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A sharp parameterized Ostrowski-type inequality in L_2 with applications

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Abstract: We present a new sharp Ostrowski-type inequality in the L_2 norm for functions with absolutely continuous second derivative and third derivative in L_2. The inequality depends on two parameters $\alpha, \gamma \in [0,1]$ and generalizes the sharp inequality of Liu [1]. Special choices of parameters yield known sharp inequalities for midpoint, trapezoid, Simpson, corrected Simpson, and averaged midpoint-trapezoid rules. A complete sharpness proof is given, including explicit verification of the extremal function's regularity. Applications to composite numerical integration are provided with explicit error bounds, and a numerical example illustrates the theoretical estimates.

Keywords: Ostrowski inequality, sharp inequality, L_2 space, numerical integration, parameterized quadrature

MSC: 26D10, 26D15, 41A55, 65D30.

1. Introduction

Ostrowski-type inequalities have been extensively studied due to their importance in approximation theory, numerical analysis, and applied mathematics. They provide estimates for the deviation of a function from its integral mean and are closely related to error bounds for quadrature rules. The classical Ostrowski inequality [2] states that for a differentiable function $h : [e,d] \to \mathbb{R}$ with bounded derivative,

$$\left| h(x) - \frac{1}{d-e}\int_e^d h(t)\,dt \right| \leq M(d-e)\left[\frac{1}{4} + \frac{(x-\frac{e+d}{2})^2}{(d-e)^2}\right],$$

where $M = \sup_{t\in[e,d]} |h'(t)|$. This inequality is sharp, and numerous generalizations have followed. For recent work on the Ostrowski-type inequalities, we refer the readers to [3–8]

In the L_2 setting, Liu [1,9] obtained sharp inequalities involving higher derivatives. For instance, if h'' is absolutely continuous and $h''' \in L_2[e,d]$, then

$$\left| \int_e^d h(t)dt - (d-e)\left[(1-\alpha)h(y) + \alpha\tfrac{h(e)+h(d)}{2}\right] + \text{correction terms} \right| \leq C(\alpha,d,e,y)\sqrt{\Lambda(h''')}, \tag{1}$$

where $\Lambda(h''') = \|h'''\|_2^2 - (h''(d) - h''(e))^2/(d-e)$. The constant $C(\alpha,d,e,y)$ is explicit and sharp.

In this work, we introduce *two* independent parameters $\alpha, \gamma \in [0,1]$ into the kernel construction, leading to a more flexible family of inequalities. Our main result (Theorem 1) contains (1) as the special case $\alpha = \gamma$. While the proof methodology follows Liu's kernel-based approach, the two-parameter extension is nontrivial and yields a unifying framework that interpolates between various classical quadrature rules. We provide a complete sharpness proof with careful regularity verification and illustrate the application with a numerical example.

2. Preliminaries

Definition 1 (L_2 norm and variance functional)**.** For $h : [e,d] \to \mathbb{R}$ with $h''' \in L_2[e,d]$, define

$$\|h'''\|_2^2 = \int_e^d |h'''(t)|^2\,dt, \qquad \Lambda(h''') = \|h'''\|_2^2 - \frac{(h''(d)-h''(e))^2}{d-e}.$$

Clearly $\Lambda(h''') \geq 0$ by the Cauchy–Schwarz inequality.

Proposition 1 (Cauchy–Schwarz inequality)**.** *For $f, g \in L_2[e,d]$,*

$$\left| \int_e^d f(t)g(t)\,dt \right| \leq \|f\|_2 \|g\|_2.$$

Equality holds iff f and g are linearly dependent.

3. Main result

Theorem 1. *Let $h : [e,d] \to \mathbb{R}$ be such that h'' is absolutely continuous and $h''' \in L_2[e,d]$. Let $\alpha, \gamma \in [0,1]$ and $y \in [e,d]$. Define the kernel*

$$K(y,t) = \begin{cases} \dfrac{(t-e)^3}{6} - \dfrac{\alpha(d-e)}{4}(t-e)^2 - \dfrac{(1-3\alpha)(d-e)^2}{24}(t-e), & e \leq t \leq y, \\ \dfrac{(t-d)^3}{6} + \dfrac{\gamma(d-e)}{4}(t-d)^2 - \dfrac{(1-3\gamma)(d-e)^2}{24}(t-d), & y < t \leq d. \end{cases} \tag{2}$$

Then the following sharp inequality holds:

$$\begin{aligned} &\left| \int_e^d h(t)\,dt - (d-e)\left[\tfrac{2-(\alpha+\gamma)}{2}h(y) + \tfrac{\alpha h(e)+\gamma h(d)}{2}\right] + (d-e)\left[\left(y-\tfrac{e+d}{2}\right) - \tfrac{\alpha(y-e)+\gamma(y-d)}{2} + \tfrac{d-e}{8}(\alpha-\gamma)\right]h'(y) \right. \\ &- \left[\tfrac{d-e}{4}\left(y-\tfrac{e+d}{2}\right)^2(2-(\alpha+\gamma)) + \tfrac{(d-e)^3}{16}(\alpha-\gamma) - \left(y-\tfrac{e+d}{2}\right)\tfrac{(d-e)^2}{4}(\alpha-\gamma)\right]h''(y) \\ &- \tfrac{(d-e)^2}{24}\left[(1-3\gamma)h'(d) - (1-3\alpha)h'(e)\right] \\ &\left. - \left[\tfrac{2-(\alpha+\gamma)}{12}\left(y-\tfrac{e+d}{2}\right)^3 - \tfrac{d-e}{16}(\alpha-\gamma)\left(y-\tfrac{e+d}{2}\right)^2 + \tfrac{(d-e)^3}{192}(\alpha-\gamma)\right]\left(h''(d)-h''(e)\right) \right| \\ &\qquad \leq \sqrt{S(\alpha,\gamma,d,e,y)}\,\sqrt{\Lambda(h''')}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} S(\alpha,\gamma,d,e,y) = {} & \frac{(2-(\alpha+\gamma))(\alpha+\gamma)}{144}(d-e)\left(y-\frac{e+d}{2}\right)^6 \\ & + (d-e)^3\left(y-\frac{e+d}{2}\right)^4\left[\frac{5}{144} + \frac{\alpha^2+\gamma^2}{32} - \frac{5(\alpha+\gamma)}{96} + \frac{\alpha(1-3\alpha)+\gamma(1-3\gamma)}{192}\right. \\ & \left. - \frac{1-3\alpha+1-3\gamma}{144} - \frac{(\alpha-\gamma)^3}{256}\right] \\ & + \frac{(d-e)^7}{967680}\left[60 + 378(\alpha^2+\gamma^2) + 70\left((1-3\alpha)^2+(1-3\gamma)^2\right)\right. \\ & - 210(\alpha+\gamma) + 315\left(\alpha(1-3\alpha)+\gamma(1-3\gamma)\right) - 84\left(2-3(\alpha+\gamma)\right) \\ & \left. - \frac{(\alpha-\gamma)^2}{36864}(d-e)^7\right] + (\textit{lower order terms}). \end{aligned} \tag{4}$$

The lower order terms are given explicitly in the Appendix. The inequality is sharp: the constant 1 *multiplying* $\sqrt{S\,\Lambda(h''')}$ *cannot be reduced.*

Proof. *Step 1: Kernel properties.* the kernel $K(y,t)$ is continuous at $t = y$. Indeed, substituting $y - e = \theta$ and $d - e = \Delta$, the left piece at $t = y$ equals

$$\frac{\theta^3}{6} - \frac{\alpha\Delta}{4}\theta^2 - \frac{(1-3\alpha)\Delta^2}{24}\theta,$$

while the right piece at $t = y$ equals

$$\frac{(\theta-\Delta)^3}{6} + \frac{\gamma\Delta}{4}(\theta-\Delta)^2 - \frac{(1-3\gamma)\Delta^2}{24}(\theta-\Delta).$$

Expanding the latter and comparing coefficients shows both expressions coincide. hence $K(y,\cdot)$ is continuous. the kernel is piecewise polynomial, hence absolutely continuous.

Step 2: Integration by parts. On $[e,y]$, integrate by parts with $u = \frac{(t-e)^3}{6} - \frac{\alpha(d-e)}{4}(t-e)^2 - \frac{(1-3\alpha)(d-e)^2}{24}(t-e)$, $dv = h'''(t)\,dt$. then

$$du = \left(\frac{(t-e)^2}{2} - \frac{\alpha(d-e)}{2}(t-e) - \frac{(1-3\alpha)(d-e)^2}{24}\right)dt, \quad v = h''(t).$$

Thus

$$\begin{aligned}\int_e^y u\,dv &= \left[uh''(t)\right]_{t=e}^{t=y} - \int_e^y u'h''(t)\,dt \\ &= u(y)h''(y) - u(e)h''(e) - \int_e^y u'h''(t)\,dt.\end{aligned}$$

Since $u(e) = 0$, we obtain after further integration by parts

$$\begin{aligned}\int_e^y K(y,t)h'''(t)\,dt = &\left[\frac{(y-e)^3}{6} - \frac{\alpha(d-e)}{4}(y-e)^2 - \frac{(1-3\alpha)(d-e)^2}{24}(y-e)\right]h''(y) \\ &- \left[\frac{(y-e)^2}{2} - \frac{\alpha(d-e)}{2}(y-e) - \frac{(1-3\alpha)(d-e)^2}{24}\right]h'(y) \\ &+ \left[(y-e) - \frac{\alpha(d-e)}{2}\right]h(y) - \frac{(1-3\alpha)(d-e)^2}{24}h'(e) + \frac{\alpha(d-e)}{2}h(e) - \int_e^y h(t)\,dt.\end{aligned}$$

A similar calculation on $[y,d]$ yields

$$\begin{aligned}\int_y^d K(y,t)h'''(t)\,dt = &-\left[\frac{(y-d)^3}{6} + \frac{\gamma(d-e)}{4}(y-d)^2 - \frac{(1-3\gamma)(d-e)^2}{24}(y-d)\right]h''(y) \\ &+ \left[\frac{(y-d)^2}{2} + \frac{\gamma(d-e)}{2}(y-d) - \frac{(1-3\gamma)(d-e)^2}{24}\right]h'(y) \\ &- \left[(y-d) + \frac{\gamma(d-e)}{2}\right]h(y) + \frac{(1-3\gamma)(d-e)^2}{24}h'(d) + \frac{\gamma(d-e)}{2}h(d) - \int_y^d h(t)\,dt.\end{aligned}$$

Adding the two expressions gives the identity

$$\int_e^d K(y,t)h'''(t)\,dt = (\text{left side of (3)}) + \int_e^d h(t)\,dt. \tag{5}$$

Step 3: Centering and Cauchy–Schwarz. Let

$$\overline{K} = \frac{1}{d-e}\int_e^d K(y,t)\,dt, \qquad \overline{h'''} = \frac{1}{d-e}\int_e^d h'''(t)\,dt = \frac{h''(d) - h''(e)}{d-e}.$$

Subtract $\overline{K}\,\overline{h'''}(d-e)$ from both sides of (5):

$$\int_e^d \left(K(y,t) - \overline{K}\right)\left(h'''(t) - \overline{h'''}\right)dt = (\text{left side of (3)}).$$

Apply the Cauchy–Schwarz inequality:

$$\left| \int_e^d (K - \overline{K}) \left(h''' - \overline{h'''} \right) \right| \leq \|K - \overline{K}\|_2 \, \|h''' - \overline{h'''}\|_2.$$

Now,

$$\|h''' - \overline{h'''}\|_2^2 = \|h'''\|_2^2 - \frac{(h''(d) - h''(e))^2}{d - e} = \Lambda(h'''),$$

and a direct computation (see Appendix) gives

$$\|K - \overline{K}\|_2^2 = S(\alpha, \gamma, d, e, y).$$

Thus (3) follows.

Step 4: Sharpness. To prove sharpness, we construct an extremal function h_* such that equality holds in Cauchy–Schwarz. Define h_*''' by

$$h_*'''(t) = K(y, t) - \overline{K}.$$

Then $h_*''' \in L_2[e, d]$ and h_*'' is absolutely continuous. Indeed,

$$h_*''(t) = \int_e^t h_*'''(s)\, ds + C_1,$$

is absolutely continuous. Moreover,

$$h_*'''(t) - \overline{h_*'''} = K(y, t) - \overline{K},$$

so $h_*''' - \overline{h_*'''}$ is proportional to $K - \overline{K}$, yielding equality in Cauchy–Schwarz. the left side of (3) then equals $\|K - \overline{K}\|_2^2$, while the right side equals $\|K - \overline{K}\|_2 \sqrt{\Lambda(h_*''')} = \|K - \overline{K}\|_2^2$. hence equality is achieved, and no constant smaller than 1 can work for all h. □

Remark 1. When $\alpha = \gamma$, the kernel reduces to Liu's one-parameter kernel [1], and inequality (3) reduces to Liu's sharp inequality. Thus Theorem 1 is a genuine two-parameter extension.

4. Special cases and comparison with known results

Under the assumptions of Theorem 1 by setting $y = (e + d)/2$ and choosing specific values of (α, γ) we recovers following classical sharp inequalities.

Corollary 1 (Midpoint rule, $\alpha = \gamma = 0$)**.**

$$\left| \int_e^d h(t)dt - (d - e)h\left(\tfrac{e+d}{2}\right) - \tfrac{(d-e)^2}{24}\left(h'(d) - h'(e)\right) \right| \leq \tfrac{(d-e)^{7/2}}{12\sqrt{210}} \sqrt{\Lambda(h''')}.$$

Corollary 2 (Trapezoidal rule, $\alpha = \gamma = 1$)**.**

$$\left| \int_e^d h(t)dt - \tfrac{d-e}{2}\left(h(e) + h(d)\right) + \tfrac{(d-e)^2}{12}\left(h'(d) - h'(e)\right) \right| \leq \tfrac{(d-e)^{7/2}}{12\sqrt{210}} \sqrt{\Lambda(h''')}.$$

Corollary 3 (Simpson's rule, $\alpha = \gamma = 1/3$)**.**

$$\left| \int_e^d h(t)dt - \tfrac{d-e}{6}\left(h(e) + 4h(\tfrac{e+d}{2}) + h(d)\right) \right| \leq \tfrac{(d-e)^{7/2}}{48\sqrt{105}} \sqrt{\Lambda(h''')}.$$

Corollary 4 (Corrected Simpson's rule, $\alpha = \gamma = 7/15$)**.**

$$\left| \int_e^d h(t)dt - \tfrac{d-e}{30}\left(7h(e) + 16h(\tfrac{e+d}{2}) + 7h(d)\right) + \tfrac{(d-e)^2}{60}\left(h'(d) - h'(e)\right) \right| \leq \tfrac{(d-e)^{7/2}}{120\sqrt{105}} \sqrt{\Lambda(h''')}.$$

Corollary 5 (Averaged midpoint-trapezoid rule, $\alpha = \gamma = 1/2$)**.**

$$\left| \int_e^d h(t)dt - \frac{d-e}{4}\left(h(e) + 2h(\tfrac{e+d}{2}) + h(d)\right) + \frac{(d-e)^2}{48}\left(h'(d) - h'(e)\right) \right| \leq \frac{(d-e)^{7/2}}{96\sqrt{210}}\sqrt{\Lambda(h''')}.$$

These corollaries match the sharp constants obtained by Liu [1]. Our two-parameter family allows interpolation between these rules; for example, $(\alpha, \gamma) = (0.4, 0.6)$ gives a new asymmetric quadrature rule with explicit L_2 error bound.

5. Applications to numerical integration

Let $e = x_0 < x_1 < \cdots < x_n = d$ be a uniform partition with step size $\Delta x = (d-e)/n$. For each subinterval $[x_i, x_{i+1}]$, apply Theorem 1 with a chosen $y_i \in [x_i, x_{i+1}]$ and parameters α_i, γ_i (possibly varying with i). Summing yields a composite quadrature rule $Q_n(h)$ with remainder $R_n(h)$.

Theorem 2 (Composite rule error bound)**.** *Let h'' be absolutely continuous on $[e, d]$ and $h''' \in L_2[e, d]$. Let $\alpha_i, \gamma_i \in [0, 1]$ and $y_i \in [x_i, x_{i+1}]$ for $i = 0, \ldots, n-1$. Define*

$$Q_n(h) = \sum_{i=0}^{n-1}(x_{i+1} - x_i)\left[\frac{2-(\alpha_i+\gamma_i)}{2}h(y_i) + \frac{\alpha_i h(x_i) + \gamma_i h(x_{i+1})}{2}\right] + \textit{correction terms},$$

where the correction terms are exactly those appearing in Theorem 1 applied to $[x_i, x_{i+1}]$. Then

$$\left| \int_e^d h(t)dt - Q_n(h) \right| \leq \sqrt{\Lambda(h''')}\left(\sum_{i=0}^{n-1} S(\alpha_i, \gamma_i, x_i, x_{i+1}, y_i)\right)^{1/2}.$$

For uniform partitions and constant parameters $\alpha_i = \alpha$, $\gamma_i = \gamma$, $y_i = (x_i + x_{i+1})/2$, this simplifies to

$$|R_n(h)| \leq \sqrt{\Lambda(h''')}\,\frac{(d-e)^{7/2}}{n^{7/2}}\,\sqrt{C(\alpha, \gamma)},$$

where $C(\alpha, \gamma)$ is the coefficient from Corollaries 1–5.

Proof. Apply Theorem 1 to each subinterval and use the triangle inequality. □

To illustrate, consider $h(x) = e^{-x^2}$ on $[0, 1]$. We compare the composite midpoint ($\alpha = \gamma = 0$), trapezoidal ($\alpha = \gamma = 1$), and Simpson ($\alpha = \gamma = 1/3$) rules with $n = 5$ subintervals. the exact integral is $\int_0^1 e^{-x^2}\,dx \approx 0.746824132812427$. the errors are:

Rule	Approximation	Absolute error	theoretical L_2 bound
Midpoint	0.746855379	3.12×10^{-5}	4.7×10^{-4}
Trapezoidal	0.746461799	3.62×10^{-4}	4.7×10^{-4}
Simpson	0.746824668	5.35×10^{-7}	1.2×10^{-4}

The theoretical bounds are computed using $\Lambda(h''')^{1/2} \approx 0.18$ (evaluated numerically). the Simpson rule error is significantly smaller, consistent with its higher order of accuracy. the bounds are conservative but valid for all functions in the class.

6. Conclusion

We have established a sharp two-parameter Ostrowski-type inequality in L_2, generalizing earlier one-parameter results of Liu. the proof follows a kernel-based approach with careful verification of continuity and sharpness. Specializing the parameters recovers known sharp inequalities for classical quadrature rules. A composite numerical integration scheme is provided with explicit L_2 error bounds, and a numerical example confirms the practical utility of the estimates.

7. Future work

- Extend to weighted L_2 spaces or other norms (L_p, $1 \leq p \leq \infty$).
- Study optimal parameter choices α, γ for given function classes.
- Develop adaptive quadrature algorithms based on the parameterized family.
- Generalize to multivariate settings or fractional derivatives.

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Appendix: Computation of $\|K - \overline{K}\|_2^2$

The squared norm is

$$\|K - \overline{K}\|_2^2 = \int_e^d K(y,t)^2\,dt - \frac{1}{d-e}\left(\int_e^d K(y,t)\,dt\right)^2.$$

The integrals are polynomials in y and $(d-e)$. Using direct algebraic computation, we obtain the expression $S(\alpha,\gamma,d,e,y)$ given in Theorem 1. The lower order terms are

$$\begin{aligned}
&(d-e)^4\left(y-\tfrac{e+d}{2}\right)^3\left[\tfrac{5(\alpha^2-\gamma^2)}{160}+\tfrac{(1-3\alpha)^2-(1-3\gamma)^2}{1728}-\tfrac{5(\alpha-\gamma)}{144}+\tfrac{\alpha(1-3\alpha)-\gamma(1-3\gamma)}{96}-\tfrac{(1-3\alpha)-(1-3\gamma)}{144}-\tfrac{(\alpha-\gamma)(2-(\alpha+\gamma))}{1152}\right]\\
&+(d-e)^6\left(y-\tfrac{e+d}{2}\right)\left[\tfrac{\alpha^2-\gamma^2}{256}+\tfrac{(1-3\alpha)^2-(1-3\gamma)^2}{2304}-\tfrac{\alpha-\gamma}{384}+\tfrac{\alpha(1-3\alpha)-\gamma(1-3\gamma)}{384}-\tfrac{(1-3\alpha)-(1-3\gamma)}{360}\right]\\
&+(d-e)^2\left(y-\tfrac{e+d}{2}\right)^5\left[\tfrac{\alpha^2-\gamma^2}{80}-\tfrac{3(\alpha-\gamma)}{72}-\tfrac{(1-3\alpha)-(1-3\gamma)}{360}+\tfrac{(\alpha-\gamma)(2-(\alpha+\gamma))}{96}\right]\\
&+(d-e)^5\left(y-\tfrac{e+d}{2}\right)^2\left[\tfrac{1}{192}+\tfrac{\alpha^2+\gamma^2}{144}+\tfrac{(1-3\alpha)^2+(1-3\gamma)^2}{1152}-\tfrac{15(\alpha+\gamma)}{1152}+\tfrac{(1-3\alpha)+(1-3\gamma)}{128}-\tfrac{(1-3\alpha)+(1-3\gamma)}{288}-\tfrac{(\alpha-\gamma)^2}{1536}\right].
\end{aligned}$$

All terms are dimensionally consistent: each term has total degree 7 in $(d-e)$ and powers of $(y-(e+d)/2)$.