



Article

A strict exponential inequality for cyclic polygons and its elliptical generalization

Kia Keng Giam

Temasek Junior College, Singapore; giamkiakeng@gmail.com

Received: 27 June 2026; Accepted: 24 August 2026; Published: 22 September 2026

Abstract: We present a novel strict inequality for cyclic n -gons, establishing a relationship between their side lengths, circumradius, and generalized inradius. Specifically, we demonstrate that for any non-degenerate cyclic n -gon, the inequality $\left(\frac{a_1 + a_2 + \dots + a_{n-1}}{a_n}\right)^{R/r} > \sqrt{e}$ strictly holds. The proof leverages the Padé approximant bound for the natural logarithm and establishes the strict concavity and subadditivity of an associated trigonometric function. We further extend this result to polygons inscribed in ellipses via affine transformations, generalizing the bound in terms of the ellipse's semi-axes.

Keywords: cyclic polygon, geometric inequality, circumradius, generalized inradius, Padé approximant, trigonometric inequality, concavity, subadditivity, affine transformation, ellipse, inscribed polygon

MSC: 51M16, 52A40.

1. Introduction

In the geometric analysis of convex planar shapes, the ratio of area to semiperimeter serves as a fundamental continuous shape parameter. For tangential polygons, this ratio defines the exact inradius.

By formally adopting this isoperimetric ratio $r = \frac{A}{s}$ as the generalized inradius for non-tangential cyclic polygons, we can construct strict analytical bounds between the circumcircle and this generalized parameter without relying on the existence of a true inscribed circle. Furthermore, extending these discrete bounds into continuous functional spaces allows for the dedublation of classical algebraic relationships into strict analytical inequalities across multiple geometric forms.

Let $\mathcal{P}$ be a non-degenerate cyclic n -gon with side lengths $a_1, a_2, \dots, a_n$, circumradius R , and generalized inradius $r = \frac{A}{s}$. The primary result of this paper establishes a novel strict lower bound connecting these parameters.

Theorem 1. *For any non-degenerate cyclic n -gon $\mathcal{P}$, the following inequality strictly holds:*

$$\left(\frac{a_1 + a_2 + \dots + a_{n-1}}{a_n}\right)^{R/r} > \sqrt{e} \quad (1)$$

Isolating the natural logarithm, this theorem can be equivalently expressed as:

$$\ln\left(\frac{S - a_n}{a_n}\right) > \frac{r}{2R}, \quad (2)$$

where $S = 2s$ is the total perimeter of the polygon.

The foundational geometric inequality and primary proof methods presented in this note were originally developed by the author under the pseudonym ‘Tveltzel’ in response to a problem posed on Mathematics Stack Exchange [1].

2. Geometric preliminaries

Let O be the circumcenter of the cyclic n -gon $\mathcal{P}$. Any simple (non-intersecting) polygon inscribed in a circle is strictly convex, as its vertices lie in sequential order on the boundary of a strictly convex circular domain. This geometric reality guarantees that all interior angles are strictly less than π .

We parameterize the polygon using directed central half-angles $\theta_i \in (0, \pi)$ such that $\sum_{i=1}^n \theta_i = \pi$. Let d_i be the signed perpendicular distance (apothem) from O to the side a_i . We establish the following sequence of geometric relations:

$$d_i = R \cos \theta_i \quad (3)$$

$$A = \sum_{i=1}^n \frac{1}{2} a_i d_i \quad (4)$$

$$r = \frac{A}{s} = \frac{\sum_{i=1}^n a_i d_i}{\sum_{i=1}^n a_i}. \quad (5)$$

Consequently, r serves as the exact perimeter-weighted arithmetic mean of the signed perpendicular distances from the circumcenter to the sides. If the circumcenter lies outside the polygon, the distance to the side subtending the major arc takes a negative weight, rigorously preserving the exact geometric relationship without relying on the existence of an inscribed circle.

3. Required lemmas

To analytically bound the logarithmic term, we utilize a lower bound derived from bounds inspired by Topsøe [2] on logarithmic functions, which directly motivates the strict inequality presented below.

Lemma 1 (Logarithmic Bound). *For all real $x > 1$,*

$$\ln(x) > \frac{2(x-1)}{x+1}. \quad (6)$$

Proof. Define the auxiliary function $g(x) = \ln(x) - \frac{2(x-1)}{x+1}$. Evaluating at the boundary yields $g(1) = 0$. Computing the first derivative with respect to x yields:

$$g'(x) = \frac{1}{x} - \frac{2(x+1) - 2(x-1)}{(x+1)^2} = \frac{1}{x} - \frac{4}{(x+1)^2} = \frac{(x-1)^2}{x(x+1)^2}. \quad (7)$$

For all $x > 1$, $(x-1)^2 > 0$ and $x(x+1)^2 > 0$, strictly implying $g'(x) > 0$. Thus, $g(x)$ is strictly monotonically increasing on the interval $(1, \infty)$, establishing that $g(x) > 0$ for $x > 1$. $\square$

The following result is a standard property of non-negative concave functions. See [3].

Lemma 2 (Subadditivity). *Let $f : [0, D] \rightarrow \mathbb{R}$ be concave with $f(0) \geq 0$. For any $x, y \in [0, D]$ such that $x + y \in [0, D]$, it follows that $f(x) + f(y) \geq f(x + y)$.*

Proof. For any $x, y \geq 0$ such that $x + y \leq D$, concavity yields:

$$f(x) = f\left(\frac{x}{x+y}(x+y) + \frac{y}{x+y}(0)\right) \geq \frac{x}{x+y}f(x+y) + \frac{y}{x+y}f(0).$$

$\square$

For the convenience of the reader, a proof is included here.

The following general result follows directly by induction over k :

Corollary 1. Let $f : [0, D] \rightarrow \mathbb{R}$ be concave with $f(0) \geq 0$. For any $x_1, \dots, x_k \in [0, D]$ such that $\sum_{i=1}^k x_i \in [0, D]$, it follows that

$$\sum_{i=1}^k f(x_i) \geq f\left(\sum_{i=1}^k x_i\right).$$

4. Proof of Theorem 1

Let $x = \frac{S-a_n}{a_n}$. By the generalized polygon inequality, the sum of any $n-1$ sides of a non-degenerate n -gon is strictly greater than the remaining side, meaning $S - a_n > a_n$, and consequently $x > 1$. Applying Lemma 1 yields:

$$\ln\left(\frac{S-a_n}{a_n}\right) > \frac{2\left(\frac{S-a_n}{a_n} - 1\right)}{\frac{S-a_n}{a_n} + 1} = \frac{2(S-2a_n)}{S} = 2 - \frac{4a_n}{S}. \quad (8)$$

By transitivity, it is thus sufficient to prove the following algebraic condition:

$$2 - \frac{4a_n}{S} > \frac{r}{2R}. \quad (9)$$

Translating the perimeter S and area A into trigonometric representations using the directed central angles, we have $a_i = 2R \sin \theta_i$, $S = 2R \sum_{i=1}^n \sin \theta_i$, and $A = \frac{1}{2}R^2 \sum_{i=1}^n \sin(2\theta_i) = R^2 \sum_{i=1}^n \sin \theta_i \cos \theta_i$. The ratio $\frac{r}{2R}$ becomes:

$$\frac{r}{2R} = \frac{2A}{2SR} = \frac{2R^2 \sum_{i=1}^n \sin \theta_i \cos \theta_i}{2R \left(2R \sum_{i=1}^n \sin \theta_i\right)} = \frac{\sum_{i=1}^n \sin \theta_i \cos \theta_i}{2 \sum_{i=1}^n \sin \theta_i}. \quad (10)$$

Substituting these parameters into the algebraic inequality (9) yields:

$$2 - \frac{8R \sin \theta_n}{2R \sum_{i=1}^n \sin \theta_i} > \frac{\sum_{i=1}^n \sin \theta_i \cos \theta_i}{2 \sum_{i=1}^n \sin \theta_i}. \quad (11)$$

Because the polygon is non-degenerate ($S > 0$), multiplying through by $2 \sum_{i=1}^n \sin \theta_i$ preserves the inequality direction. Splitting the sums to isolate the n -th terms simplifies the condition to:

$$\sum_{i=1}^{n-1} \sin \theta_i (4 - \cos \theta_i) > \sin \theta_n (4 + \cos \theta_n). \quad (12)$$

We define the function $f(t) = \sin t(4 - \cos t) = 4 \sin t - \frac{1}{2} \sin(2t)$. Since $\sum_{i=1}^n \theta_i = \pi$, evaluating $f(t)$ at the supplementary angle $\pi - \theta_n$ gives:

$$f(\pi - \theta_n) = \sin(\pi - \theta_n)(4 - \cos(\pi - \theta_n)) = \sin \theta_n (4 + \cos \theta_n). \quad (13)$$

Thus, the geometric condition (12) is identically equivalent to:

$$\sum_{i=1}^{n-1} f(\theta_i) > f\left(\sum_{i=1}^{n-1} \theta_i\right). \quad (14)$$

To rigorously complete the proof, we analyze the concavity of $f(t)$. Differentiating twice yields:

$$f'(t) = 4 \cos t - \cos(2t) \quad (15)$$

$$f''(t) = -4 \sin t + 2 \sin(2t) = -4 \sin t + 4 \sin t \cos t = -4 \sin t (1 - \cos t). \quad (16)$$

Because $\sin t > 0$ and $1 - \cos t > 0$ for all $t \in (0, \pi)$, it follows immediately that $f''(t) < 0$, confirming $f(t)$ is concave on this interval. Since $f(0) = 0$ and the partial sum of angles remains entirely within $[0, \pi]$, Corollary 1 guarantees subadditivity, completing the proof of Theorem 1.

5. Elliptical generalization

This mathematical framework can be extended to polygons inscribed in ellipses by leveraging affine transformations.

Theorem 2. Let $\mathcal{E}$ be an ellipse with semi-major axis a and semi-minor axis b ($a \geq b$). For an n -gon inscribed in $\mathcal{E}$ with side lengths L_i and generalized inradius $r_{\text{ell}} = A_{\text{ell}}/s_{\text{ell}}$ (where A_{ell} represents the area of the inscribed polygon and s_{ell} represents its semiperimeter), the following holds:

$$\frac{L_1 + L_2 + \cdots + L_{n-1}}{L_n} > \frac{b}{a} \exp\left(\frac{r_{\text{ell}}}{2a}\right). \quad (17)$$

Proof. Apply the affine transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x, \frac{a}{b}y)$. This transformation maps the ellipse $\mathcal{E}$ to its auxiliary circle of radius $R = a$, and maps the inscribed n -gon to a cyclic polygon with corresponding side lengths $l_i = \sqrt{(\Delta x_i)^2 + (\frac{a}{b}\Delta y_i)^2}$, where Δx_i and Δy_i represent the differences in the x and y coordinates, respectively, between the consecutive vertices of the i -th side. Because $a \geq b$, we extract rigorous metric distortion bounds for every segment:

$$L_i \leq l_i \leq \frac{a}{b}L_i. \quad (18)$$

By analyzing the determinant of the Jacobian of T , the area transforms proportionally as $A_{\text{circ}} = \frac{a}{b}A_{\text{ell}}$. Summing the perimeter bounds yields $s_{\text{circ}} \leq \frac{a}{b}s_{\text{ell}}$, ensuring the generalized inradius bounds satisfy $r_{\text{circ}} \geq r_{\text{ell}}$. Applying Theorem 1 to this auxiliary system where $R = a$, and mapping the length ratio back to the ellipse metric using (18), we formulate the upper bound:

$$\frac{S_{\text{circ}} - l_n}{l_n} < \frac{a}{b} \left(\frac{S_{\text{ell}} - L_n}{L_n} \right). \quad (19)$$

Substituting this upper bound into the established logarithmic inequality (2) yields the generalized limit. This demonstrates that the exponential limit scales linearly via the aspect ratio $\frac{b}{a} = \sqrt{1 - \varepsilon^2}$, recovering Theorem 1 precisely as eccentricity $\varepsilon \rightarrow 0$. $\square$

6. Concluding remarks

The translation of geometric isoperimetric properties into strict algebraic inequalities provides a robust framework for isolating specific polygon components. By combining rational function approximations with the subadditivity of concave trigonometric functions, we have established a generalized exponential limit for cyclic n -gons that successfully scales across affine transformations into elliptical spaces.

Funding Information: The author declares no competing financial or non-financial interests. No external funding was received for this work. The author appreciates all reviewers for their useful comments.

Data Availability: No data was used or required in this work.

Conflicts of Interest: The author declares that there are no competing interests.

References

- [1] Tvelzel. (2026, March 6). [Answer to the question “Does every cyclic polygon satisfy $\left(\frac{a_1+a_2+\cdots+a_{n-1}}{a_n}\right)^{R/r} \geq \sqrt{e}$?]. *Mathematics Stack Exchange*. <https://math.stackexchange.com/questions/5127293/does-every-cyclic-polygon-satisfy-left-frac-a-1-a-2-cdots-a-n-1-a-n/5127568>
- [2] Topsøe, F. (2007). Some bounds for the logarithmic function. In Y. J. Cho, J. K. Kim, & S. S. Dragomir (Eds.), *Inequality Theory and Applications* (Vol. 4, pp. 137–151). Nova Science Publishers.
- [3] Schechter, E. (1997). *Handbook of Analysis and its Foundations*. Academic Press.



© 2026 by the authors; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (<http://creativecommons.org/licenses/by/4.0/>).