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A weighted seminorm framework Čebyšev-type inequalities and applications

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Abstract: This paper contributes to the study of weighted semi-norms and their role in integral inequalities, continuing our earlier investigations based on convexity techniques. By incorporating Sonin's identity into a weighted semi-norm setting, we obtain a unified extension of several classical inequalities of Čebyšev type. The proposed framework allows us to generalize and refine a number of well-known results, including inequalities associated with Čebyšev, Grüss, Ostrowski, and Lupaş, while placing previous contributions by Dragomir and others in a broader weighted context. In particular, we establish new bounds for the weighted Čebyšev functional $T_w(f,g)$ expressed in terms of the semi-norm $\Delta_p(f)$, which captures the global oscillatory behavior of the underlying function. Additional improvements are obtained through the use of a weighted Hölder–Işcan type inequality. The resulting theory not only encompasses the classical, unweighted case as a special situation but also offers greater adaptability in problems involving non-uniform weights, probabilistic measures, and weighted approximation processes. As an illustration of applicability, several consequences for numerical integration are discussed, including generalized midpoint and trapezoidal bounds.

Keywords: weighted Čebyšev functional, convex functions, Grüss inequality, Ostrowski inequality, weighted semi-norms, Sonin's identity, sharp constants, Δ-seminorms

MSC: 26D15, 26D10, 26A51, 41A55.

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1. Introduction and Motivation

1.1. From convexity to seminorms: Evolving the framework

Our earlier investigation [1] laid the groundwork by constructing a convexity-based theory for weighted Čebyšev functionals. In that paper, we introduced a powerful master inequality from which many classical and new inequalities of the Čebyšev-Grüss-Ostrowski type emerge as special cases, demonstrating the unifying power of convexity arguments in this domain.

This current work represents the next logical step in this research program, moving the focus from convexity methods toward a systematic theory built around weighted semi-norms. Where the previous paper established the value of a convexity perspective, we now demonstrate that a seminorm-based approach offers a different and highly advantageous viewpoint, particularly for quantifying function oscillation and deriving optimal bounds.

The historical context for this work is rich and spans more than a century. It begins with Čebyšev's seminal work [2], which introduced the famous inequality carrying his name. Grüss [3] later contributed fundamental bounds under different conditions, while Ostrowski [4] found important connections between these approaches. The classical functional,

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt\int_a^b g(t)\,dt,$$

has attracted continuous interest, with notable advances including Lupaş's work on sharp constants [5] and various refinements [6].

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A particularly influential development came from Dragomir [7], who systematically employed Sonin's identity [8] to derive bounds using Δ-seminorms. This was complemented by the work of Cerone and Dragomir [9–11], who further developed the Δ-seminorm theory, while more recent refinements have incorporated techniques like the Hölder-Işcan inequality [12].

1.2. Advancing beyond existing literature

Motivated by our earlier convexity-based analysis [1] and the broad development of Čebyšev-type inequalities in the literature, the present work goes well beyond existing results in several essential directions. Rather than offering isolated refinements, we introduce a coherent and systematic framework that substantially extends what is currently known.

1. We develop a rigorous and self-contained theory for weighted Δ_p-semi- norms, including their fundamental structural properties. This approach provides a natural and highly adaptable means of quantifying function oscillation, and in many instances proves more effective than classical norm-based techniques.
2. By further exploiting weighted versions of Sonin's identity, we extend our earlier methods to a fully seminorm-driven setting. This refinement leads to sharper estimates in weighted environments and significantly broadens the class of admissible applications.
3. A careful analysis of extremal functions allows us to clarify when the best-known constants from the unweighted theory remain optimal and when the presence of weights yields genuinely stronger inequalities with improved constants.
4. As a further contribution, we establish weighted variants of the Hölder–Işcan inequality. These results appear to be new and, in practice, often lead to tighter bounds than those obtained via classical Hölder inequalities or convexity-based arguments.
5. In order to demonstrate the practical relevance of the theory, we apply our results to problems in numerical integration. In particular, explicit weighted Peano kernels are derived for generalized midpoint and trapezoidal rules, thereby facilitating direct computational implementation.

1.3. Standing assumptions

Throughout this paper, we work on a finite interval $[a,b]$. Unless otherwise stated, the weight function $w : [a,b] \to [0,\infty)$ is assumed to be essentially bounded ($w \in L^\infty([a,b])$) and satisfy $\int_a^b w(x)dx = 1$. When results involve derivatives or specific kernel constructions (e.g., in §**??**), we explicitly state the required regularity of the functions and the weight.

1.4. Navigating the paper

The paper unfolds as follows. In §2, we establish the weighted seminorm framework and fundamental properties of Δ_p-seminorms. §3 presents the generalized Sonin identity within this seminorm context. Our main results—Hölder-type bounds and applications to various function classes—appear in §4 and §5. The novel Hölder-Işcan refinements are introduced in §6, while applications to numerical integration and exponential inequalities follow in §7 and §8. We show how classical results are recovered in §9 and discuss sharpness properties in §10. The paper concludes with §10, where we reflect on advantages over previous approaches and suggest promising future directions.

2. The weighted Δ-seminorm framework

2.1. Foundational definitions and properties

For completeness and self-containment, we restate the essential definitions and properties that form the basis of our theory.

Definition 1 (Weighted Δ-Seminorm)**.** Let $w : [a,b] \to [0,\infty)$ be essentially bounded with $\int_a^b w(x)dx = 1$. For $1 \le p < \infty$ and any measurable function $f : [a,b] \to \mathbb{R}$, define the weighted Δ-seminorm as:

$$\Delta_p(f) := \left(\int_a^b \int_a^b w(x)w(t)|f(x) - f(t)|^p \, dt \, dx \right)^{1/p}.$$

For $p = \infty$, define:

$$\Delta_\infty(f) := \operatorname*{ess\,sup}_{(x,t)\in[a,b]^2} |f(x) - f(t)|.$$

Proposition 1 (Fundamental properties)**.** *The weighted Δ_p-seminorm satisfies:*

(1) *Non-negativity: $\Delta_p(f) \ge 0$ for all f*

(2) *Absolute homogeneity: $\Delta_p(\alpha f) = |\alpha|\Delta_p(f)$ for $\alpha \in \mathbb{R}$*

(3) *Triangle inequality: $\Delta_p(f + g) \le \Delta_p(f) + \Delta_p(g)$*

(4) *Null space: $\Delta_p(f) = 0$ if and only if f equals a constant c for w-almost every $x \in [a,b]$ (equivalently: for μ-almost every (x,t) with $\mu(dx,dt) = w(x)w(t)\,dx\,dt$).*

Thus Δ_p is a seminorm that becomes a norm on the quotient space modulo constants.

Proof. Properties (1) and (2) are immediate. For (3), consider $F_f(x,t) := f(x) - f(t)$ as an element of $L^p([a,b]^2, \mu)$ where $\mu(dx,dt) = w(x)w(t)dxdt$. Then $\Delta_p(f) = \|F_f\|_{L^p(\mu)}$ and the result follows from the triangle inequality in $L^p(\mu)$.

For property (4), note that $\Delta_p(f) = 0$ implies $f(x) = f(t)$ for μ-almost every (x,t). By Fubini's theorem, this means there exists a constant c such that $f(x) = c$ for almost every x in the support of w. □

Example 1 (Connection to Variance)**.** For $p = 2$, the weighted Δ-seminorm relates directly to variance:

$$\begin{aligned}\Delta_2(f)^2 &= 2\,\mathrm{Var}_w(f) \\ &= 2\left(\int_a^b w(x)f(x)^2 \, dx - \left(\int_a^b w(x)f(x)\,dx \right)^2 \right).\end{aligned}$$

This demonstrates that $\Delta_2(f)$ measures oscillation in a statistically natural way.

Remark 1 (Connection to classical theory)**.** When $w(x) = \frac{1}{b-a}$, our $\Delta_p(f)$ reduces to the classical Δ-seminorms studied in [7,9,11]:

$$\Delta_p(f) = \left(\frac{1}{(b-a)^2} \int_a^b \int_a^b |f(x) - f(t)|^p \, dx \, dt \right)^{1/p} = (b-a)^{-2/p} \|f\|_p^\Delta,$$

where $\|f\|_p^\Delta = \left(\int_a^b \int_a^b |f(x) - f(t)|^p \, dx \, dt \right)^{1/p}$. This demonstrates that our framework properly generalizes the classical theory.

3. Generalized Sonin identity and weighted Čebyšev functional

The weighted Čebyšev functional serves as the natural generalization of its classical counterpart:

Definition 2 (Weighted Čebyšev Functional)**.** Let $w : [a,b] \to [0,\infty)$ be essentially bounded with $\int_a^b w = 1$. For measurable functions $f, g : [a,b] \to \mathbb{R}$, define:

$$T_w(f,g) := \int_a^b w(x)f(x)g(x)\,dx - \left(\int_a^b w(x)f(x)\,dx \right) \left(\int_a^b w(x)g(x)\,dx \right).$$

The following generalization of Sonin's identity [8] is fundamental to our approach:

Theorem 1 (Generalized Sonin identity). *For any $\lambda \in \mathbb{R}$, we have:*

$$T_w(f,g) = \int_a^b w(x)(g(x)-\lambda)\left(f(x) - \int_a^b w(t)f(t)\,dt\right)dx.$$

Proof. Direct computation reveals:

$$\begin{aligned} T_w(f,g) &= \int_a^b w(x)f(x)g(x)\,dx - \left(\int_a^b wf\right)\left(\int_a^b wg\right) \\ &= \int_a^b w(x)(g(x)-\lambda)\left(f(x) - \int_a^b w(t)f(t)\,dt\right)dx + \lambda\left[\int_a^b w(x)f(x)\,dx - \int_a^b w(x)f(x)\,dx\right], \end{aligned}$$

where the second term vanishes identically. □

Remark 2 (Historical Context). The classical Sonin identity [8], utilized extensively by Dragomir [7], corresponds to the special case $w(x) = \frac{1}{b-a}$. Our weighted version enables the treatment of problems involving non-uniform distributions, making it particularly valuable in statistical applications and weighted approximation theory [13].

4. Main results: Weighted semi-norm bounds

4.1. Hölder-type inequalities

Our first main result generalizes the classical Hölder approach to the weighted setting:

Theorem 2 (Weighted Hölder bound). *Let $1 \le p < \infty$ and q be its conjugate exponent. For measurable functions f, g with finite weighted norms:*

$$|T_w(f,g)| \le \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{q,w}\Delta_p(f),$$

where $\|h\|_{q,w} = \left(\int_a^b w(x)|h(x)|^q\,dx\right)^{1/q}$.

Proof. Applying Theorem 1 and Hölder's inequality:

$$\begin{aligned} |T_w(f,g)| &\le \int_a^b w(x)|g(x)-\lambda|\left|f(x) - \int_a^b w(t)f(t)\,dt\right|dx \\ &\le \|g-\lambda\|_{q,w}\left(\int_a^b w(x)\left|f(x) - \int_a^b w(t)f(t)\,dt\right|^p dx\right)^{1/p}. \end{aligned}$$

Jensen's inequality yields:

$$\left|f(x) - \int_a^b w(t)f(t)\,dt\right|^p \le \int_a^b w(t)|f(x)-f(t)|^p\,dt,$$

so:

$$\int_a^b w(x)\left|f(x) - \int_a^b w(t)f(t)\,dt\right|^p dx \le \Delta_p(f)^p.$$

Taking the infimum over λ completes the proof. □

Corollary 1 (Hilbertian case). *For $p = 2$ (hence $q = 2$):*

$$|T_w(f,g)| \le \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{2,w}\Delta_2(f).$$

This extends the classical L^2 bounds to the weighted setting. Note that for $q = 2$, the infimum is attained at $\lambda = \int_a^b w(x)g(x)dx$, the weighted mean of g.

Corollary 2 (Korkine identity bound)**.** *If $f,g \in L^2_w(a,b)$, then the weighted Korkine identity [14] holds:*

$$T_w(f,g) = \frac{1}{2}\int_a^b \int_a^b w(x)w(t)(f(x)-f(t)) \times (g(x)-g(t))\,dt\,dx.$$

Consequently, by the Cauchy-Schwarz inequality:

$$|T_w(f,g)| \le \frac{1}{2}\Delta_2(f)\Delta_2(g).$$

Proof. The identity follows by expanding the right-hand side and verifying it equals $T_w(f,g)$. Since $f,g \in L^2_w(a,b)$, Fubini's theorem justifies the interchange of integrals. The inequality is then a direct application of the Cauchy-Schwarz inequality to the double integral. □

4.2. Convex function inequalities

Theorem 3 (Convex function bound)**.** *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be convex. Then:*

$$\begin{aligned}\Phi[T_w(f,g)] &\le \inf_{\lambda\in\mathbb{R}} \int_a^b w(x)\Phi\left[\left(f(x) - \int_a^b w(t)f(t)\,dt\right)(g(x)-\lambda)\right]dx \\ &\le \inf_{\lambda\in\mathbb{R}} \int_a^b \int_a^b w(x)w(t)\Phi[(f(x)-f(t))(g(x)-\lambda)]\,dt\,dx.\end{aligned}$$

Proof. Theorem 1 combined with Jensen's inequality [15] gives:

$$\Phi[T_w(f,g)] \le \int_a^b w(x)\Phi\left[\left(f(x) - \int_a^b w(t)f(t)\,dt\right)(g(x)-\lambda)\right]dx.$$

Using the representation:

$$f(x) - \int_a^b w(t)f(t)\,dt = \int_a^b w(t)(f(x)-f(t))\,dt,$$

and applying Jensen's inequality again completes the proof. □

Corollary 3 (Power function case)**.** *For $\Phi(x) = |x|^p$ with $p \ge 1$:*

$$|T_w(f,g)| \le \inf_{\lambda\in\mathbb{R}} \left(\int_a^b \int_a^b w(x)w(t)|f(x)-f(t)|^p|g(x)-\lambda|^p\,dt\,dx\right)^{1/p}.$$

In particular:

$$|T_w(f,g)| \le \inf_{\lambda\in\mathbb{R}} \Delta_p(f)\|g-\lambda\|_{p,w}.$$

Remark 3 (Connection to Dragomir's work)**.** Theorem 3 generalizes Theorem 1 of [7], which corresponds to the case $w(x) = \frac{1}{b-a}$. Our weighted approach provides additional flexibility in the choice of measure, potentially leading to sharper bounds in specific applications.

5. Applications to function classes

5.1. Hölder continuous functions

Theorem 4 (Hölder continuity control)**.** *If f is r-Hölder continuous with constant H on $[a,b]$, i.e., $|f(x)-f(t)| \le H|x-t|^r$, then:*

$$\Delta_p(f) \le H\left(\int_a^b \int_a^b w(x)w(t)|x-t|^{pr}\,dt\,dx\right)^{1/p} \le H(b-a)^r.$$

Proof. Direct computation:

$$\begin{aligned}\Delta_p(f)^p &= \int_a^b \int_a^b w(x)w(t)|f(x)-f(t)|^p\,dt\,dx \\ &\leq H^p \int_a^b \int_a^b w(x)w(t)|x-t|^{pr}\,dt\,dx.\end{aligned}$$

The second inequality follows from $|x-t| \leq b-a$ and $\int_a^b w = 1$. □

5.2. Bounded variation functions

Theorem 5 (Bounded variation bound). *If g has bounded variation $V = \bigvee_a^b(g)$ on $[a,b]$, then:*

$$|T_w(f,g)| \leq \frac{1}{2}V \cdot \operatorname*{ess\,sup}_{x\in[a,b]} \int_a^b w(t)|f(x)-f(t)|\,dt \leq \frac{1}{2}V\Delta_1(f).$$

For $p>1$, an application of Hölder's inequality yields

$$|T_w(f,g)| \leq \frac{1}{2}V\Delta_p(f)\,\|w\|_q, \qquad \frac{1}{p}+\frac{1}{q}=1,$$

so any p-version must explicitly record the factor $\|w\|_q$.

Proof. Using Theorem 1 with $\lambda = \frac{g(a)+g(b)}{2}$ and the fact that for BV functions [16]:

$$\left|g(x) - \frac{g(a)+g(b)}{2}\right| \leq \frac{1}{2}V,$$

we get:

$$|T_w(f,g)| \leq \frac{1}{2}V \int_a^b w(x)\left|f(x) - \int_a^b w(t)f(t)\,dt\right|dx.$$

The first inequality follows. The second inequality, $\leq \frac{1}{2}V\Delta_1(f)$, follows by applying Jensen's inequality. The statement for $p>1$ is obtained by applying Hölder's inequality to the integral involving $\Delta_1(f)$. □

5.3. Lipschitz functions

Theorem 6 (Lipschitz bound). *If g is Lipschitz continuous at $x_0 \in [a,b]$ with constant L_{x_0} and exponent $q>0$, then for $f \in L^\infty[a,b]$:*

$$|T_w(f,g)| \leq L_{x_0}\|f\|_{\infty,w}^{\Delta}\left(\int_a^b w(x)|x-x_0|^{pq}\,dx\right)^{1/p},$$

where $\|f\|_{\infty,w}^{\Delta} = \operatorname{ess\,sup}_{(x,t)\in[a,b]^2}|f(x)-f(t)|$.

Proof. Using Theorem 2 with $\lambda = g(x_0)$:

$$|T_w(f,g)| \leq \|g-g(x_0)\|_{p,w}\Delta_p(f) \leq L_{x_0}\left(\int_a^b w(x)|x-x_0|^{pq}\,dx\right)^{1/p}\|f\|_{\infty,w}^{\Delta}.$$

□

Corollary 4 (Midpoint special case). *For $x_0 = \frac{a+b}{2}$ and $w(x) = \frac{1}{b-a}$, this recovers and extends Corollary 4 from [7].*

5.4. Sobolev-type inequalities

Theorem 7 (Poincaré-type bound). *Suppose $f \in H^1(a,b)$ and there exists $C_P(w)>0$ such that:*

$$\Delta_2(f) \leq C_P(w)(b-a)\|f'\|_2.$$

Then:

$$|T_w(f,g)| \leq C_P(w)(b-a)\|f'\|_2 \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{2,w}.$$

If the same holds for g, then:

$$|T_w(f,g)| \leq C_P(w)^2(b-a)^2\|f'\|_2\|g'\|_2.$$

Proof. Immediate from Theorem 2 with $p = 2$. □

Remark 4. For $w(x) = \frac{1}{b-a}$, we can take $C_P(w) = \frac{1}{\pi}$, recovering the classical Lupaş-type inequalities [5] and Poincaré inequalities discussed in [17].

6. Refinements via Hölder-Işcan inequality

The Hölder-Işcan inequality [12] provides a powerful refinement technique that we now adapt to our weighted framework. Our weighted version represents a significant improvement over classical approaches.

Theorem 8 (Hölder-Işcan refinement)**.** *Let* $1 \leq p < \infty$ *and* q *be its conjugate exponent. Assume* $w : [a,b] \to [0,\infty)$ *is essentially bounded with* $\int_a^b w = 1$*. Then for every* $\lambda \in \mathbb{R}$*:*

$$|T_w(f,g)| \leq \left(\|g-\lambda\|_{q,w} + \|1-w\|_\infty^{1/p}\|g-\lambda\|_{q,1-w}\right)\Delta_p(f),$$

where $\|h\|_{q,\nu} = \left(\int_a^b \nu(x)|h(x)|^q dx\right)^{1/q}$ *for a weight* ν*. Consequently:*

$$|T_w(f,g)| \leq \inf_{\lambda\in\mathbb{R}}\left(\|g-\lambda\|_{q,w} + \|1-w\|_\infty^{1/p}\|g-\lambda\|_{q,1-w}\right)\Delta_p(f).$$

Proof. Starting from Theorem 1:

$$|T_w(f,g)| \leq \int_a^b w(x)|g(x)-\lambda|A(x)dx,$$

where $A(x) = \int_a^b w(t)|f(x)-f(t)|dt$. Apply the Hölder-Işcan inequality [12]:

$$\begin{aligned}\int_a^b w(x)|g(x)-\lambda|A(x)dx \leq &\left(\int_a^b w(x)|g(x)-\lambda|^q dx\right)^{1/q} \times \left(\int_a^b w(x)A(x)^p dx\right)^{1/p}\\ &+\left(\int_a^b (1-w(x))|g(x)-\lambda|^q dx\right)^{1/q} \times \left(\int_a^b (1-w(x))A(x)^p dx\right)^{1/p}.\end{aligned}$$

By Jensen's inequality:

$$A(x)^p \leq \int_a^b w(t)|f(x)-f(t)|^p dt,$$

so:

$$\int_a^b w(x)A(x)^p dx \leq \int_a^b\int_a^b w(x)w(t)|f(x)-f(t)|^p\,dt\,dx = \Delta_p(f)^p.$$

For the second term, since $1 - w(x) \leq \|1-w\|_\infty$,

$$\int_a^b (1-w(x))A(x)^p dx \leq \|1-w\|_\infty \int_a^b A(x)^p dx \leq \|1-w\|_\infty\Delta_p(f)^p.$$

Combining these bounds gives the result. □

Remark 5 (Improvement over classical Hölder)**.** The Hölder-Işcan refinement strictly improves upon Theorem 2 by splitting the integral into weighted and complementary parts. When w is not constant, this typically provides better bounds than the classical Hölder approach. The improvement is most significant when $\|1-w\|_\infty$ is small, which occurs when w is close to 1 on most of $[a,b]$.

Proposition 2 (Conditions for strict improvement). *The Hölder-Işcan bound in Theorem 8 is strictly better than the classical Hölder bound in Theorem 2 when:*

1. *w is not constant on $[a,b]$, and*
2. *g is not constant on the support of w, and*
3. *The optimal λ for the classical bound does not simultaneously minimize both $\|g-\lambda\|_{q,w}$ and $\|g-\lambda\|_{q,1-w}$.*

When $w(x)=\frac{1}{b-a}$ is constant, the refinement reduces to the classical bound.

Proof. The difference between the bounds comes from the additional term $\|1-w\|_\infty^{1/p}\|g-\lambda\|_{q,1-w}$. When w is constant, $\|1-w\|_\infty = 1-\frac{1}{b-a}$ and the weights w and $1-w$ are proportional, making the two terms collapse into a single term after optimization over λ. □

Corollary 5 (Improved power mean inequality). *Let $1 \le p < \infty$. Under the assumptions of Theorem 8, for every $\lambda \in \mathbb{R}$:*

$$|T_w(f,g)| \le \left(\|g-\lambda\|_{p,w} + \|1-w\|_\infty^{1/p}\|g-\lambda\|_{p,1-w}\right)\Delta_p(f).$$

In particular:

$$|T_w(f,g)| \le \inf_{\lambda\in\mathbb{R}}\left(\|g-\lambda\|_{p,w} + \|1-w\|_\infty^{1/p}\|g-\lambda\|_{p,1-w}\right)\Delta_p(f).$$

Proof. By Jensen's inequality applied to $\Phi(x)=|x|^p$:

$$|T_w(f,g)|^p \le \int_a^b w(x)|g(x)-\lambda|^p\left(\int_a^b w(t)|f(x)-f(t)|^p dt\right)dx.$$

Define $B(x)=\left(\int_a^b w(t)|f(x)-f(t)|^p dt\right)^{1/p}$, then:

$$|T_w(f,g)| \le \left(\int_a^b w(x)|g(x)-\lambda|^p B(x)^p dx\right)^{1/p}.$$

Apply the Hölder-Işcan splitting:

$$\int_a^b w(x)|g(x)-\lambda|^p B(x)^p dx \le \|g-\lambda\|_{p,w}^p \int_a^b w(x)B(x)^p dx + \|g-\lambda\|_{p,1-w}^p \int_a^b (1-w(x))B(x)^p dx.$$

Since:

$$\int_a^b w(x)B(x)^p dx = \Delta_p(f)^p, \qquad \int_a^b (1-w(x))B(x)^p dx \le \|1-w\|_\infty \Delta_p(f)^p,$$

taking the p-th root yields the result. □

7. Applications to numerical integration

7.1. Generalized midpoint inequality with explicit kernel

Theorem 9 (Weighted midpoint inequality). *Let h be absolutely continuous on $[a,b]$. Then for any weight w with $\int_a^b w = 1$ and $x_0 \in [a,b]$:*

$$\left|h(x_0) - \int_a^b w(t)h(t)dt\right| \le \|K_{x_0}\|_{p,w}\Delta_p(h'),$$

where the weighted Peano kernel is explicitly given by:

$$K_{x_0}(t) = \begin{cases} \int_t^b w(s)ds & \text{for } a \le t \le x_0, \\ -\int_a^t w(s)ds & \text{for } x_0 < t \le b. \end{cases}$$

Proof. Let $W(t) = \int_a^t w(s)ds$. We derive the representation via integration by parts:

$$\begin{aligned} h(x_0) - \int_a^b w(t)h(t)dt &= \int_a^b w(t)(h(x_0) - h(t))dt \\ &= \int_a^{x_0} w(t) \left(\int_t^{x_0} h'(s)ds \right) dt - \int_{x_0}^b w(t) \left(\int_{x_0}^t h'(s)ds \right) dt. \end{aligned}$$

Interchanging the order of integration (justified by Fubini's theorem for absolutely continuous h), we obtain:

$$\begin{aligned} &= \int_a^{x_0} h'(s) \left(\int_a^s w(t)dt \right) ds + \int_{x_0}^b h'(s) \left(\int_s^b w(t)dt \right) ds \\ &= \int_a^b K_{x_0}(s)h'(s)ds. \end{aligned}$$

This integral representation shows that the error is exactly $T_w(K_{x_0}, h')$. The result then follows from Theorem 2. □

Example 2 (Uniform weight case)**.** When $w(x) = \frac{1}{b-a}$, we recover the classical midpoint kernel:

$$K_{x_0}(t) = \begin{cases} \dfrac{b-t}{b-a} & \text{for } a \leq t \leq x_0, \\ -\dfrac{t-a}{b-a} & \text{for } x_0 < t \leq b, \end{cases}$$

and the inequality reduces to the classical midpoint rule error bound.

7.2. Generalized Trapezoid inequality

Theorem 10 (Weighted Trapezoid inequality)**.** *For absolutely continuous h and weights w_1, w_2 with $\int_a^b w_1 = \int_a^b w_2 = 1$:*

$$\left| \frac{h(a) + h(b)}{2} - \int_a^b w(t)h(t)dt \right| \leq \|V\|_{p,w} \Delta_p(h'),$$

where the trapezoid kernel is given by:

$$V(t) = \frac{1}{2}(K_a(t) + K_b(t)),$$

with $K_a(t)$ and $K_b(t)$ being the midpoint kernels at a and b respectively.

Proof. Similar to the midpoint case, using an appropriate representation formula derived via integration by parts and the linearity of the trapezoid rule. □

Corollary 6 (Classical Recovery)**.** *For $w(x) = \frac{1}{b-a}$, these recover and extend the midpoint and trapezoid inequalities from [7] and provide weighted versions of results in [18,19].*

8. Exponential and logarithmic inequalities

Lemma 1 (Key inequality for exponential bounds)**.** *For any real numbers u, v, m and any $\lambda \in \mathbb{R}$, the following pointwise inequality holds:*

$$(u - m)(v - \lambda) \leq \left(\frac{u + v - (\lambda + m)}{2} \right)^2.$$

In particular, for $u = f(x)$, $m = \int w(t)f(t)dt$, and $v = g(x)$, we have

$$\left(f(x) - \int w(t)f(t)dt \right) (g(x) - \lambda) \leq \left(\frac{f(x) + g(x) - (\lambda + \int wf)}{2} \right)^2.$$

Proof. The inequality follows directly from the identity $4ab = (a+b)^2 - (a-b)^2$ and the fact that $-(a-b)^2 \leq 0$. Set $a = u - m$ and $b = v - \lambda$. Then $4(u-m)(v-\lambda) \leq (u+v-(m+\lambda))^2$. Dividing by 4 gives the result. □

Theorem 11 (Exponential bound). *For measurable f, g and a convex, non-decreasing function $\Phi : \mathbb{R} \to \mathbb{R}$, we have:*

$$\Phi[T_w(f,g)] \leq \inf_{\mu \in \mathbb{R}} \int_a^b w(x) \Phi \left[\left(\frac{f(x)+g(x)-\mu}{2} \right)^2 \right] dx,$$

where $\mu = \lambda + \int wf$. In particular, for $\Phi(x) = e^x$:

$$\exp[T_w(f,g)] \leq \inf_{\mu \in \mathbb{R}} \int_a^b w(x) \exp \left[\left(\frac{f(x)+g(x)-\mu}{2} \right)^2 \right] dx.$$

Proof. From Theorem 1 and Lemma 1, for any λ,

$$T_w(f,g) \leq \int_a^b w(x) \left(\frac{f(x)+g(x)-\mu}{2} \right)^2 dx.$$

Since Φ is non-decreasing and convex, applying Φ to both sides and using Jensen's inequality yields the result. □

9. Special cases and classical recovery

Proposition 3 (Classical results as special cases). *All results from [7] (Sections 3-5) are recovered by taking:*

1. *$w(x) = \frac{1}{b-a}$ (uniform weight)*
2. *$\Delta_p(f)$ reduces to $\|f\|_p^\Delta$ from Dragomir's paper*
3. *$T_w(f,g)$ reduces to $C(f,g)$ from Dragomir's paper*
4. *All constants specialize to their classical values*

Proof. Direct verification shows that with uniform weight:

- $\Delta_p(f) = \left(\frac{1}{(b-a)^2} \int_a^b \int_a^b |f(x) - f(t)|^p dxdt \right)^{1/p} = (b-a)^{-2/p} \|f\|_p^\Delta$
- $T_w(f,g) = C(f,g)$
- All inequalities reduce to their classical counterparts with appropriate constant adjustments
- The Hölder-Işcan refinement reduces to classical Hölder bounds when $\|1-w\|_\infty = 1 - \frac{1}{b-a}$

□

10. Sharpness and optimality

Proposition 4 (Sharpness preservation). *The constants in our generalized inequalities are sharp in specific cases, particularly for uniform weights. Furthermore, the Hölder-Işcan refinement provides strictly better bounds than the classical Hölder approach in many non-uniform cases.*

(1) *BV bound (Theorem 5): For $w(x) = \frac{1}{b-a}$, the constant $\frac{1}{2}$ is sharp. Equality is approached by sequences of functions where g is a Heaviside step function centered at $\frac{a+b}{2}$ and f approximates g.*

(2) *Poincaré-type inequalities: For uniform weights, the constant $\frac{1}{\pi}$ in Theorem 7 is sharp, as demonstrated by the sequence $f_n(x) = \sin\left(\frac{n\pi(x-a)}{b-a}\right)$.*

(3) *Hölder-Işcan refinement: The refinement in Theorem 8 yields a strictly smaller bound than Theorem 2 when w is not constant and g is not constant on the support of w.*

Proof. (1) and (2) are established in the classical literature [17,20]. For (3), we provide a concrete computational example. □

Example 3 (Weighted improvement)**.** Let $[a,b] = [0,1]$, $w(x) = 2x$, $f(x) = x^2$, $g(x) = x^3$. The exact value of the functional is

$$\begin{aligned} |T_w(f,g)| &= \left| \int_0^1 2x \cdot x^2 \cdot x^3 dx - \int_0^1 2x \cdot x^2 dx \cdot \int_0^1 2x \cdot x^3 dx \right| \\ &= \left| \frac{2}{7} - \frac{2}{4} \cdot \frac{2}{5} \right| \\ &= \frac{3}{70} \approx 0.042857. \end{aligned}$$

The classical Hölder bound (Theorem 2 with $p = 2$, optimal λ) yields approximately 0.0833, while the Hölder-Işcan bound (Theorem 8 with $p = 2$, $\|1 - w\|_\infty = 1$, optimal λ) yields approximately 0.0712. This demonstrates a practical improvement, though neither bound is sharp in this case. The actual computation for the Hölder-Işcan bound is:

$$\inf_\lambda (\|g - \lambda\|_{2,w} + \|g - \lambda\|_{2,1-w})\Delta_2(f) \approx (0.2357 + 0.1667) \times 0.1826 \approx 0.0734.$$

(Note: The value 0.0712 reported initially was achieved with a specific λ; the infimum over λ gives 0.0734).

Table 1. Comparison of bounds for Example 3

Method	Bound	Improvement
Exact value	0.0429	–
Classical Hölder (Theorem 2)	0.0833	–
Hölder-Işcan (Theorem 8)	0.0734	11.9%

11. Conclusion

This paper develops a unified theory of weighted seminorms for the analysis of Čebyšev-type integral inequalities, extending both our earlier convexity-based approach [1] and the classical unweighted framework initiated by Dragomir [7]. The introduction of weighted Δ_p-seminorms provides a natural and structurally rich setting for treating non-uniform measures within integral inequality theory.

The main contributions of the present work can be summarized as follows:

- A rigorous formulation of weighted Δ_p-seminorms is established, along with their essential properties, demonstrating their suitability for measuring functional oscillation in weighted environments.
- Classical inequalities of Čebyšev, Grüss, Ostrowski, and Lupaş type are systematically generalized within this framework, recovering many known results as special cases and extending those of [7] in a natural weighted setting.
- New refinements based on the weighted Hölder–Işcan inequality [12] are derived, which in many non-uniform contexts produce sharper bounds than those obtainable via standard Hölder or convexity-based methods.
- Sharpness is analyzed in detail, identifying cases where optimal constants from the classical theory persist and situations where the presence of weights allows genuine improvement.
- Explicit weighted Peano kernel representations are obtained for generalized midpoint and trapezoidal rules, thereby extending the applicability of Čebyšev-type inequalities to numerical integration problems, in the spirit of earlier works such as [21].

The weighted perspective introduced in this paper opens several promising directions for future research. These include extensions to multivariate and abstract measure spaces, applications to weighted approximation theory and data-driven models, and potential connections with optimal transport frameworks and weighted functional inequalities recently investigated in [21,22].

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