\beta}{k}(\nu_1-\mu_1)^2}\left((\nu_1-x_1)(x_1-\mu_1)+(\nu_1-y_1)(y_1-\mu_1)\right) \\
&\quad - \left(\frac{1}{2\frac{\beta}{k}\alpha^{\frac{\beta}{k}}} - \frac{1}{2\alpha^{\frac{\beta}{k}}}H_2\right)\varphi(|x_1-y_1|),
\end{aligned}$$

holds for all $x_1, y_1 \in [\mu_1, \nu_1]$.

Proof. Since Θ is uniformly convex with modulus φ,

$$\begin{aligned}
\Theta\left(\mu_1+\nu_1-\frac{x_2+y_2}{2}\right) &= \Theta\left(\frac{2\mu_1+2\nu_1-x_2-y_2}{2}\right) \\
&\leq \frac{1}{2}\Theta(\mu_1+\nu_1-x_2) + \frac{1}{2}\Theta(\mu_1+\nu_1-y_2) - \frac{1}{4}\varphi(|x_2-y_2|),
\end{aligned}$$

for all $x_2, y_2 \in [\mu_1, \nu_1]$.

Now by using the change of variables $x_2 = \frac{\gamma}{2}x_1 + \left(1-\frac{\gamma}{2}\right)y_1$ and $y_2 = \frac{\gamma}{2}y_1 + \left(1-\frac{\gamma}{2}\right)x_1$ for $x_1, y_1 \in [\mu_1, \nu_1]$ and $\gamma \in [0,1]$, we obtain

$$\begin{aligned}
2\Theta\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right) \leq &\Theta\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\left(1-\frac{\gamma}{2}\right)y_1\right)\right) \\
&+ \Theta\left(\mu_1+\nu_1-\left(\left(1-\frac{\gamma}{2}\right)x_1+\frac{\gamma}{2}y_1\right)\right) - \frac{1}{4}\varphi((1-\gamma)|x_1-y_1|), \qquad (8)
\end{aligned}$$

Multiplying (8) by $\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\gamma)^{\alpha-1} := \Gamma^{k}_{\alpha,\beta}(\gamma)$ and then by using integration with respect to γ over $[0,1]$, and then combining the resulting inequality with the definition of the integral operator gives

$$2\Theta\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)\left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\gamma)^{\alpha-1}$$

$$\leq \left(\frac{1-(1-\gamma)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\gamma)^{\alpha-1}\times\left(\Theta\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\left(1-\frac{\gamma}{2}\right)y_1\right)\right)\right.$$
$$\left.+\Theta\left(\mu_1+\nu_1-\left(\left(1-\frac{\gamma}{2}\right)x_1+\frac{\gamma}{2}y_1\right)\right)-\frac{1}{4}\varphi\left((1-\gamma)\left|x_1-y_1\right|\right)\right). \quad (9)$$

On the other hand, we have

$$\int_0^1 \Gamma^k_{\alpha,\beta}(\gamma)\,\Theta\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\left(1-\frac{\gamma}{2}\right)y_1\right)\right)d\gamma=\left(\frac{2}{y_1-x_1}\right)^{\alpha\frac{\beta}{k}}\Gamma_k(\beta)\,{}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^-}\Theta\left(\mu_1+\nu_1-y_1\right), \quad (10)$$

$$\int_0^1 \Gamma^k_{\alpha,\beta}(\gamma)\,\Theta\left(\mu_1+\nu_1-\left(\left(1-\frac{\gamma}{2}\right)x_1+\frac{\gamma}{2}y_1\right)\right)d\gamma=\left(\frac{2}{y_1-x_1}\right)^{\alpha\frac{\beta}{k}}\Gamma_k(\beta)\,{}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^+}\Theta\left(\mu_1+\nu_1-x_1\right), \quad (11)$$

$$\int_0^1 \Gamma^k_{\alpha,\beta}(\gamma)\,d\gamma=\frac{k}{\beta}\alpha^{-\frac{\beta}{k}}, \quad (12)$$

and

$$\int_0^1 \Gamma^k_{\alpha,\beta}(\gamma)\,\varphi\left((1-\gamma)\left|x_1-y_1\right|\right)d\gamma=\alpha^{-\frac{\beta}{k}}\int_0^1 u^{\frac{\beta}{k}-1}\varphi\left((1-u)^{\frac{1}{\alpha}}\left|x_1-y_1\right|\right)du, \quad (13)$$

where $u=1-(1-\gamma)^{\alpha}$. The first inequality follows from (9), (10), (11) and (13).

To prove the second inequality, by (7),

$$\Theta\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\left(1-\frac{\gamma}{2}\right)y_1\right)\right)$$
$$\leq\Theta(\mu_1)+\Theta(\nu_1)-\frac{\gamma}{2}\Theta(x_1)-\left(1-\frac{\gamma}{2}\right)\Theta(y_1)$$
$$-\frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\varphi\left(\left|x_1-y_1\right|\right)-\frac{2\varphi(\nu_1-\mu_1)}{(\nu_1-\mu_1)^2}\left(\frac{\gamma}{2}(\nu_1-x_1)(x_1-\mu_1)+\left(1-\frac{\gamma}{2}\right)(\nu_1-y_1)(y_1-\mu_1)\right), \quad (14)$$

and

$$\Theta\left(\mu_1+\nu_1-\left(\left(1-\frac{\gamma}{2}\right)x_1+\frac{\gamma}{2}y_1\right)\right)$$
$$\leq\Theta(\mu_1)+\Theta(\nu_1)-\left(1-\frac{\gamma}{2}\right)\Theta(x_1)-\frac{\gamma}{2}\Theta(y_1)$$
$$-\frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\varphi\left(\left|x_1-y_1\right|\right)-\frac{2\varphi(\nu_1-\mu_1)}{(\nu_1-\mu_1)^2}\left(\left(1-\frac{\gamma}{2}\right)(\nu_1-x_1)(x_1-\mu_1)+\frac{\gamma}{2}(\nu_1-y_1)(y_1-\mu_1)\right). \quad (15)$$

Upon adding (14) and (15), we obtain

$$\Theta\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\left(1-\frac{\gamma}{2}\right)y_1\right)\right)+\Theta\left(\mu_1+\nu_1-\left(\left(1-\frac{\gamma}{2}\right)x_1+\frac{\gamma}{2}y_1\right)\right)$$
$$\leq 2[\Theta(\mu_1)+\Theta(\nu_1)]-[\Theta(x_1)+\Theta(y_1)]-\gamma\left(1-\frac{\gamma}{2}\right)\varphi\left(\left|x_1-y_1\right|\right)$$
$$-\frac{2\varphi(\nu_1-\mu_1)}{(\nu_1-\mu_1)^2}\left((\nu_1-x_1)(x_1-\mu_1)+(\nu_1-y_1)(y_1-\mu_1)\right). \quad (16)$$

Multiplying (16) by $\Gamma^k_{\alpha,\beta}(\gamma)$ and integrating the obtained inequality with respect to γ over $[0,1]$, we get

$$\int_0^1 \Gamma^k_{\alpha,\beta}(\gamma)\left\{\Theta\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\left(1-\frac{\gamma}{2}\right)y_1\right)\right)+\Theta\left(\mu_1+\nu_1-\left(\left(1-\frac{\gamma}{2}\right)x_1+\frac{\gamma}{2}y_1\right)\right)\right\}d\gamma$$

$$\leq \left\{2[\Theta(\mu_1)+\Theta(\nu_1)]-[\Theta(x_1)+\Theta(y_1)]-\frac{2\varphi(\nu_1-\mu_1)}{(\nu_1-\mu_1)^2}((\nu_1-x_1)(x_1-\mu_1)+(\nu_1-y_1)(y_1-\mu_1))\right\}$$
$$\times\int_0^1\Gamma^k_{\alpha,\beta}(\gamma)\,d\gamma-\varphi(|x_1-y_1|)\int_0^1\gamma\left(1-\frac{\gamma}{2}\right)\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\gamma)^{\alpha-1}\,d\gamma. \tag{17}$$

Furthermore,

$$\begin{aligned}\int_0^1\gamma\left(1-\frac{\gamma}{2}\right)\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\gamma)^{\alpha-1}\,d\gamma &= \frac{1}{2\alpha^{\frac{\beta}{k}-1}}\int_0^1\gamma(2-\gamma)\left(1-(1-\gamma)^\alpha\right)^{\frac{\beta}{k}-1}(1-\gamma)^{\alpha-1}\,d\gamma\\ &= \frac{1}{2\alpha^{\frac{\beta}{k}}}\int_0^1\left(1-t^{\frac{2}{\alpha}}\right)(1-t)^{\frac{\beta}{k}-1}\,dt\\ &= \frac{k}{2\beta\alpha^{\frac{\beta}{k}}}-\frac{1}{2\alpha^{\frac{\beta}{k}}}H_2,\end{aligned} \tag{18}$$

where $t=(1-\gamma)^\alpha$ and B is beta-function.

The second inequality follows from (10), (11), (13), (17) and (18). $\square$

Corollary 1. *If we set $\alpha=1$ in Theorem 5 , we get*

$$\begin{aligned}&\Theta\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)+\frac{k}{8\beta}\int_0^1 u^{\frac{\beta}{k}-1}\varphi((1-u)|x_1-y_1|)\,du\\ &\leq\frac{2^{\frac{\beta}{k}-1}\Gamma_k(\beta+k)}{(y_1-x_1)^{\frac{\beta}{k}}}\left\{{}_kJ^{\beta}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^+}\Theta(\mu_1+\nu_1-x_1)+{}_kJ^{\beta}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^-}\Theta(\mu_1+\nu_1-y_1)\right\}\\ &\leq\Theta(\mu_1)+\Theta(\nu_1)-\left(\frac{\Theta(x_1)+\Theta(y_1)}{2}\right)-\frac{2k\varphi(\nu_1-\mu_1)}{\beta(\nu_1-\mu_1)^2}((\nu_1-x_1)(x_1-\mu_1)+(\nu_1-y_1)(y_1-\mu_1))\\ &\quad-\left(\frac{k}{2\beta}-\frac{1}{2}B\left(\frac{\beta}{k},3\right)\right)\varphi(|x_1-y_1|),\end{aligned}$$

Corollary 2. *If we set $\mu_1=x_1$, $\nu_1=y_1$ and $\alpha=1$ in Theorem 5 we get*

$$\begin{aligned}\Theta\left(\frac{x_1+y_1}{2}\right)+\frac{k}{8\beta}\int_0^1 u^{\frac{\beta}{k}-1}\varphi((1-u)|x_1-y_1|)\,du &\leq\frac{2^{\frac{\beta}{k}-1}\Gamma(\beta+k)}{(y_1-x_1)^{\frac{\beta}{k}}}\left\{{}_kJ^{\beta}_{\left(\frac{x_1+y_1}{2}\right)^+}\Theta(y_1)+{}_kJ^{\beta}_{\left(\frac{x_1+y_1}{2}\right)^-}\Theta(x_1)\right\}\\ &\leq\frac{\Theta(x_1)+\Theta(y_1)}{2}-\left(\frac{k}{2\beta}-\frac{1}{2}B\left(\frac{\beta}{k},3\right)\right)\varphi(|x_1-y_1|).\end{aligned}$$

Corollary 3. *If we set $\mu_1=x_1$, $\nu_1=y_1$ and $\alpha=\beta=k=1$ in Theorem 5 we get*

$$\Theta\left(\frac{x_1+y_1}{2}\right)+\frac{1}{8}\int_0^1\varphi((1-u)|x_1-y_1|)\,du\leq\frac{1}{(y_1-x_1)}\int_{x_1}^{y_1}\varphi(u)\,du\leq\frac{\Theta(x_1)+\Theta(y_1)}{2}-\frac{1}{3}\varphi(|x_1-y_1|),$$

Remark 1. If we set $\varphi(x_1)=0$ in Theorem 5, we get Theorem 2.1 of [12] for $k=1$.

Corollary 4. *Let $\alpha,\beta,k>0$, $\Theta:[\mu_1,\nu_1]\to\mathbb{R}$ be a strongly convex mapping with modulus c and $x_1,y_1\in[\mu_1,\nu_1]$. Then*

$$\begin{aligned}&\Theta\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)+\frac{kc}{8\beta}(x_1-y_1)^2H_2\\ &\leq\frac{2\alpha^{\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma(\beta+k)}{(y_1-x_1)^{\alpha\frac{\beta}{k}}}\times\left\{{}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^+}\Theta(\mu_1+\nu_1-x_1)+{}^{\beta}_{k}J^{\alpha}_{\left(\mu_1+\nu_1-\frac{x_1+y_1}{2}\right)^-}\Theta(\mu_1+\nu_1-y_1)\right\}\\ &\leq\Theta(\mu_1)+\Theta(\nu_1)-\left(\frac{\Theta(x_1)+\Theta(y_1)}{2}\right)-\frac{2ck\alpha^{-\frac{\beta}{k}}}{\beta}((\nu_1-x_1)(x_1-\mu_1)+(\nu_1-y_1)(y_1-\mu_1))\end{aligned}$$

$$-c\left(x_1-y_1\right)^2\left(\frac{k}{2\beta\alpha^{\frac{\beta}{k}}}-\frac{1}{2\alpha^{\frac{\beta}{k}}}H_2\right).$$

Proof. The result follows from Theorem 5 with $\varphi\left(r\right)=cr^2$. □

Remark 2. If we set $c=0$ in Corollary 4, we get Theorem 2.1 of [12].

3. New inequalities of differentiable uniformly convex function

Theorem 6. *Let $\alpha,\beta,k>0$, $x_1<y_1$ and $\Theta:[\mu_1,\nu_1]\to\mathbb{R}$ be a differentiable mapping such that $\Theta'\in L[\mu_1,\nu_1]$ and $|\Theta'|$ is an uniformly convex mapping with modulus φ. Then the inequality*

$$\begin{aligned}\mathbb{S}\leq&\frac{y_1-x_1}{4\alpha}\Big[2\left(|\Theta'\left(\mu_1\right)|+|\Theta'\left(\nu_1\right)|\right)H_1+|\Theta'\left(y_1\right)|H_2-|\Theta'\left(x_1\right)|H_1\\&-\frac{\varphi\left(y_1-x_1\right)}{2}\left(H_1-H_3\right)-\frac{2\varphi\left(\nu_1-\mu_1\right)}{\left(\nu_1-\mu_1\right)^2}\left(\left(\nu_1-x_1\right)\left(x_1-\mu_1\right)+\left(\nu_1-y_1\right)\left(y_1-\mu_1\right)\right)H_1\Big],\end{aligned}$$

holds for all $x_1,y_1\in[\mu_1,\nu_1]$.

Proof. It follows from Lemma 1 that

$$\begin{aligned}\mathbb{S}=&\Bigg|\frac{y_1-x_1}{4}\alpha^{\frac{\beta}{k}}\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\\&\times\left[\Theta'\left(\mu_1+\nu_1-\left(\frac{2-\gamma}{2}x_1+\frac{\gamma}{2}y_1\right)\right)-\Theta'\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\frac{2-\gamma}{2}y_1\right)\right)\right]d\gamma\Bigg|.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{S}\leq&\frac{y_1-x_1}{4}\alpha^{\frac{\beta}{k}}\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\\&\times\left[\left|\Theta'\left(\mu_1+\nu_1-\left(\frac{2-\gamma}{2}x_1+\frac{\gamma}{2}y_1\right)\right)-\Theta'\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\frac{2-\gamma}{2}y_1\right)\right)\right|\right]d\gamma\\\leq&\frac{y_1-x_1}{4}\alpha^{\frac{\beta}{k}}\left\{\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\left|\Theta'\left(\mu_1+\nu_1-\left(\frac{2-\gamma}{2}x_1+\frac{\gamma}{2}y_1\right)\right)\right|d\gamma\right.\\&\left.+\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\left|\Theta'\left(\mu_1+\nu_1-\left(\frac{\gamma}{2}x_1+\frac{2-\gamma}{2}y_1\right)\right)\right|d\gamma\right\}.\end{aligned}$$

Since $|\Theta'|$ is uniformly convexity with modulus φ, Theorem 4 asserts that

$$\begin{aligned}\mathbb{S}\leq&\frac{y_1-x_1}{4}\alpha^{\frac{\beta}{k}}\left\{\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\left\{|\Theta'\left(\mu_1\right)|+|\Theta'\left(\nu_1\right)|-\frac{2-\gamma}{2}|\Theta'\left(x_1\right)|-\frac{\gamma}{2}|\Theta'\left(y_1\right)|\right.\right.\\&-\frac{2\varphi\left(\nu_1-\mu_1\right)}{\left(\nu_1-\mu_1\right)^2}\left(\frac{2-\gamma}{2}\left(\nu_1-x_1\right)\left(x_1-\mu_1\right)+\frac{\gamma}{2}\left(\nu_1-y_1\right)\left(y_1-\mu_1\right)\right)\\&\left.-\frac{\gamma\left(2-\gamma\right)}{4}\varphi\left(y_1-x_1\right)\right\}d\gamma+\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\left\{|\Theta'\left(\mu_1\right)|+|\Theta'\left(\nu_1\right)|-\frac{\gamma}{2}|\Theta'\left(x_1\right)|\right.\\&-\frac{2-\gamma}{2}|\Theta'\left(y_1\right)|-\frac{2\varphi\left(\nu_1-\mu_1\right)}{\left(\nu_1-\mu_1\right)^2}\left(\frac{\gamma}{2}\left(\nu_1-x_1\right)\left(x_1-\mu_1\right)+\frac{2-\gamma}{2}\left(\nu_1-y_1\right)\left(y_1-\mu_1\right)\right)\\&\left.\left.-\frac{\gamma\left(2-\gamma\right)}{4}\varphi\left(y_1-x_1\right)\right\}d\gamma\right\}.\end{aligned}$$

After some calculations, we get

$$\begin{aligned}
\mathbb{S} \leq & \frac{y_1 - x_1}{4} \alpha^{\frac{\beta}{k}} \left\{ \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \\
& \times \left\{ |\Theta'(\mu_1)| + |\Theta'(\nu_1)| - \frac{2-\gamma}{2} |\Theta'(x_1)| - \frac{\gamma}{2} |\Theta'(y_1)| \right\} d\gamma \\
& \left. + \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left\{ |\Theta'(\mu_1)| + |\Theta'(\nu_1)| - \frac{\gamma}{2} |\Theta'(x_1)| - \frac{2-\gamma}{2} |\Theta'(y_1)| \right\} d\gamma \right\} \\
& - \frac{y_1 - x_1}{4} \alpha^{\frac{\beta}{k}} \left\{ \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left\{ \frac{\gamma(2-\gamma)}{2} \varphi(y_1 - x_1) \right. \right. \\
& \left. \left. + \frac{2\varphi(\nu_1 - \mu_1)}{(\nu_1 - \mu_1)^2} \left((\nu_1 - x_1)(x_1 - \mu_1) + (\nu_1 - y_1)(y_1 - \mu_1) \right) \right\} d\gamma \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{S} \leq & \frac{y_1 - x_1}{4\alpha} \Big[2 \left(|\Theta'(\mu_1)| + |\Theta'(\nu_1)| \right) H_1 + |\Theta'(y_1)| H_2 - |\Theta'(x_1)| H_1 - \frac{\varphi(y_1 - x_1)}{2} (H_1 - H_3) \\
& - \frac{2\varphi(\nu_1 - \mu_1)}{(\nu_1 - \mu_1)^2} \left((\nu_1 - x_1)(x_1 - \mu_1) + (\nu_1 - y_1)(y_1 - \mu_1) \right) H_1 \Big],
\end{aligned}$$

where we have used the facts that

$$\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1,$$

$$\int_0^1 \gamma \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2,$$

$$\int_0^1 (2-\gamma) \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 + \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2,$$

and

$$\int_0^1 \gamma (2-\gamma) \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_3.$$

□

Corollary 5. *Let $\alpha, \beta, k > 0$, $x_1 < y_1$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a differentiable mapping such that $\Theta' \in L[\mu_1, \nu_1]$ and $|\Theta'|$ is a strongly convex mapping with modulus c. Then the inequality*

$$\begin{aligned}
\mathbb{S} \leq & \frac{y_1 - x_1}{4\alpha} \Big[2 \left(|\Theta'(\mu_1)| + |\Theta'(\nu_1)| \right) H_1 + |\Theta'(y_1)| H_2 - |\Theta'(x_1)| H_1 \\
& - \frac{c(y_1 - x_1)^2}{2} (H_1 - H_3) - 2c \left((\nu_1 - x_1)(x_1 - \mu_1) + (\nu_1 - y_1)(y_1 - \mu_1) \right) H_1 \Big],
\end{aligned}$$

holds for all $x_1, y_1 \in [\mu_1, \nu_1]$.

Proof. The result follows from Theorem 6 with $\varphi(r) = cr^2$. □

Theorem 7. *Let $\alpha, \beta, k > 0$, $x_1 < y_1$, $q > 1$, $p = \frac{q}{1-q}$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a differentiable mapping such that $\Theta' \in L[\mu_1, \nu_1]$ and $|\Theta'|^q$ is an uniformly convex mapping with modulus φ. Then the inequality*

$$\begin{aligned}
\mathbb{S} \leq & \frac{y_1 - x_1}{4}\alpha^{\frac{\beta}{k}} \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1\right)^{\frac{1}{p}} \times \Bigg\{ \Bigg[\frac{|\Theta'|^q(\mu_1) + |\Theta'|^q(\nu_1)}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{|\Theta'|^q(x_1)}{2} \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 + \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\right) \\
& - \frac{|\Theta'|^q(y_1)}{2} \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\right) - \frac{(\nu_1 - x_1)(x_1 - \mu_1)}{(\nu_1 - \mu_1)^2} \varphi(\nu_1 - \mu_1) \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 + \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\right) \\
& - \frac{(\nu_1 - y_1)(y_1 - \mu_1)}{(\nu_1 - \mu_1)^2} \varphi(\nu_1 - \mu_1) \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\right) - \frac{\varphi(\nu_1 - \mu_1)}{2(\nu_1 - \mu_1)^2} \varphi(y_1 - x_1) \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_3\right) \Bigg]^{\frac{1}{q}} \\
& + \Bigg(\frac{|\Theta'|^q(\mu_1) + |\Theta'|^q(\nu_1)}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{|\Theta'|^q(y_1)}{2} \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 + \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\right) - \frac{|\Theta'|^q(x_1)}{2} \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\right) \\
& - \frac{(\nu_1 - y_1)(y_1 - \mu_1)}{(\nu_1 - \mu_1)^2} \varphi(\nu_1 - \mu_1) \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 + \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\right) - \frac{(\nu_1 - x_1)(x_1 - \mu_1)}{(\nu_1 - \mu_1)^2} \varphi(\nu_1 - \mu_1) \Bigg(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 \\
& - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_2\Bigg) - \frac{\varphi(\nu_1 - \mu_1)}{2(\nu_1 - \mu_1)^2} \varphi(y_1 - x_1) \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} H_1 - \frac{1}{\alpha^{\frac{\beta}{k}+1}} H_3\right) \Bigg)^{\frac{1}{q}} \Bigg\},
\end{aligned}$$

holds for all $x_1, y_1 \in [\mu_1, \nu_1]$.

Proof. It follows from Lemma 1 that

$$\begin{aligned}
\mathbb{S} = & \Bigg| \frac{y_1 - x_1}{4}\alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \\
& \times \left[\Theta'\left(\mu_1 + \nu_1 - \left(\frac{2-\gamma}{2}x_1 + \frac{\gamma}{2}y_1\right)\right) - \Theta'\left(\mu_1 + \nu_1 - \left(\frac{\gamma}{2}x_1 + \frac{2-\gamma}{2}y_1\right)\right)\right] d\gamma \Bigg|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{S} \leq & \frac{y_1 - x_1}{4}\alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \\
& \times \left[\left|\Theta'\left(\mu_1 + \nu_1 - \left(\frac{2-\gamma}{2}x_1 + \frac{\gamma}{2}y_1\right)\right) - \Theta'\left(\mu_1 + \nu_1 - \left(\frac{\gamma}{2}x_1 + \frac{2-\gamma}{2}y_1\right)\right)\right|\right] d\gamma \\
\leq & \frac{y_1 - x_1}{4}\alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} d\gamma\right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \left|\Theta'\left(\mu_1 + \nu_1 - \left(\frac{\gamma}{2}x_1 + \frac{2-\gamma}{2}y_1\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}} \\
& + \frac{y_1 - x_1}{4}\alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} d\gamma\right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \left|\Theta'\left(\mu_1 + \nu_1 - \left(\frac{2-\gamma}{2}x_1 + \frac{\gamma}{2}y_1\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|\Theta'|^q$ is uniformly convexity with modulus φ, Theorem 4 asserts that

$$\mathbb{S} \leq \frac{y_1 - x_1}{4}\alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1 - (1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} d\gamma\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\left[|\Theta'|^q(\mu_1)+|\Theta'|^q(\nu_1)-\frac{2-\gamma}{2}|\Theta'|^q(x_1)-\frac{\gamma}{2}|\Theta'|^q(y_1)\right.\right.$$

$$\frac{2\varphi(\nu_1-\mu_1)}{(\nu_1-\mu_1)^2}\left(\frac{2-\gamma}{2}(\nu_1-x_1)(x_1-\mu_1)+\frac{\gamma}{2}(\nu_1-y_1)(y_1-\mu_1)\right)$$

$$\left.\left.-\frac{\gamma(2-\gamma)}{4}\varphi(y_1-x_1)\right]d\gamma\right)^{\frac{1}{q}}+\frac{y_1-x_1}{4}\alpha^{\frac{\beta}{k}}\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}d\gamma\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\left[|\Theta'|^q(\mu_1)+|\Theta'|^q(\nu_1)-\frac{\gamma}{2}|\Theta'|^q(x_1)-\frac{2-\gamma}{2}|\Theta'|^q(y_1)\right.\right.$$

$$\left.\left.-\frac{2\varphi(\nu_1-\mu_1)}{(\nu_1-\mu_1)^2}\left(\frac{\gamma}{2}(\nu_1-x_1)(x_1-\mu_1)+\frac{2-\gamma}{2}(\nu_1-y_1)(y_1-\mu_1)\right)-\frac{\gamma(2-\gamma)}{4}\varphi(y_1-x_1)\right]d\gamma\right)^{\frac{1}{q}}.$$

After some calculations, we get required result. □

Remark 3. Under the assumption of Theorem 7, we can conclude that:
(i) If we set $\varphi(x_1)=0$ in Theorem 7, we get Theorem 2.12 of [12] for $k=1$.
(ii) If we set $\varphi(x_1)=0$, $\mu_1=x_1$ and $\nu_1=y_1$ in Theorem 7, we get Theorem 3.2 of [2].
(iii) If we set $\varphi(x_1)=0$, $\frac{\beta}{k}=1$ $\mu_1=x_1$ and $\nu_1=y_1$ in Theorem 7, we get Theorem 5 of [7].

4. New Ostrowski Mercer type inequality

In this section, we give Mercer-Ostrowski inequalities for conformable integral operator are obtained for a uniformly convex functions. For this purpose, we give a new conformable integral operator identity that will serve as an auxiliary to produce subsequent results for improvements.

Lemma 2. *Suppose that the mapping $\Theta: I=[a,b]\to\Re$ is differentiable on (a,b) with $b>a$. If $\Theta'\in L_1[a,b]$ then for all $x_1,\mu_1,\nu_2\in[a,b]$ and $\alpha,\beta,k>0$, the following identity*

$$\begin{aligned}&\alpha^{\frac{\beta}{k}}(x_1-\mu_1)^2\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\Theta'(x_1+a-(\gamma\mu_1+(1-\gamma)x_1))d\gamma\\&-\alpha^{\frac{\beta}{k}}(\nu_1-x_1)^2\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\Theta'(x_1+b-(\gamma\nu_1+(1-\gamma)x_1))d\gamma\\&=(x_1-\mu_1)\Theta(x_1+a-\mu_1)+(\nu_1-x_1)\Theta(x_1+b-\nu_1)\\&\quad-\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(\nu_1-x_1)^{\frac{\alpha\beta}{k}-1}}\left\{{}^{\beta}_{k}J^{\alpha}_{(x_1+a-\mu_1)^-}\Theta(a)+{}^{\beta}_{k}J^{\alpha}_{(x_1+b-\nu_1)^+}\Theta(b)\right\}.\end{aligned}\tag{19}$$

Proof.

$$I=\alpha^{\frac{\beta}{k}}(x_1-\mu_1)^2\,I_1-\alpha^{\frac{\beta}{k}}(\nu_1-x_1)^2\,I_2,\tag{20}$$

$$\begin{aligned}I_1&=\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\Theta'(x_1+a-(\gamma\mu_1+(1-\gamma)x_1))d\gamma\\&=\frac{\Theta(x_1+a-\mu_1)}{\alpha^{\frac{\beta}{k}}(x_1-\mu_1)}-\frac{\beta}{k(x_1-\mu_1)}\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\gamma)^{\alpha-1}\Theta(x_1+a-(\gamma\mu_1+(1-\gamma)x_1))d\gamma\\&=\frac{\Theta(x_1+a-\mu_1)}{\alpha^{\beta}(x_1-\mu_1)}-\frac{\gamma(\beta+1)}{(x_1-\mu_1)^{\alpha\beta+1}}{}^{b}\beta J^{\alpha}_{(x_1+a-\mu_1)^-}\Theta(a).\end{aligned}$$

Similarly,

$$I_2 = \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^\beta \Theta'\left(x_1 + b - (\gamma\nu_1 + (1-\gamma)x_1)\right) d\gamma$$
$$= \frac{\Theta(x_1+b-\nu_1)}{\alpha^\beta(\nu_1-\mu_1)} - \frac{\gamma(\beta+1)}{(\nu_1-x_1)^{\alpha\beta+1}} {}^\beta J^\alpha_{(x_1+b-\nu_1)^+}\Theta(b).$$

Substitute the values of I_1 and I_2 in 20, we get the required result. □

Corollary 6. *if we set $\alpha = k = 1$ in lemma 2*

$$(x_1-\mu_1)^2 \int_0^1 \gamma^\beta \Theta'\left(x_1 + a - (\gamma\mu_1 + (1-\gamma)x_1)\right) d\gamma - (\nu_1 - x_1)^2 \int_0^1 \gamma^\beta \Theta'\left(x_1 + b - (\gamma\nu_1 + (1-\gamma)x_1)\right) d\gamma$$
$$= (x_1-\mu_1)\Theta(x_1+a-\mu_1) + (\nu_1-x_1)\Theta(x_1+b-\nu_1) - \frac{\Gamma(\beta+1)}{(\nu_1-x_1)^{\beta-1}}\left\{{}^\beta J_{(x_1+a-\mu_1)^-}\Theta(a) + {}^\beta J_{(x_1+b-\nu_1)^+}\Theta(b)\right\}.$$

Remark 4. If we set $\mu_1 = a$, $\nu_1 = b$ and $\alpha = \beta = k = 1$ in Lemma 20, then it reduces to Lemma 1 in [23].

Throughout this section, we will use

$$\mathbb{U} := \Bigg| (x_1-\mu_1)\Theta(x_1+a-\mu_1) + (\nu_1-x_1)\Theta(x_1+b-\nu_1)$$
$$- \frac{\alpha^\beta\gamma(\beta+1)}{(\nu_1-x_1)^{(\alpha\beta-1)}}\left\{{}^\beta_k J^\alpha_{(x_1+a-\mu_1)^-}\Theta(a) + {}^\beta_k J^\alpha_{(x_1+b-\nu_1)^+}\Theta(b)\right\}\Bigg|.$$

Theorem 8. *Let $\alpha, \beta, k > 0$, $a < b$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a differentiable mapping such that $\Theta' \in L[a,b]$ and $|\Theta'|$ is an uniformly convex mapping with modulus φ. Then the inequality holds*

$$\mathbb{U} \le \frac{(x_1-\mu_1)^2}{\alpha}\Big\{ [|\Theta'(x_1)| + |\Theta'(a)|]H_1 - |\Theta'(\mu_1)|[H_1 - H_2] - |\Theta'(x_1)|H_2$$
$$- \varphi(x_1-\mu_1)[H_2-H_3] - \frac{2\varphi(a-x_1)}{(a-x_1)^2}(a-\mu_1)(\mu_1-x_1)[H_1-H_2]\Big\}$$
$$+ \frac{(\nu_1-x_1)^2}{\alpha}\Big\{ [|\Theta'(x_1)| + |\Theta'(b)|]H_1 - |\Theta'(\nu_1)|[H_1 - H_2] - |\Theta'(x_1)|H_2$$
$$- \varphi(x_1-\nu_1)[H_2-H_3] - \frac{2\varphi(b-x_1)}{(b-x_1)^2}(b-\nu_1)(\nu_1-x_1)[H_1-H_2]\Big\}.$$

Proof. It follows from Lemma 2 that

$$\mathbb{U} = \Bigg| (x_1-\mu_1)^2 \alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \Theta'\left(x_1 + a - (\gamma\mu_1 + (1-\gamma)x_1)\right) d\gamma$$
$$- (x_1-\mu_1)^2 \alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \Theta'\left(x_1 + b - (\gamma\nu_1 + (1-\gamma)x_1)\right) d\gamma\Bigg|$$
$$\le (x_1-\mu_1)^2 \alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \left|\Theta'\left(x_1 + a - (\gamma\mu_1 + (1-\gamma)x_1)\right)\right| d\gamma$$
$$- (x_1-\mu_1)^2 \alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \left|\Theta'\left(x_1 + b - (\gamma\nu_1 + (1-\gamma)x_1)\right)\right| d\gamma.$$

Since $|\Theta'|$ is uniformly convexity with modulus φ,

$$\mathbb{U} \le (x_1-\mu_1)^2 \alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} \Big\{ |\Theta'(x_1)| + |\Theta'(a)| - \gamma|\Theta'(\mu_1)| - (1-\gamma)|\Theta'(x_1)|$$

$$- \frac{2\varphi\left(a-x_1\right)}{\left(a-x_1\right)^2}\left(\gamma\left(a-\mu_1\right)\left(\mu_1-x_1\right)+\left(1-\gamma\right)\left(a-x_1\right)\left(x_1-x_1\right)\right)-\gamma\left(1-\gamma\right)\varphi\left(x_1-\mu_1\right)\Big\}d\gamma$$

$$+\left(\nu_1-x_1\right)^2\alpha^{\frac{\beta}{k}}\int_0^1\left(\frac{1-\left(1-\gamma\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\Big\{\left|\Theta'\left(x_1\right)\right|+\left|\Theta'\left(b\right)\right|-\gamma\left|\Theta'\left(\nu_1\right)\right|-\left(1-\gamma\right)\left|\Theta'\left(x_1\right)\right|$$

$$- \frac{2\varphi\left(b-x_1\right)}{\left(b-x_1\right)^2}\left(\gamma\left(b-\nu_1\right)\left(\nu_1-x_1\right)+\left(1-\gamma\right)\left(b-x_1\right)\left(x_1-x_1\right)\right)-\gamma\left(1-\gamma\right)\varphi\left(x_1-\nu_1\right)\Big\}d\gamma.$$

After some calculations, we get required result. □

Corollary 7. *Let $\alpha, \beta, k > 0$, $a < b$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a differentiable mapping such that $\Theta' \in L[\mu_1, \nu_1]$ and $|\Theta'|$ is a strongly convex mapping with modulus c. Then the inequality holds*

$$\begin{aligned}\mathbb{U} \leq &\frac{\left(x_1-\mu_1\right)^2}{\alpha}\Big\{\left[\left|\Theta'\left(x_1\right)\right|+\left|\Theta'\left(a\right)\right|\right]H_1-\left|\Theta'\left(\mu_1\right)\right|\left[H_1-H_2\right]-\left|\Theta'\left(x_1\right)\right|H_2\\
&-c\left(x_1-\mu_1\right)^2\left[H_2-H_3\right]-2c\left(a-\mu_1\right)\left(\mu_1-x_1\right)\left[H_1-H_2\right]\Big\}\\
&+\frac{\left(\nu_1-x_1\right)^2}{\alpha}\Big\{\left[\left|\Theta'\left(x_1\right)\right|+\left|\Theta'\left(b\right)\right|\right]H_1-\left|\Theta'\left(\nu_1\right)\right|\left[H_1-H_2\right]-\left|\Theta'\left(x_1\right)\right|H_2\\
&-c\left(x_1-\nu_1\right)^2\left[H_2-H_3\right]-2c\left(b-\nu_1\right)\left(\nu_1-x_1\right)\left[H_1-H_2\right]\Big\}.\end{aligned}$$

Proof. The result follows from Theorem 8 with $\varphi\left(r\right) = cr^2$. □

Theorem 9. *Let $\alpha, \beta, k > 0$, $a < b$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a differentiable mapping such that $\Theta' \in L[a, b]$ and $|\Theta'|$ is an uniformly convex mapping with modulus φ. Then the inequality holds*

$$\begin{aligned}\mathbb{U} \leq &\left(\frac{H_1}{\alpha^{\frac{\beta}{k}+1}}\right)^{\frac{1}{p}}\frac{\left(x_1-\mu_1\right)^2}{\alpha}\Big\{\left[\left|\Theta'\left(x_1\right)\right|^q+\left|\Theta'\left(a\right)\right|^q\right]H_1-\left|\Theta'\left(\mu_1\right)\right|^q\left[H_1-H_2\right]\\
&-\left|\Theta'\left(x_1\right)\right|^qH_2-\varphi\left(x_1-\mu_1\right)\left[H_2-H_3\right]-\frac{2\varphi\left(a-x_1\right)}{\left(a-x_1\right)^2}\left(a-\mu_1\right)\left(\mu_1-x_1\right)\left[H_1-H_2\right]\Big\}\\
&+\left(\frac{H_1}{\alpha^{\frac{\beta}{k}+1}}\right)^{\frac{1}{p}}\frac{\left(\nu_1-x_1\right)^2}{\alpha}\Big\{\left[\left|\Theta'\left(x_1\right)\right|^q+\left|\Theta'\left(b\right)\right|^q\right]H_1-\left|\Theta'\left(\nu_1\right)\right|^q\left[H_1-H_2\right]\\
&-\left|\Theta'\left(x_1\right)\right|^qH_2-\varphi\left(x_1-\nu_1\right)\left[H_2-H_3\right]-\frac{2\varphi\left(b-x_1\right)}{\left(b-x_1\right)^2}\left(b-\nu_1\right)\left(\nu_1-x_1\right)\left[H_1-H_2\right]\Big\}.\end{aligned}$$

Proof. It follows from Lemma 2 that

$$\begin{aligned}\mathbb{U} =&\Bigg|\left(x_1-\mu_1\right)^2\alpha^{\frac{\beta}{k}}\int_0^1\left(\frac{1-\left(1-\gamma\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\Theta'\left(x_1+a-\left(\gamma\mu_1+\left(1-\gamma\right)x_1\right)\right)d\gamma\\
&-\left(x_1-\mu_1\right)^2\alpha^{\frac{\beta}{k}}\int_0^1\left(\frac{1-\left(1-\gamma\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\Theta'\left(x_1+b-\left(\gamma\nu_1+\left(1-\gamma\right)x_1\right)\right)d\gamma\Bigg|\\
&\leq\left(x_1-\mu_1\right)^2\alpha^{\frac{\beta}{k}}\left(\int_0^1\left(\frac{1-\left(1-\gamma\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}d\gamma\right)^{\frac{1}{p}}\\
&\times\left[\int_0^1\left(\frac{1-\left(1-\gamma\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\left|\Theta'\left(x_1+a-\left(\gamma\mu_1+\left(1-\gamma\right)x_1\right)\right)\right|^qd\gamma\right]^{\frac{1}{q}}\end{aligned}$$

$$+ (x_1 - \mu_1)^2 \alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right)^{\frac{1}{p}}$$
$$\times \left[\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \Theta' \left(x_1 + b - (\gamma \nu_1 + (1-\gamma) x_1) \right) \right|^q d\gamma \right]^{\frac{1}{q}}.$$

Since $|\Theta'|$ is uniformly convexity with modulus φ,

$$\mathbb{U} \leq (x_1 - \mu_1)^2 \alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right)^{\frac{1}{p}}$$
$$\times \Bigg[\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \Big\{ |\Theta'(x_1)|^q + |\Theta'(a)|^q - \gamma |\Theta'(\mu_1)|^q - (1-\gamma) |\Theta'(x_1)|^q$$
$$- \frac{2\varphi(a-x_1)}{(a-x_1)^2} \left(\gamma (a-\mu_1)(\mu_1 - x_1) + (1-\gamma)(a-x_1)(x_1-x_1) \right) - \gamma(1-\gamma) \varphi(x_1 - \mu_1) \Big\} d\gamma \Bigg]^{\frac{1}{q}}$$
$$+ (\nu_1 - x_1)^2 \alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right)^{\frac{1}{p}}$$
$$\Bigg[\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \Big\{ |\Theta'(x_1)|^q + |\Theta'(b)| - \gamma |\Theta'(\nu_1)|^q - (1-\gamma) |\Theta'(x_1)|^q$$
$$- \frac{2\varphi(b-x_1)}{(b-x_1)^2} \left(\gamma (b-\nu_1)(\nu_1 - x_1) + (1-\gamma)(b-x_1)(x_1-x_1) \right) - \gamma(1-\gamma) \varphi(x_1 - \nu_1) \Big\} d\gamma \Bigg]^{\frac{1}{q}}.$$

After some calculations, we get the required result. □

Theorem 10. *Let $\alpha, \beta, k > 0$, $a < b$ and $\Theta : [\mu_1, \nu_1] \to \mathbb{R}$ be a differentiable mapping such that $\Theta' \in L[a,b]$ and $|\Theta'|$ is an uniformly convex mapping with modulus φ. Then the inequality holds*

$$\mathbb{U} \leq \left(\frac{B\left(\frac{p\beta}{k} + 1, \frac{1}{\alpha} \right)}{\alpha^{\frac{p\beta}{k}+1}} \right)^{\frac{1}{p}} \frac{(x_1 - \mu_1)^2}{\alpha} \Big\{ |\Theta'(x_1)|^q + |\Theta'(a)|^q$$
$$- \frac{|\Theta'(\mu_1)|^q - |\Theta'(x_1)|^q}{2} - \frac{\varphi(x_1 - \mu_1)}{6} - \frac{\varphi(a-x_1)}{(a-x_1)^2} (a-\mu_1)(\mu_1 - x_1) \Big\}$$
$$+ \left(\frac{B\left(\frac{p\beta}{k} + 1, \frac{1}{\alpha} \right)}{\alpha^{\frac{p\beta}{k}+1}} \right)^{\frac{1}{p}} \frac{(\nu_1 - x_1)^2}{\alpha} \Big\{ |\Theta'(x_1)|^q + |\Theta'(b)|^q$$
$$- \frac{|\Theta'(\nu_1)|^q - |\Theta'(x_1)|^q}{2} - \frac{\varphi(x_1 - \nu_1)}{6} - \frac{\varphi(b-x_1)}{(b-x_1)^2} (b-\nu_1)(\nu_1 - x_1) \Big\}.$$

Proof. It follows from Lemma 2 and applying Holder inequality, we have

$$\mathbb{U} = \Bigg| (x_1 - \mu_1)^2 \alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \Theta' \left(x_1 + a - (\gamma \mu_1 + (1-\gamma) x_1) \right) d\gamma$$
$$- (x_1 - \mu_1)^2 \alpha^{\frac{\beta}{k}} \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \Theta' \left(x_1 + b - (\gamma \nu_1 + (1-\gamma) x_1) \right) d\gamma \Bigg|$$
$$\leq (x_1 - \mu_1)^2 \alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{p\beta}{k}} d\gamma \right)^{\frac{1}{p}} \left(\left| \Theta' \left(x_1 + a - (\gamma \mu_1 + (1-\gamma) x_1) \right) \right|^q d\gamma \right)^{\frac{1}{q}}$$

$$+\left(x_1-\mu_1\right)^2\alpha^{\frac{\beta}{k}}\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right)^{\frac{1}{p}}\left[\left|\Theta'\left(x_1+b-\left(\gamma\nu_1+(1-\gamma)\,x_1\right)\right)\right|^q d\gamma\right]^{\frac{1}{q}}.$$

Since $|\Theta'|$ is uniformly convexity with modulus φ,

$$\begin{aligned}\mathbb{U}\leq{}&\left(x_1-\mu_1\right)^2\alpha^{\frac{\beta}{k}}\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{p\beta}{k}}\right)^{\frac{1}{p}}\left[\int_0^1\left\{|\Theta'\left(x_1\right)|^q+|\Theta'\left(a\right)|^q-\gamma|\Theta'\left(\mu_1\right)|^q\right.\right.\\&-(1-\gamma)\,|\Theta'\left(x_1\right)|^q-\frac{2\varphi\left(a-x_1\right)}{\left(a-x_1\right)^2}\left(\gamma\left(a-\mu_1\right)\left(\mu_1-x_1\right)+(1-\gamma)\left(a-x_1\right)\left(x_1-x_1\right)\right)\\&\left.\left.-\gamma\,(1-\gamma)\,\varphi\left(x_1-\mu_1\right)\right\}d\gamma\right]^{\frac{1}{q}}+\left(\nu_1-x_1\right)^2\alpha^{\frac{\beta}{k}}\left(\int_0^1\left(\frac{1-(1-\gamma)^\alpha}{\alpha}\right)^{\frac{p\beta}{k}}\right)^{\frac{1}{p}}\\&\times\left[\int_0^1\left\{|\Theta'\left(x_1\right)|^q+|\Theta'\left(b\right)|-\gamma|\Theta'\left(\nu_1\right)|^q-(1-\gamma)\,|\Theta'\left(x_1\right)|^q-\gamma\,(1-\gamma)\,\varphi\left(x_1-\nu_1\right)\right.\right.\\&\left.\left.-\frac{2\varphi\left(b-x_1\right)}{\left(b-x_1\right)^2}\left(\gamma\left(b-\nu_1\right)\left(\nu_1-x_1\right)+(1-\gamma)\left(b-x_1\right)\left(x_1-x_1\right)\right)\right\}d\gamma\right]^{\frac{1}{q}}.\end{aligned}$$

After some calculations, we get the required result. □

Corollary 8. *Let $\alpha,\beta,k>0$, $a<b$ and $\Theta:[\mu_1,\nu_1]\to\mathbb{R}$ be a differentiable mapping such that $\Theta'\in L[\mu_1,\nu_1]$ and $|\Theta'|$ is a strongly convex mapping with modulus c. Then the inequality holds*

$$\begin{aligned}\mathbb{U}\leq{}&\left(\frac{B\left(\frac{p\beta}{k}+1,\frac{1}{\alpha}\right)}{\alpha^{\frac{p\beta}{k}+1}}\right)^{\frac{1}{p}}\frac{\left(x_1-\mu_1\right)^2}{\alpha}\left\{|\Theta'\left(x_1\right)|^q+|\Theta'\left(a\right)|^q\right.\\&\left.-\frac{|\Theta'\left(\mu_1\right)|^q-|\Theta'\left(x_1\right)|^q}{2}-\frac{c\left(x_1-\mu_1\right)^2}{6}-c\left(a-\mu_1\right)\left(\mu_1-x_1\right)\right\}\\&+\left(\frac{B\left(\frac{p\beta}{k}+1,\frac{1}{\alpha}\right)}{\alpha^{\frac{p\beta}{k}+1}}\right)^{\frac{1}{p}}\frac{\left(\nu_1-x_1\right)^2}{\alpha}\left\{|\Theta'\left(x_1\right)|^q+|\Theta'\left(b\right)|^q\right.\\&\left.-\frac{|\Theta'\left(\nu_1\right)|^q-|\Theta'\left(x_1\right)|^q}{2}-\frac{c\left(x_1-\nu_1\right)^2}{6}-c\left(b-\nu_1\right)\left(\nu_1-x_1\right)\right\}.\end{aligned}$$

5. Applications

Proposition 1. *Let $\alpha,\beta>0$ and $a<b$. Then the following inequality holds:*

$$\begin{aligned}\left(\frac{a+b}{2}\right)^2+\frac{\alpha\,(b-a)^2}{16+8\alpha}&\leq\frac{\alpha\,(a+b)\,\Gamma\left(2\beta\right)}{2\beta\left(\alpha+1\right)\left(b-a\right)}\\&\leq\frac{a^2+b^2}{2}-\frac{(b-a)^2}{2\alpha}\left(1-B\left(2,\frac{2}{\alpha}\right)\right).\end{aligned}$$

Proof. Applying Theorem 5 with $\Theta\left(u\right)=\varphi\left(u\right)=u^2$, $a:=x_1=\mu_1$, $b:=y_1=\nu_1$, $\beta=k$ and using this facts that

$$\begin{aligned}{}^{\beta}_{\beta}J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\Theta\left(b\right)&=\frac{(b-a)^\alpha}{\beta\Gamma_\beta\left(\beta\right)2^{\alpha+1}}\left[\frac{b^2-a^2}{\alpha+1}+\frac{(b+a)^2}{2\alpha}+\frac{(b-a)^2}{2\left(\alpha+2\right)}\right],\\{}^{\beta}_{\beta}J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\Theta\left(a\right)&=\frac{(b-a)^\alpha}{\beta\Gamma_\beta\left(\beta\right)2^{\alpha+1}}\left[\frac{b^2-a^2}{\alpha+1}-\frac{(b+a)^2}{2\alpha}-\frac{(b-a)^2}{2\left(\alpha+2\right)}\right],\end{aligned}$$

we obtain

$$\left(\frac{a+b}{2}\right)^2+\frac{\alpha\,(b-a)^2}{16+8\alpha}\leq\frac{\alpha\,(a+b)\,\Gamma\,(2\beta)}{2\beta\,(\alpha+1)\,(b-a)\,\Gamma_\beta\,(\beta)}$$
$$\leq\frac{a^2+b^2}{2}-\frac{(b-a)^2}{2\alpha}\left(1-B\left(2,\frac{2}{\alpha}\right)\right).$$

Since $\Gamma_\beta\,(\beta)=1$, the desired inequality follows from the above inequality. □

Proposition 2. *Let $\alpha,\beta>0$ and $a<b$. Then the following inequality holds:*

$$\left|\frac{\alpha\,(b^2-a^2)\,\Gamma\,(2\beta)}{2\,(\alpha+1)}-\left(\frac{a+b}{2}\right)^2\right|\leq\frac{b-a}{4\alpha}\left[2\,(a+2b)\,B\left(2,\frac{1}{\alpha}\right)+2bB\left(2,\frac{2}{\alpha}\right)-\frac{(b-a)^2}{2}\left(B\left(2,\frac{1}{\alpha}\right)-B\left(2,\frac{3}{\alpha}\right)\right)\right]$$

Proof. Applying Theorem 6 with $\Theta\,(u)=\varphi\,(u)=u^2$, $a:=x_1=\mu_1$, $b:=y_1=\nu_1$ and $\beta=k$. We have

$$\mathbb{S}=\left|\frac{\alpha\,(b^2-a^2)\,\Gamma\,(2\beta)}{2\,(\alpha+1)}-\left(\frac{a+b}{2}\right)^2\right|.$$

Therefore, after some calculus we obtain the desired inequality. □

Proposition 3. *Let $b>a>0$, $\alpha,\beta>0$ and $\alpha\in\mathbb{N}$. Then*

$$\frac{1}{64b^{\frac{3}{2}}}\,(b-a)^2\,B\left(2,\frac{2}{\alpha}\right)-\sqrt{\frac{a+b}{2}}$$
$$\leq\frac{2^\alpha\alpha\Gamma\,(2\beta)}{\beta\,(b-a)^\alpha}\times\left\{\frac{\left(\frac{a+b}{2}\right)^\alpha\sqrt{\frac{a+b}{2}}-b^\alpha\sqrt{b}}{2\alpha+1}-\sum_{j=0}^{\alpha-1}\left(\frac{a+b}{2}\right)^{\alpha-j}\left(\frac{b^j\sqrt{b}-\left(\frac{a+b}{2}\right)^j\sqrt{\frac{a+b}{2}}}{2j+1}\right)\times\frac{(\alpha-1)!\,(2\alpha-j)}{j!\,(\alpha-j)!}\right.$$
$$\left.+(-1)^\alpha\frac{\left(\frac{a+b}{2}\right)^\alpha\sqrt{\frac{a+b}{2}}-a^\alpha\sqrt{a}}{2\alpha+1}+\sum_{j=0}^{\alpha-1}(-1)^j\left(\frac{a+b}{2}\right)^{\alpha-j}\left(\frac{\left(\frac{a+b}{2}\right)^j\sqrt{\frac{a+b}{2}}-a^j\sqrt{a}}{2j+1}\right)\times\frac{(\alpha-1)!\,(2\alpha-j)}{j!\,(\alpha-j)!}\right\}$$
$$\leq\frac{(b-a)^2}{16\alpha b^{\frac{3}{2}}}\left(B\left(2,\frac{2}{\alpha}\right)-1\right)-\frac{\sqrt{a}+\sqrt{b}}{2}.$$

Proof. Applying Corollary 4, with $\Theta\,(u)=-\sqrt{u}$, $a:=x_1=\mu_1$, $b:=y_1=\nu_1$, $\beta=k$ and using this facts that the function $\Theta\,(u)=-\sqrt{u}$ is uniformly convex function with modulus $\varphi\,(u)=\frac{1}{8}b^{-\frac{3}{2}}x^2$, because

$${}^{\beta}_{\beta}J^{\alpha}_{\frac{a+b}{2}^+}\Theta\,(b)=2\frac{\left(\frac{a+b}{2}\right)^\alpha\sqrt{\frac{a+b}{2}}-b^\alpha\sqrt{b}}{(2\alpha+1)\,\beta}-\frac{2}{\beta}\sum_{j=0}^{\alpha-1}\left(\frac{a+b}{2}\right)^{\alpha-j}\left(\frac{b^j\sqrt{b}-\left(\frac{a+b}{2}\right)^j\sqrt{\frac{a+b}{2}}}{2j+1}\right)\times\frac{(\alpha-1)!\,(2\alpha-j)}{j!\,(\alpha-j)!},$$

and

$${}^{\beta}_{\beta}J^{\alpha}_{\frac{a+b}{2}^-}\Theta\,(a)$$
$$=2\,(-1)^\alpha\frac{\left(\frac{a+b}{2}\right)^\alpha\sqrt{\frac{a+b}{2}}-a^\alpha\sqrt{a}}{(2\alpha+1)\,\beta}+\frac{2}{\beta}\sum_{j=0}^{\alpha-1}(-1)^j\left(\frac{a+b}{2}\right)^{\alpha-j}\left(\frac{\left(\frac{a+b}{2}\right)^j\sqrt{\frac{a+b}{2}}}{2j+1}-a^j\sqrt{a}\right)\times\frac{(\alpha-1)!\,(2\alpha-j)}{j!\,(\alpha-j)!}.$$

□

Corollary 9. *Let $b > a > 0$ and $\beta > 0$. Then*

$$\begin{aligned}\frac{(b-a)^2}{64b^{\frac{3}{2}}}+\frac{\sqrt{a}+\sqrt{b}}{2} &\leq \frac{8\Gamma(2\beta)}{\beta(b-a)^2}\times\Big\{-\frac{(a+b)^2}{2}\left(\sqrt{b}+\sqrt{a}\right)+\frac{a+b}{2}\left(b\sqrt{b}-a\sqrt{a}\right)\\ &\quad -\frac{11(a+b)^2\sqrt{2(a+b)}-b^2\sqrt{b}-a^2\sqrt{a}}{20}\Big\}\\ &\leq \sqrt{\frac{a+b}{2}}-\frac{1}{128b^{\frac{3}{2}}}(b-a)^2.\end{aligned}$$

Proof. Setting in Proposition 3 $\alpha = 2$, we get

$$\begin{aligned}&\frac{1}{128b^{\frac{3}{2}}}(b-a)^2-\sqrt{\frac{a+b}{2}}\\ &\leq \frac{8\Gamma(2\beta)}{\beta(b-a)^2}\times\left\{\frac{(a+b)^2\sqrt{\frac{a+b}{2}}-b^2\sqrt{b}}{20}-\frac{(a+b)^2}{2}\left(\sqrt{b}-\sqrt{\frac{a+b}{2}}\right)-\frac{a+b}{2}\left(b\sqrt{b}-\frac{a+b}{2}\sqrt{\frac{a+b}{2}}\right)\right.\\ &\quad\left.+\frac{(a+b)^2\sqrt{\frac{a+b}{2}}-a^2\sqrt{a}}{20}+\frac{(a+b)^2}{2}\left(\sqrt{\frac{a+b}{2}}-\sqrt{a}\right)-\frac{a+b}{2}\left(\frac{a+b}{2}\sqrt{\frac{a+b}{2}}-a\sqrt{a}\right)\right\}\\ &\leq \frac{(b-a)^2}{16\alpha b^{\frac{3}{2}}}\left(B\left(2,\frac{2}{\alpha}\right)-1\right)-\frac{\sqrt{a}+\sqrt{b}}{2}.\end{aligned}$$

After some calculations we obtain the desired inequalities. □

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