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On a linear transformation of the Cauchy and related inequalities

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Abstract: By examining the properties of a certain linear transformation of functionals, we present applications of Cauchy, Aczel, Callebaut, and Beckenbach type inequalities. Additionally, we provide results for complex functionals.

Keywords: Cauchy inequality, Callebaut inequality, Aczel inequality, Beckenbach inequality, Hölder's inequality

MSC: 26D15, 26D07, 26A51.

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1. Introduction and preliminary results

Let E be a nonempty set and L a linear class of real-valued functions on E. We consider positive linear functionals $B : L \to \mathbb{R}$ such that

$$B(a\mathrm{f} + b\mathrm{g}) = aB(\mathrm{f}) + bB(\mathrm{g}), \quad f \geq 0 \ \Rightarrow \ B(\mathrm{f}) \geq 0, \quad B(1) = 1.$$

The Cauchy inequality [1, p. 113] is our starting point.

Theorem 1. *If $B : L \to \mathbb{R}$ is a positive linear functional, then*

$$B^2(\mathrm{fg}) \leq B\left(\mathrm{f}^2\right) B\left(\mathrm{g}^2\right), \tag{1}$$

for all real functions $\mathrm{f}^2, \mathrm{g}^2, f\mathrm{g} \in L$.

2. Main results

Lemma 1. *If $B : L \to \mathbb{R}$ is a positive linear functional, then for $u : E \to \mathbb{R}, v : E \to \mathbb{R}, u, v, uv \in L$, we have*

$$B[(2B(u) - u)(2B(v) - v)] = B(uv). \tag{2}$$

Proof.

$$\begin{aligned} B[(2B(u) - u)(2B(v) - v)] &= B[4B(u)B(v) - 2B(u)v - 2B(v)u + uv] \\ &= 4B(u)B(v) - 2B(u)B(v) - 2B(v)B(u) + B(uv) \\ &= B(uv). \end{aligned}$$

□

Lemma 2. *Let $B : L \to \mathbb{R}$ is a positive linear functional.*

(i) *For* $\mathrm{g} : E \to \mathbb{R}$, g, $\mathrm{g}^2 \in L$, *we have*

$$B\left((2B(\mathrm{g})-\mathrm{g})^2\right)=B(\mathrm{g}^2). \tag{3}$$

(ii) *Cauchy inequality is invariant under substitutions*

$$f \to 2B(\mathrm{f})-\mathrm{f},\ g \to 2B(\mathrm{g})-g.$$

Theorem 2. *If $B : L \to \mathbb{R}$ is a positive linear functional, then for all real functions $\mathrm{f}, \mathrm{g}, \mathrm{fg}, \mathrm{f}^2, \mathrm{g}^2 \in L$, we have*

$$|2B(\mathrm{f})B(\mathrm{g})-B(\mathrm{fg})| \le B^{\frac{1}{2}}\left(\mathrm{f}^2\right)B^{\frac{1}{2}}\left(\mathrm{g}^2\right). \tag{4}$$

Proof. We use the Cauchy inequality (1) and identity (23)

$$\begin{aligned}|2B(\mathrm{f})B(\mathrm{g})-B(\mathrm{fg})| &= |B(\mathrm{f}(2B(\mathrm{g})-\mathrm{g}))| \\ &\le B^{\frac{1}{2}}\left(\mathrm{f}^2\right)B^{\frac{1}{2}}\left((2B(\mathrm{g})-\mathrm{g})^2\right) \\ &= B^{\frac{1}{2}}\left(\mathrm{f}^2\right)B^{\frac{1}{2}}\left(\mathrm{g}^2\right).\end{aligned}$$

□

Remark 1. A direct consequence of inequality (2) is

$$B(\mathrm{f})B(\mathrm{g}) \le \frac{B^{\frac{1}{2}}\left(\mathrm{f}^2\right)B^{\frac{1}{2}}\left(\mathrm{g}^2\right)+B(\mathrm{fg})}{2}.$$

Corollary 1. *Assume $p_i \in \mathbb{R}_+$, $\sum\limits_{i=1}^{n}\frac{1}{p_i}=\frac{1}{2}$, and $g, \mathrm{g}^2, f_i, f_i^2, f_i^{p_i} \in L$, $i=1,\dots,n, g\prod_{i=1}^{n} f_i \in L$. Then we have*

$$\left|2B\left(\prod_{i=1}^{n} f_i\right)B(\mathrm{g})-B\left(\prod_{i=1}^{n} f_i g\right)\right| \le \prod_{i=1}^{n} B^{\frac{1}{p_i}}\left(\mathrm{f}_i^{p_i}\right)B^{\frac{1}{2}}\left(\mathrm{g}^2\right).$$

Proof. We use (2) and Hölder inequality [2, p. 103] with $p_i \to \frac{2}{p_i}$

$$\begin{aligned}\left|2B\left(\prod_{i=1}^{n} f_i\right)B(\mathrm{g})-B\left(\prod_{i=1}^{n} f_i g\right)\right| &\le B^{\frac{1}{2}}\left(\left(\prod_{i=1}^{n} f_i\right)^2\right)B^{\frac{1}{2}}\left(\mathrm{g}^2\right) \\ &= B^{\frac{1}{2}}\left(\prod_{i=1}^{n} f_i^2\right)B^{\frac{1}{2}}\left(\mathrm{g}^2\right) \\ &\le \prod_{i=1}^{n} B^{\frac{1}{p_i}}\left(\mathrm{f}_i^{p_i}\right)B^{\frac{1}{2}}\left(\mathrm{g}^2\right).\end{aligned}$$

□

2.1. Conversions

The next Theorem contains conversions of inequality (2), similar to Cauchy conversions for isotonic linear functionals (see [3, p. 14]).

Theorem 3. *Under the assumptions of Theorem 2, if $0 < m < M < \infty$ and*

$$0 < m(2B(\mathrm{g})-\mathrm{g}(x)) \le f(x) \le M(2B(\mathrm{g})-\mathrm{g}(x)),\ x \in E.$$

Then

$$(m+M)(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg}))\geq B\left(\mathsf{f}^2\right)+mMB\left(\mathsf{g}^2\right), \tag{i}$$

$$2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg})\geq 2\frac{\sqrt{mM}}{m+M}B^{\frac{1}{2}}\left(\mathsf{f}^2\right)B^{\frac{1}{2}}\left(\mathsf{g}^2\right), \tag{ii}$$

$$\frac{(\sqrt{M}-\sqrt{m})^2}{2\sqrt{mM}}(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg}))\geq B^{\frac{1}{2}}\left(\mathsf{f}^2\right)B^{\frac{1}{2}}\left(\mathsf{g}^2\right)-(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg})). \tag{iii}$$

$$\frac{(M-m)^2}{4mM}(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg}))^2\geq B\left(\mathsf{f}^2\right)B\left(\mathsf{g}^2\right)-(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg}))^2 \tag{iv}$$

$$B\left(\mathsf{f}^2\right)B\left(\mathsf{g}^2\right)-(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg}))^2\leq\frac{1}{4}(M-m)^2B\left(\mathsf{g}^2\right)^2. \tag{v}$$

Proof. (i) With substitution $g_1=2B(\mathsf{g})-\mathsf{g}$ we have

$$mg_1(x)\leq f(x)\leq Mg_1(x),\ x\in E, \tag{5}$$

and we get

$$\left(Mg_1(x)-\mathsf{f}(x)\right)\left(\mathsf{f}(x)-mg_1(x)\right)\geq 0,\ x\in E, \tag{6}$$

and then

$$Mf(x)g_1(x)-mMg_1^2(x)+mf(x)g_1(x)-\mathsf{f}^2(x)\geq 0,$$

and

$$(m+M)B\left(\mathsf{fg}_1\right)-mMB\left(\mathsf{g}_1^2\right)-B\left(\mathsf{f}^2\right)\geq 0.$$

Since $B(\mathsf{g}_1^2)=B(\mathsf{g}^2)$, $B(\mathsf{g}_1 f)=2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg})$ we proved (i).

(ii) The inequality we can write as

$$2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg})\geq\frac{B\left(\mathsf{f}^2\right)+mMB\left(\mathsf{g}^2\right)}{m+M}, \tag{7}$$

and (ii) follows after application AM-GM inequality on the numerator in (7).

(iii) From (ii) we have $\frac{m+M}{2\sqrt{mM}}(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg}))\geq B^{\frac{1}{2}}\left(\mathsf{f}^2\right)B^{\frac{1}{2}}\left(\mathsf{g}^2\right)$ and after subtracting $2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg})$ we get (iii).

(iv) After subtraction $(2B(\mathsf{f})B(\mathsf{g})-B(\mathsf{fg}))^2$ in (7) we get result.

(v) If we act with A on (5) we get

$$mB\left(\mathsf{g}_1^2\right)\leq B\left(\mathsf{fg}_1\right)\leq MB\left(\mathsf{g}_1^2\right),$$

and then

$$I_1:=\left(MB\left(\mathsf{g}_1^2\right)-B\left(\mathsf{fg}_1\right)\right)\left(B\left(\mathsf{fg}_1\right)-mB\left(\mathsf{g}_1^2\right)\right)\geq 0.$$

On the other hand, if we act with A on (6) we get

$$\begin{aligned}0\leq I_2&:=B\left(\mathsf{g}_1^2\right)B\left[\left(Mg_1-\mathsf{f}\right)\left(\mathsf{f}-mg_1\right)\right]\\&=B\left(\mathsf{g}_1^2\right)\left(MB\left(\mathsf{fg}_1\right)-MmB\left(\mathsf{g}_1^2\right)-B\left(\mathsf{f}^2\right)+mB\left(\mathsf{fg}_1\right)\right).\end{aligned}$$

Now

$$I_1-I_2=B\left(\mathsf{f}^2\right)B\left(\mathsf{g}_1^2\right)-B^2\left(\mathsf{fg}_1\right).$$

If we denote

$$U=MB\left(\mathsf{g}_1^2\right)-B\left(\mathsf{fg}_1\right)\ \text{and}\ V=B\left(\mathsf{fg}_1\right)-mB\left(\mathsf{g}_1^2\right),$$

we conclude

$$I_1 - I_2 \leq I_1 = UV \leq \frac{(U+V)^2}{4},$$

and the result follows.

□

2.2. Inequalities of Aczel, Callebaut and Beckenbach type

In the proof of next result, we need the following lemma (see [1, p, 118]).

Lemma 3. *If* $1 \leq \alpha \leq \beta \leq 2$ *or* $\beta \leq \alpha \leq 1$, *then*

$$B(\mathsf{fg})^2 \leq B\left(\mathsf{f}^\alpha \mathsf{g}^{2-\alpha}\right) B\left(\mathsf{f}^{2-\alpha}\mathsf{g}^\alpha\right) \leq B\left(\mathsf{f}^\beta \mathsf{g}^{2-\beta}\right) B\left(\mathsf{f}^{2-\beta}\mathsf{g}^\beta\right) \leq B\left(\mathsf{f}^2\right) B\left(\mathsf{g}^2\right).$$

Theorem 4. *Let* $B : L \to \mathbb{R}$ *is a positive linear functional. Let* $\mathsf{f}, \mathsf{g} \geq 0$ *on* E *and* $\mathsf{fg}, \mathsf{f}^\gamma, \mathsf{g}^\gamma, \in L$ *for* $0 \leq \gamma \leq 2$. *Then for either* $1 \leq \alpha \leq \beta \leq 2$ *or* $\beta \leq \alpha \leq 1$, *we have*

$$[2B(\mathsf{f})B(\mathsf{g}) - B(\mathsf{fg})]^2 \leq B\left(\mathsf{f}^2\right) B\left(\mathsf{g}^2\right).$$

Proof. By using Lemma 3, we have

$$\begin{aligned}
[2B(\mathsf{f})B(\mathsf{g}) - B(\mathsf{fg})]^2 =& [B(\mathsf{f}(2B(\mathsf{g}) - \mathsf{g}))]^2 \\
\leq& B\left(\mathsf{f}^\alpha (2B(\mathsf{g}) - \mathsf{g})^{2-\alpha}\right) B\left(\mathsf{f}^{2-\alpha}(2B(\mathsf{g}) - \mathsf{g})^\alpha\right) \\
\leq& B\left(\mathsf{f}^\beta (2B(\mathsf{g}) - \mathsf{g})^{2-\beta}\right) B\left(\mathsf{f}^{2-\beta}(2B(\mathsf{g}) - \mathsf{g})^\beta\right) \\
=& B(\mathsf{f}^2)B((2B(\mathsf{g}) - \mathsf{g})^2) = B(\mathsf{f}^2)B(\mathsf{g}^2).
\end{aligned}$$

□

Remark 2. Inequality in Lemma 3 is known as Callebaut's inequality and its first published version is derived for sequences (see [4]).

Similarly, the proof of the next theorem follows from the $B((2B(\mathsf{g})\mathsf{g})^2) = B(\mathsf{g}^2)$ and use the Lemma (Aczel's inequality for isotonic functionals, [1, p. 125]) .

Lemma 4 (Aczél). *Let* $B : L \to \mathbb{R}$ *is a positive linear functional. If* $\mathsf{f}^2, \mathsf{g}^2, \mathsf{fg} \in L$ *and* $g_0^2 - B\left(\mathsf{g}^2\right) > 0$(*or* $f_0^2 - B\left(\mathsf{f}^2\right) > 0$), *where* g_0, f_0 *are real numbers, then*

$$\left(\mathsf{f}_0 g_0 - B(\mathsf{fg})\right)^2 \geq \left(\mathsf{f}_0^2 - B\left(\mathsf{f}^2\right)\right)\left(\mathsf{g}_0^2 - B\left(\mathsf{g}^2\right)\right).$$

Theorem 5. *Let* $B : L \to \mathbb{R}$ *is a positive linear functional and let* $\mathsf{f}, \mathsf{g} \geq 0$ *on* E. *If* $\mathsf{f}^2, \mathsf{g}^2, f\mathsf{g} \in L$ *and*

$$g_0^2 - B\left(\mathsf{g}^2\right) > 0 \text{ or } f_0^2 - B\left(\mathsf{f}^2\right) > 0,$$

where $f_0, g_0 \in \mathbb{R}$ *then the inequality*

$$\left(\mathsf{f}_0 g_0 - 2B(\mathsf{f})B(\mathsf{g}) + B(\mathsf{fg})\right)^2 \geq \left(\mathsf{f}_0^2 - B\left(\mathsf{f}^2\right)\right)\left(\mathsf{g}_0^2 - B\left(\mathsf{g}^2\right)\right),$$

holds.

2.3. Further refinements

Let E be a non-empty set, $\mathcal{A}$ be an algebra of subsets of E and L be a linear class of real-valued functions $\mathsf{g} : E \to \mathbb{R}$ having additional property:

$$f \in L, E_1 \in \mathcal{A} \Rightarrow fC_{E_1} \in L,$$

where C_{E_1} is the indicator function of E_1.

Let $B : L \to \mathbb{R}$ is a positive linear functional. Then for every $E_1 \in \mathcal{A}$ such that $B\left(C_{E_1}\right) > 0$, the functional A_1 defined for all $g \in L$ by $A_1(g) = \frac{B(gC_{E_1})}{B(C_{E_1})}$ is an isotonic functional with $A_1(1) = 1$.

Theorem 6. *Let $B : L \to \mathbb{R}$ is a positive linear functional and suppose that the f, f^2, g, $g^2 \in L$.*

(i) If $E_1 \in \mathcal{A}$ has $B(f(2B(g)g)C_{E_j}) > 0,\ j = 1, 2$ where $E_2 = E \backslash E_1$, then

$$\frac{\sqrt{B\left(g^2\right)}}{B(g)} \geq \frac{\sqrt{B\left(\bar{g}_{E_1}^2\right)}}{B\left(\bar{g}_{E_1}\right)},$$

where

$$\bar{g}_{E_1}(t) = 2B(g)C_{E_1}(t)g(t)C_{E_1}(t) + \frac{B\left((2B(g)g)^2C_{E_1}\right)}{2B\left(g\right)B(C_{E_1}) - B\left(gC_{E_1}\right)}C_{E_2}(t).$$

(ii) If $E_1 \in \mathcal{A}$ has $B(f(2B(g)g)C_{E_j}) > 0,\ j = 1, 2$ where $E_2 = E \backslash E_1$, then

$$\frac{\sqrt{B(f^2)}}{2B(f)B(g) - B(fg)} \geq \frac{\sqrt{B(\bar{f}_{E_1}^2)}}{2B(\bar{f}_{E_1})B(g) - B(\bar{f}_{E_1}g)}, \tag{8}$$

where

$$\bar{f}_{E_1}(t) = f(t)C_{E_1}(t) + \frac{(2B(g)g(t))B(f^2C_{E_1})}{2B(fC_{E_1})B(g) - B(fgC_{E_1})}C_{E_2}(t). \tag{9}$$

(iii) If $E_1 \in \mathcal{A}$ has $B(g(2B(f) - f)C_{E_j}) > 0,\ j = 1, 2$ where $E_2 = E \backslash E_1$, then

$$\frac{\sqrt{B(f^2)}}{2B(f)B(g) - B(fg)} \geq \frac{\sqrt{B(\hat{f}_{E_1}^2)}}{2B(\hat{f}_{E_1})B(g) - B(\hat{f}_{E_1}g)}, \tag{10}$$

where

$$\hat{f}_{E_1}(t) = (2B(f) - f(t))C_{E_1}(t) + \frac{g(t)B((2B(f) - f)^2C_{E_1})}{2B(fC_{E_1})B(g) - B(fgC_{E_1})}C_{E_2}(t). \tag{11}$$

(iv) If $E_1 \in \mathcal{A}$ and $B(C_{E_1}) > 0$ then

$$B\left(g^2\right) - B(g) \geq \sqrt{B\left((2B(g) - g)^2C_{E_1}\right)} - 2B\left(g\right)B\left(C_{E_1}\right) + B\left(gC_{E_1}\right). \tag{12}$$

(v) If $E_1 \in \mathcal{A}$ and $B(C_{E_1}) > 0$ then

$$\sqrt{B\left(f^2\right)B\left(g^2\right)} - 2B(f)B(g) + B(fg) \geq \sqrt{B\left(f^2C_{E_1}\right)B\left((2B(g) - g)^2C_{E_1}\right)} - 2B(g)B\left(fC_{E_1}\right) + B\left(fgC_{E_1}\right).$$

We need several lemmas to proof the Theorem 6.

Lemma 5. (See [5] and [1, p. 122]). *Let $B : L \to \mathbb{R}$ is a positive linear functional. Suppose $E_1 \in \mathcal{A}$ has $B\left(C_{E_2}\right) > 0$ where $E_2 = E \backslash E_1$. Then for $g \in L$ such that g is non-negative and $g^p \in L (p > 1)$ and $B\left(gC_{E_1}\right) > 0$, we have*

$$B\left(g^p\right)^{1/p} / B(g) \geq B\left(g_{E_1}^p\right)^{1/p} / B\left(g_{E_1}\right), \tag{13}$$

where

$$g_{E_1}(t) = g(t)C_{E_1}(t) + \left(B\left(g^pC_{E_1}\right) / B\left(gC_{E_1}\right)\right)^{1/(p-1)} C_{E_2}(t). \tag{14}$$

Lemma 6. (See [5] and [1, p. 122]). *Let the non-negative functions* $\mathrm{f}, \mathrm{g} : E \to \mathbb{R}$ *along with* $\mathrm{f}^p, \mathrm{g}^q, \mathrm{fg} \in L$ *and* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. *Also, let* $E_1 \in \mathcal{A}$ *and positive linear functional* $B : L \to \mathbb{R}$ *satisfies* $B\left(\mathrm{fg}C_{E_1}\right) > 0$ *and* $B\left(\mathrm{g}^q C_{E_2}\right) > 0$ *where* $E_2 = E \backslash E_1$. *Then one have*

$$\frac{B\left(\mathrm{f}^p\right)^{\frac{1}{p}}}{B(\mathrm{fg})} \geq \frac{B\left(\widetilde{f}_{E_1}^p\right)^{\frac{1}{p}}}{B(\widetilde{f}_{E_1}\mathrm{g})}, \tag{15}$$

where

$$\widetilde{f}_{E_1}(t) = f(t)C_{E_1}(t) + \left[\frac{g(t)A\left(\mathrm{f}^p C_{E_1}\right)}{B\left(\mathrm{fg}C_{E_1}\right)}\right]^{\frac{q}{p}} C_{E_2}(t). \tag{16}$$

Lemma 7. (See [5, p. 639]). *Under assumption of the Lemma 6 the following is valid*

$$B\left(\mathrm{g}^p\right)^{1/p} - B(\mathrm{g}) \geq B\left(\mathrm{g}^p C_{E_1}\right)^{1/p} B\left(C_{E_1}\right)^{1/q} - B\left(\mathrm{g}C_{E_1}\right), \tag{17}$$

and

$$B\left(\mathrm{f}^p\right)^{1/p} B\left(\mathrm{g}^q\right)^{1/q} - B(\mathrm{fg}) \geq B\left(\mathrm{f}^p C_{E_1}\right)^{1/p} B\left(\mathrm{g}^q C_{E_1}\right)^{1/q} - B\left(\mathrm{fg}C_{E_1}\right). \tag{18}$$

Proof of Theorem 6. (i) If we substitute $g \to 2B(\mathrm{g}) - g$ in (13) then the function g_{E_1} in (14) becomes

$$\bar{g}_{E_1}(t) = 2B(\mathrm{g})C_{E_1}(t) - g(t)C_{E_1}(t) + \frac{B\left((2B(\mathrm{g})-\mathrm{g})^2 C_{E_1}\right)}{2B\left(\mathrm{g}\right)B(C_{E_1}) - B\left(\mathrm{g}C_{E_1}\right)} C_{E_2}(t).$$

(ii) If we substitute $g \to 2B(\mathrm{g}) - g$ in (16), for $p = q = 2$. Then the function $\widetilde{f}_{E_1}$ becomes

$$\bar{f}_{E_1}(t) = f(t)C_{E_1}(t) + \frac{(2B(\mathrm{g}) - g(t))B(\mathrm{f}^2 C_{E_1})}{2B(\mathrm{f}C_{E_1})B(\mathrm{g}) - B(\mathrm{fg}C_{E_1})} C_{E_2}(t), \tag{19}$$

is well defined since $2B(\mathrm{f}C_{E_1})B(\mathrm{g}) - B(\mathrm{fg}C_{E_1}) = B(\mathrm{f}(2B(\mathrm{g})-\mathrm{g})C_{E_1}) > 0$. Also $2B(\mathrm{f})B(\mathrm{g}) - B(\mathrm{fg}) = B(\mathrm{f}(2B(\mathrm{g})-\mathrm{g})C_{E_1}) + B(\mathrm{f}(2B(\mathrm{g})-\mathrm{g})C_{E_2}) > 0$.

(iii) First observe $2B(\mathrm{f})B(\mathrm{g}) - B(\mathrm{fg}) = B(\mathrm{g}(2B(\mathrm{f})-\mathrm{f})C_{E_1}) + B(\mathrm{g}(2B(\mathrm{f})-\mathrm{f})C_{E_2}) > 0$. If we substitute $f \to 2B(\mathrm{f}) - \mathrm{f}$ in (16), for $p = q = 2$, the function $\widetilde{f}_{E_1}$ becomes

$$\hat{f}_{E_1}(t) = (2B(\mathrm{f}) - \mathrm{f}(t))C_{E_1}(t) + \frac{g(t)B((2B(\mathrm{f})-\mathrm{f})^2 C_{E_1})}{2B(\mathrm{f}C_{E_1})B(\mathrm{g}) - B(\mathrm{fg}C_{E_1})} C_{E_2}(t).$$

(iv) We put $p = q = 2$ in (17).

(v) We put $g \to 2B(\mathrm{g}) - g$ in (18).

□

3. Results for complex functionals

Here we let positive linear functional $B : L \to \mathbb{R}$ and consider the class ([1, p. 128])

$$\bar{L} = \{\mathrm{f} : E \to \mathbf{C}, \mathrm{Re}\,\mathrm{f} \in L, \mathrm{Im}\,\mathrm{f} \in L\},$$

and $\bar{B} : \bar{L} \to \mathbf{C}$ the functional given by

$$\bar{B}(\mathrm{f}) = B(\mathrm{Re}\,\mathrm{f}) + iB(\mathrm{Im}\,\mathrm{f}) = \mathrm{Re}(\bar{B}(\mathrm{f})) + i\,\mathrm{Im}(\bar{B}(\mathrm{f})). \tag{20}$$

Lemma 8. (See [1, p. 130]). *Let* $\mathrm{g} : E \to \mathbb{R}$ *and* $\mathrm{f} : E \to \mathbb{C}$ *be the functions along with* $\mathrm{g}^2, |\mathrm{f}|^2 \in L, \mathrm{f}^2, \mathrm{fg} \in \bar{L}$. *Then one have*

$$|\bar{B}(\mathrm{fg})|^2 \leq \frac{1}{2} B\left(\mathrm{g}^2\right)\left\{B\left(|\mathrm{f}|^2\right) + \left|\bar{B}\left(\mathrm{f}^2\right)\right|\right\}. \tag{21}$$

Lemma 9. *If* $\bar{B} : \bar{L} \to \mathbf{C}$ *is linear functional defined with* (20), *then for* $\mathrm{f} : E \to \mathbb{C}$, $\mathrm{f}, \mathrm{f}^2 \in \bar{L}$, *we have*

$$\bar{B}\left((2\bar{B}(\mathsf{f})-\mathsf{f})^2\right)=\bar{B}(\mathsf{f}^2), \tag{22}$$

and

$$B\left(|2\bar{B}(\mathsf{f})-\mathsf{f}|^2\right)=B(|\mathsf{f}|^2). \tag{23}$$

If $k: E \to \mathbb{C}$, $k \in \bar{L}$, $\mathsf{f}k \in \bar{L}$, *then*

$$\bar{B}\left((2\bar{B}(\mathsf{f})-\mathsf{f})(2\bar{B}(k)-k)\right)=\bar{B}(\mathsf{f}k). \tag{24}$$

Proof. From

$$\mathsf{f}=u+iv, \quad \text{where } u=\operatorname{Re}\mathsf{f} \in L, v=\operatorname{Im}\mathsf{f}\in L.$$

Then

$$\bar{B}(\mathsf{f})=B(u)+iB(v)$$

and

$$2\bar{B}(\mathsf{f})-\mathsf{f}=2(B(u)+iB(v))-(u+iv)=(2B(u)-u)+i(2B(v)-v). \tag{25}$$

If we denote

$$\mathsf{g}=2B(u)-u, \quad h=2B(v)-v,$$

then

$$2\bar{B}(\mathsf{f})-\mathsf{f}=g+ih.$$

Using (1) and (2), we now conclude that

$$B(\mathsf{g}^2)=B(u^2),\ B(h^2)=B(v^2),\ B(\mathsf{g}h)=B(uv). \tag{26}$$

Now

$$\begin{aligned}\bar{B}((2\bar{B}(\mathsf{f})-\mathsf{f})^2) =&\bar{B}(\mathsf{g}^2-h^2+2i\mathsf{g}h)\\ =&B(\mathsf{g}^2-h^2)+iB(\mathsf{g}h)\\ =&B(u^2)-B(v^2)+iB(uv).\end{aligned}$$

Since $\mathsf{f}^2=u^2-v^2+2iuv$, we conclude

$$\bar{B}((2\bar{B}(\mathsf{f})-\mathsf{f})^2=\bar{B}(\mathsf{f}^2).$$

Also $|2\bar{B}(\mathsf{f})-\mathsf{f}|^2=\mathsf{g}^2+h^2$ and using (26) we conclude

$$B(|2\bar{B}(\mathsf{f})-\mathsf{f}|^2)=B(\mathsf{g}^2)+B(h^2)=B(u^2)+B(v^2)=B(|\mathsf{f}|^2).$$

The Eq. (24) can be proved with the same arguments. □

Theorem 7. *Let* $\mathsf{g}: E\to\mathbb{R}$ *and* $\mathsf{f}: E\to\mathbb{C}$ *be the functions along with* $\mathsf{g}^2, |\mathsf{f}|^2 \in L$, *and* $\mathsf{f}^2, f\mathsf{g}\in\bar{L}$. *Then one have*

$$|2\bar{B}(\mathsf{f})B(\mathsf{g})-\bar{B}(\mathsf{fg})|^2 \le \frac{1}{2}B\left(\mathsf{g}^2\right)\left\{B\left(|\mathsf{f}|^2\right)+\left|\bar{B}\left(\mathsf{f}^2\right)\right|\right\}. \tag{27}$$

Proof. We use Cauchy inequality (21) and

$$\begin{aligned}|2\bar{B}(\mathsf{f})B(\mathsf{g})-\bar{B}(\mathsf{fg})|^2 &= |\bar{B}(\mathsf{f}(2B(\mathsf{g})-\mathsf{g}))|^2\\ &\le \frac{1}{2}B\left((2B(\mathsf{g})-\mathsf{g})^2\right)\left\{B\left(|\mathsf{f}|^2\right)+\left|\bar{B}\left(\mathsf{f}^2\right)\right|\right\}\\ &=\frac{1}{2}B\left(\mathsf{g}^2\right)\left\{B\left(|\mathsf{f}|^2\right)+\left|\bar{B}\left(\mathsf{f}^2\right)\right|\right\}.\end{aligned}$$

Alternatively, we could take a complex point of view: we could use (22) and (23).

$$\begin{aligned}|2\bar{B}(\mathrm{f})B(\mathrm{g})-\bar{B}(\mathrm{fg})|^2 &= |\bar{B}((2\bar{B}(\mathrm{f})-\mathrm{f})\mathrm{g})|^2\\ &\le \frac{1}{2}B\left(\mathrm{g}^2\right)\left\{B\left(|2B(\mathrm{f})-\mathrm{f}|^2\right)+\left|\bar{B}\left((2B(\mathrm{f})-\mathrm{f})^2\right)\right|\right\}\\ &= \frac{1}{2}B\left(\mathrm{g}^2\right)\left\{B\left(|\mathrm{f}|^2\right)+\left|\bar{B}\left(\mathrm{f}^2\right)\right|\right\}.\end{aligned}$$

□

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