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L^∞- error estimate for impulse control QVIs with vanishing control cost

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Abstract: This paper deals with the finite element approximation of the elliptic impulse control quasi-variational inequality (QVI), when the impulse control cost goes to zero. By means of the concepts of subsolutions for QVIs and a Lipschitz dependence property with respect to the impulse cost, an L^∞ error estimate is derived for both the impulse control QVI and the correponding asymptotic problem.

Keywords: Quasi variational inequalities, impulse control cost, finite elements, subsolutions, l^∞- error estimate

MSC: 35J86, 65N30, 65N15.

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1. Introduction

This paper deals with the standard finite element approximation in the L^∞- norm of the impulse quasi-variational inequality (QVI): find $u_k \in H^1(\Omega)$ such that

$$\begin{cases} a(u_k, v - u_k) \geqq (f,\ v - u_k) & \forall v \in H^1(\Omega), \\ u_k \leq k + Mu_k\,,\ v \leq k + Mu_k\,, & u_k \geq 0, \end{cases} \tag{1}$$

as the impulse cost k goes to zero.

Here Ω is a bounded convex domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary, $(.,.)$ corresponds to the inner product in $L^2(\Omega)$, $f \geq 0$ is a function in $L^\infty(\Omega)$, $a(.,.)$ is the bilinear form

$$a(u\,,v) = \int_\Omega \nabla u \nabla v(x) + cuvdx, \tag{2}$$

where c is a positive constant, and M is the nonlinear operator, defined from $L^\infty(\Omega)$ into itself, by

$$M\varphi(x) = \inf \varphi(x+\xi);\, x \in \Omega,\, \xi \geq 0;\, x + \xi \in \bar{\Omega}, \tag{3}$$

and k is a positive constant.

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h > 0$ is the mesh size.We assume that the family τ_h is regular and quasi-uniform; let also $\mathbb{V}_h$ denote the finite element space consisting of continuous piecewise linear functions, $\{\varphi_s\}$, $s = 1, 2, ...m(h)$ the basis of $\mathbb{V}_h$, and r_h the usual corresponding Lagrange interpolation operator.

The discrete counterpart of (1) consists of seeking $u_{k,h} \in \mathbb{V}_h$ such that

$$\begin{cases} a(u_{k,h}, v - u_{k,h}) \geqq (f, v - u_{k,h})\ \forall v \in \mathbb{V}_h, \\ v \leq k + r_h M u_{k,h},\ u_{k,h} \leq k + r_h M u_{k,h}. \end{cases} \tag{4}$$

In this paper, under a $W^{1,\infty}$- uniform regularity assumption, we derive error estimate for the QVI (1)

$$\left\| u_k - u_{k,h} \right\|_\infty \leq Ch,$$

and for the asymptotic problem

$$\|u_0 - u_{0,h}\|_\infty \leq Ch, \tag{5}$$

where u_0 is the solution of the continuous asymptotic QVI (as $k \to 0$), that is

$$\begin{cases} a(u_0, v - u_0) \geqq (f,\ v - u_0)\ \forall v \in H^1(\Omega), \\ u_0 \leq Mu_0\,,\ v \leq Mu_0\,,\ u_0 \geq 0, \end{cases} \tag{6}$$

(see [1]), $u_{0,h}$ is the solution of the discrete asymptotic QVI (as $k \to 0$), that is,

$$\begin{cases} a(u_{0,h}, v - u_{0,h}) \geqq (f,\ v - u_{0,h})\ \forall v \in H^1(\Omega), \\ u_{0,h} \leq r_h M u_{0,h}\,,\ v \leq r_h M u_{0,h},\ u_{0,h} \geq 0, \end{cases} \tag{7}$$

and C is a constant independent of both h and k.

To achieve this, we develop an approach that, in both the continuous and discrete settings, combines two key ingredients: a characterization of the QVI solution as the upper envelope of the set of subsolutions, and a Lipschitz continuity property with respect to the impulse cost.

It's worth mentioning that the approximation approach developped in this paper is completely different from the one in [2].

The remainder of the paper is organized as follows. §2 recalls several qualitative properties of the solution to the continuous QVI (1), establishes its Lipschitz dependence with respect to the impulse cost, and introduces the continuous asymptotic QVI. §3 presents the corresponding discrete theory: assuming a discrete maximum principle, we derive analogous qualitative properties for the solution of the discrete QVI (4), and state the discrete asymptotic QVI. Finally, in §4, we give a detailed analysis of the approximation, and convergence order.

2. Preliminaries

2.1. Elliptic variational inequality (VI)

Let $g \in L^\infty(\Omega)$ and $\psi \in W^{1,\infty}(\Omega)$ such that $\partial\psi/\partial n \geq 0$ on $\partial\Omega$. The following problem is called an elliptic variational inequality (VI): find $\omega = \sigma(\psi) \in H^1(\Omega)$ such that

$$\begin{cases} a(\omega, v - \omega) \geqq (g, v - \omega)\ \forall v \in H^1(\Omega), \\ v \leq \psi\,,\ \omega \leq \psi\,. \end{cases} \tag{8}$$

Theorem 1. [3] *Let ψ and $\tilde{\psi}$ in $W^{1,\infty}(\Omega)$.Then, we have*

$$\|\omega - \tilde{\omega}\|_\infty \leq \|\psi - \tilde{\psi}\|_\infty\,,$$

The discrete analog of VI (8) is the VI: find $\omega_h = \sigma_h(\psi) \in \mathbb{V}_h$ such that

$$\begin{cases} a(\omega_h, v - \omega_h) \geqq (g, v - \omega_h)\ \ \forall v \in \mathbb{V}_h, \\ v \leq r_h\psi\,,\ \omega_h \leq r_h\psi. \end{cases} \tag{9}$$

We assume that the matrix with generic coefficients $a(\varphi_l, \varphi_s)$ is an M-Matrix.This is the so-called discrete maximum principle (**d.m.p**) [4].

Theorem 2. *Under the d.m.p, we have*

$$\|\omega_h - \tilde{\omega}_h\|_\infty \leq \|\psi - \tilde{\psi}\|_\infty\,.$$

Theorem 3. [5] *Let $\psi \in W^{1,\infty}(\Omega)$ and $\omega \in W^{1,\infty}(\Omega)$, then*

$$\|\omega - \omega_h\|_\infty \leq Ch.$$

3. The continuous QVI

Next, we give some qualitative results for the solution of the QVI (1). Indeed, starting from $\hat{u}^0 \in H^1(\Omega)$,the unique solution of the equation

$$a(\hat{u}^0, v) = (f, v) \ \forall v \in H^1(\Omega). \tag{10}$$

We define the continuous sequence $\left(\hat{u}_k^n\right)$ such that $\hat{u}_k^n \in H^1(\Omega)$ solves the variational inequality (VI)

$$\begin{cases} a(\hat{u}_k^n, v - \hat{u}_k^n) \geq (f, v - \hat{u}_k^n) \ \forall v \in H^1(\Omega), \\ \hat{u}_k^n \leq k + M\hat{u}_k^{n-1}, v \leq k + M\hat{u}_k^{n-1}. \end{cases} \tag{11}$$

Similarly, starting from $\breve{u}^0 = 0$, we define the continuous sequence $\left(\breve{u}_k^n\right)$ such that $\breve{u}_k^n \in H^1(\Omega)$ solves the variational inequality (VI)

$$\begin{cases} a(\breve{u}_k^n, v - \breve{u}_k^n) \geq (f, v - \breve{u}_k^n) \ \forall v \in H^1(\Omega), \\ \breve{u}_k^n \leq k + M\breve{u}_k^{n-1}, v \leq k + M\breve{u}_k^{n-1}. \end{cases} \tag{12}$$

Note that due to standard comparison results in elliptic variational inequalities, the sequences $\left(\hat{u}_k^n\right)$ and $\left(\breve{u}_k^n\right)$ are decreasing and increasing, respectively.

Theorem 4. [6] *Assume that $f \geq f_0 > 0$. Then, the sequences$\left(\hat{u}_k^n\right)$ and $\left(\breve{u}_k^n\right)$ defined in (11) and (12), respectively, converge decreasingly and increasingly to the unique solution u_k of QVI. (1).*

Definition 1. [6] A function $w_k \in H^1(\Omega)$ is said to be a subsolution for QVI (1) if

$$\begin{cases} a(w_k, v) \ \leq \ (f, v) \ \ \forall v \in H^1(\Omega), \ v \geq 0, \\ w_k \leq k + Mw_k, \ \ v \leq k + Mw_k. \end{cases} \tag{13}$$

Let $\mathbb{X}$ denote the set of such subsolutions.

Theorem 5. [6] *The solution of QVI (1) is the maximum element of* $\mathbb{X}$.

Notation 1. Let k, $\tilde{k}$ be two positive parameters and $u_k = \partial(k)$, $u_{\tilde{k}} = \partial(\tilde{k})$ be the corresponding solutions to QVI (1), respectively.

Lemma 1 (Monotonicity property). *$k \geq \tilde{k}$ implies $u_k \geq u_{\tilde{k}}$.*

Proof. We proceed by induction. For $n = 0$, since M is nondecreasing and $k \geq \tilde{k}$, we have $k + M\hat{u}_k^0 \geq \tilde{k} + M\hat{u}_k^0$, which implies that $\hat{u}_k^1 \geq \hat{u}_{\tilde{k}}^1$.Let us now assume that $\hat{u}_k^{n-1} \geq \hat{u}_{\tilde{k}}^{n-1}$. Then, as $k \geq \tilde{k}$, we have $k + M\hat{u}_k^{n-1} \geq \tilde{k} + M$ $\hat{u}_{\tilde{k}}^{n-1}$, and making use of standard comparison results in elliptic VIs, we get $\hat{u}_k^n \geq \hat{u}_{\tilde{k}}^n$. Hence, due to Theorem 1, passing to the limit, as $n \to \infty$, it follows that $u_k \geq u_{\tilde{k}.}$ □

Theorem 6 (Lipschitz property). *Under conditions of Lemma 1, we have*

$$\left\| u_k - u_{\tilde{k}} \right\|_\infty \leq \left| k - \tilde{k} \right|. \tag{14}$$

Proof. Set

$$\Phi = \left| k - \tilde{k} \right|.$$

Then, since

$$\begin{aligned} k &\leq \tilde{k} + \left| k - \tilde{k} \right| \\ &\leq \tilde{k} + \Phi, \end{aligned}$$

applying Lemma 1, we get

$$\partial(k) \leq \partial(\tilde{k} + \Phi).$$

But, $\partial(\tilde{k} + \Phi)$ being the solution of QVI with source term f and impulse cost $\tilde{k} + \Phi$, and $\partial(\tilde{k}) + \Phi$ being the solution of QVI with source term $f + c.\Phi$ and impulse cost $\tilde{k} + \Phi$, we have, $f + c\Phi \geq f$, and hence, by standard comparison result for elliptic QVIs, $\partial(\tilde{k} + \Phi) \leq \partial(\tilde{k}) + \Phi$. Thus,

$$\partial(k) \leq \partial(\tilde{k}) + \Phi.$$

In a similar way we can also get

$$\partial(\tilde{k}) \leq \partial(k) + \Phi,$$

which completes the proof. □

Corollary 1. *we have*

$$\|u_k - u_0\|_\infty \leq k. \tag{15}$$

Proof. It suffices to take the limit on (14), as $\tilde{k} \to 0$. □

4. The discrete QVI

4.1. Qualitative properties

Next, we shall give analog qualitative properties for the discrete QVI (4).Their respective proofs will be omitted as they are similar to those of their continuous analogues. Indeed, starting from u_h^0, the unique solution to the equation

$$a(\hat{u}_h^0, v) = (f, v)\ \forall v \in \mathbb{V}_h,$$

we define the continuous sequence $\left(\hat{u}_{k,h}^n\right)$ such that $\hat{u}_{k,h}^n \in \mathbb{V}_h$ solves the discrete variational inequality (VI)

$$\begin{cases} a(\hat{u}_{k,h}^n, v - u_{k,h}^n) \geq (f, v - \hat{u}_{k,h}^n)\ \forall v \in \mathbb{V}_h, \\ \hat{u}_{k,h}^n \leq k + r_h M\hat{u}_{k,h}^{n-1}, v \leq k + r_h M\hat{u}_{k,h}^{n-1}. \end{cases} \tag{16}$$

Likewise, starting from $\check{u}_h^0 = 0$, we define the continuous sequence $(\check{u}_{k,h}^n)$ such that $\check{u}_{k,h}^n \in \mathbb{V}_h$ solves the discrete variational inequality (VI)

$$\begin{cases} a(\check{u}_{k,h}^n, v - \check{u}_h^n) \geq (f, v - \check{u}_{k,h}^n)\ \forall v \in \mathbb{V}_h, \\ \check{u}_{k,h}^n \leq k + r_h M\check{u}_{k,n-1h}^n, v \leq k + r_h M\check{u}_{k,n-1h}^n. \end{cases} \tag{17}$$

Note that, due to standard comparison results in discrete elliptic variational inequalities, the sequences $(\hat{u}_h^n)$ and $(\check{u}_h^n)$ are decreasing and increasing, respectively.

Theorem 7. [2] *Let the d.m.p hold. Then, the sequences* $\left(\hat{u}_{k,h}^n\right)$ *and* $(\check{u}_{k,h}^n)$ *defined in* (16) *and* (17)*, respectively, converge decreasingly and increasingly to the unique solution* u_h *of QVI* (4)*.*

Definition 2. A function $w_h \in \mathbb{V}_h$ is said to be a subsolution for the QVI (4) if

$$\begin{cases} a(w_{k,h}, v)\ \leq\ (f, v)\ \ \forall \varphi_s, \\ w_{k,h} \leq k + r_h M w_h,\ v \leq k + r_h M w_h. \end{cases} \tag{18}$$

Let $\mathbb{X}_h$ denote the set of such subsolutions.

Theorem 8. [2] *The solution of QVI* (4) *is the maximum element of* $\mathbb{X}_h$*.*

Notation 2. Let $k, \tilde{k}$ be two positive parameters and $u_{k,h} = \partial_h(k)$, $u_{\tilde{k},h} = \partial_h(\tilde{k})$ be the corresponding solutions to (4), respectively.

Lemma 2 (Monotonicity property)**.** *Let the d.m.p hold.Then $k \geq \tilde{k}$ implies that $u_{k,h} \geq u_{\tilde{k},h}$.*

Theorem 9 (Lipschitz property)**.** *Uder conditions of Lemma 2, we have*

$$\left\| u_{k,h} - u_{\tilde{k},h} \right\|_{\infty} \leq \left| k - \tilde{k} \right|. \tag{19}$$

Corollary 2. *We have*

$$\left\| u_{k,h} - u_{0,h} \right\|_{\infty} \leq k. \tag{20}$$

5. Error analysis

This section is devoted to demonstrate the proposed approximation method converges in the $L^{\infty}(\Omega)$ norm. The proof stands on the construction of an appropriate subsolution for each of the continuous and discrete QVIs, respectively. From now on, C will denote a constant independent of both h and k).

5.1. Error estimate for the QVI (1)

Theorem 10. *We have*

$$\left\| u_k - u_{k,h} \right\|_{\infty} \leq Ch.$$

The proof stands on the following two fundamental lemmas.

Lemma 3 (Construction of a discrete subsolution)**.** *Assume that*

$$\max\left\{ \|Mu_k\|_{W^{1,\infty}(\Omega)},\ \|u_k\|_{W^{1,\infty}(\Omega)} \right\} \leq C. \tag{21}$$

Then, there exists a function $\alpha_{k,h}$ such that:

$$\begin{cases} \alpha_{k,h} \leq u_{k,h}, & \text{and} \\ \left\| \alpha_{k,h} - u_k \right\|_{\infty} \leq Ch. \end{cases}$$

Proof. Let us first, just for the sake of simplicity, adopt the notations:

$$u = u_k,\ u_h = u_{k,h}.$$

Now, let $\bar{u}_h$ be the solution of the discrete variational inequality (VI)

$$\begin{cases} a(\bar{u}_h, v - \bar{u}_h) \geqq (f, v - \bar{u}_h) \ \ \forall v \in \mathbb{V}_h, \\ \bar{u}_h \leq k + r_h Mu,\ v \leq k + r_h Mu. \end{cases} \tag{22}$$

where u is the solution of (1).Then, under $W^{1,\infty}$-regularity (21)of u and Mu, using standard L^{∞} error estimate for VIs [5], we get

$$\|u - \bar{u}_h\|_{\infty} \leq Ch. \tag{23}$$

Now, since $\bar{u}_h$ is solution to the discrete VI (22), it is also a discrete subsolution, that is,

$$\begin{cases} a(\bar{u}_h, \varphi_s) \leq (f, \varphi_s) \ \ \forall s = 1, 2, ..., m(h), \\ \bar{u}_h \leq k + r_h Mu. \end{cases}$$

On the other hand, since M and r_h are both Lipschitz with constant equal to 1, using (23), yields

$$\begin{aligned}\bar{u}_h &\leq k + r_h M u - r_h M \bar{u}_h + r_h M \bar{u}_h \\ &\leq \|k + r_h M u - r_h M \bar{u}_h\|_\infty + r_h M \bar{u}_h \\ &\leq k + r_h M \bar{u}_h \; + \|u - \bar{u}_h\|_\infty \\ &\leq k + Ch \; + r_h M \bar{u}_h \,.\end{aligned}$$

So, $\bar{u}_h$ is a subsolution for the QVI with impulse cost $\tilde{k} = k + Ch$

Let $\bar{U}_h$ be the solution of such a QVI, that is, $\bar{U}_h = \partial_h(k + Ch)$.Then, as $u_h = \partial_h(k)$, making use of Theorem 9, we get

$$\begin{aligned}\|u_h - \bar{U}_h\|_\infty &\leq C\,|k - (k + Ch)| \\ &\leq Ch.\end{aligned}$$

Hence, due to Theorem 8, we have

$$\bar{u}_h \leq \bar{U}_h \leq u_h + Ch,$$

and putting

$$\alpha_h = \bar{u}_h - Ch,$$

we get

$$\alpha_h \leq u_h.$$

Finally, using (23), we obtain

$$\begin{aligned}\|\alpha_h - u\|_\infty &\leq \|\bar{u}_h - u\|_\infty + Ch \\ &\leq Ch.\end{aligned}$$

□

Lemma 4 (Construction of a continuous subsolution). *Assume that*

$$\max\left\{\left\|M u_{k,h}\right\|_{W^{1,\infty}(\Omega)},\ \left\|u_{k,h}\right\|_{W^{1,\infty}(\Omega)}\right\} \leq C. \tag{24}$$

Then, there exists a function $\beta_k^{(h)}$ *such that:*

$$\begin{cases} \beta_k^{(h)} \leq u_k, \qquad \textit{and} \\ \left\|\beta_k^{(h)} - u_{k,h}\right\|_\infty \leq Ch. \end{cases}$$

Proof. Let $\bar{u}$, be the solution of the continuous variational inequality

$$\begin{cases} a(\bar{u}, v - \bar{u}) \geqq (f, v - \bar{u})\,\forall v \in H^1(\Omega)\,, \\ \bar{u} \leq k + M u_h,\, v \leq k + M u_h, \end{cases} \tag{25}$$

with obstacle $\psi = k + M u_h$. It is clear that $u_h = \sigma_h(k + M u_h)$ is nothing but the approximation of $\bar{u} = \sigma(k + M u_h)$. Hence, as $\|\psi\|_{W^{1,\infty}(\Omega)} \leq C$, making use of standard L^∞ - error estimate for VIs [5], we get,

$$\|\bar{u} - u_h\|_\infty \leq Ch. \tag{26}$$

Now, as the solution of VI (25) is a subsolution, we have

$$\begin{cases} a(\bar{u}, v) \leq (f, v) \;\; \forall v \in H^1(\Omega), v > 0, \\ \bar{u} \leq k + M u_h\,,\;\; v \leq k + M u_h, \end{cases}$$

but, as

$$\begin{aligned}\bar{u} &\leq k + Mu_h - M\bar{u} + M\bar{u} \\ &\leq k + \|Mu_h - M\bar{u}\|_\infty + M\bar{u} \\ &\leq k + \|u_h - \bar{u}\|_\infty + M\bar{u} \\ &\leq k + Ch + M\bar{u},\end{aligned}$$

it follows that $\bar{u}$ is a subsolution for the QVI with impulse cost $\tilde{k} = k + Ch$.

Denote by $\bar{U}$ the solution of such a QVI, that is, $\bar{U} = \partial\,(k + Ch)$. Then, as $u = \partial(k)$, making use of Theorem 6, we have

$$\begin{aligned}\|u - \omega\|_\infty &\leq C\,|k - (k + Ch)| \\ &\leq Ch\end{aligned}$$

Hence, making use of Theorem 5, we obtain

$$\bar{u} \leq \bar{U} \leq u + Ch.$$

Now, putting

$$\beta^{(h)} = \bar{u} - Ch,$$

we clearly have

$$\beta^{(h)} \leq u.$$

Finally, using (26), we obtain

$$\begin{aligned}\left\|\beta^{(h)} - u_h\right\|_\infty &\leq \|\bar{u} - Ch - u_h\|_\infty \\ &\leq Ch,\end{aligned}$$

which completes the proof. □

Now, combining Lemmas 3 and 4, we are in position to derive the main result, we are in a position to prove Theorem 10.

Proof. Indeed, making use of both Lemmas 3 and 4, we have

$$\begin{aligned}u_{k,h} &\leq \beta_k^{(h)} + Ch \\ &\leq u_k + Ch \\ &\leq \alpha_{k,h} + Ch\end{aligned}$$

Thus

$$\left\|u_k - u_{k,h}\right\|_\infty \leq Ch,$$

which completes the proof. □

Corollary 3. *We have*

$$\left\|u_0 - u_{0,h}\right\|_\infty \leq Ch.$$

Proof. Using Theorem 10, and Corollaries 1 and 2, we get

$$\begin{aligned}\left\|u_0 - u_{0,h}\right\|_\infty &\leq \|u_0 - u_k\|_\infty + \left\|u_k - u_{k,h}\right\|_\infty + \left\|u_{k,h} - u_{0,h}\right\|_\infty \\ &\leq Ck + Ch + Ck.\end{aligned}$$

Thus, passing to the limit, as $k \to 0$, the desired result follows. □

6. Conclusion

We have derived maximum error estimates of the standard finite element approximation of elliptic impulse control quasi variational inequalities (QVIs) with vanishing control cost.For this purpose, we developped a method that stands, in both the continuous settings, on intrinsic qualitative properties of the QVI like the continuous dependence with respect to the control cost and the characterization of the solution of the QVI as the least upper bound of the set of subsolutions.

References

[1] Perthame, B. (1988).Vanishing impulse cost in the quasi-variational inequality for ergodic impulse control. *Asymptotic Analysis*, *1*(1), 13-21.

[2] Cortey-Dumont, P. (1980). Approximation numérique d'une inéquation quasi variationnelle liée à des problèmes de gestion de stock. RAIRO. *Analyse Numérique*, *14*(4), 335-346.

[3] Bensoussan, A., & Lions, J. L. (2011). *Applications of Variational Inequalities in Stochastic Control* (Vol. 12). Elsevier.

[4] Lu, C., Huang, W., & Qiu, J. (2014). Maximum principle in linear finite element approximations of anisotropic diffusion–convection–reaction problems. *Numerische Mathematik*, *127*(3), 515-537.

[5] Cortey-Dumont, P. (1985). On the finite element approximation in the L^∞ norm of variational inequalities with nonlinear operators. *Numerish Mathematik*, *47*(1), 45-57.

[6] Bensoussan, A., & Lions, J. L. (1984). *Impulse Control and Quasi-Variational Inequalities*. Gauthiers-Villars, Paris.