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Chebyshev–Cauchy-type inequalities with applications to Fibonacci weighted sums

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Abstract: We study finite weighted versions of Cauchy-type inequalities and their relation to the Chebyshev functional. The main elementary device is the reflection of a weighted sequence with respect to its weighted mean. This reflection preserves the total weighted mean and the weighted second mixed moment. We apply the resulting estimates to functions sampled at Fibonacci nodes and to several Fibonacci and Lucas choices of weights and moments.

Keywords: Fibonacci numbers, Lucas numbers, weighted sums, Cauchy inequality, Chebyshev inequality, Callebaut inequality, Aczél inequality, Ozeki inequality, reflection transformation

MSC: 26D15, 11B39.

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1. Introduction

1.1. Weighted sequence inequalities

Let $n \in \mathbb{N}$. All sequences considered in this paper are real finite sequences unless stated otherwise. For

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

and for a positive weight vector

$$\mathbf{p} = (p_1, p_2, \dots, p_n) \in (0, \infty)^n,$$

put

$$P_n = \sum_{i=1}^{n} p_i.$$

We use the normalized weighted mean

$$A(\mathbf{x}) = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i. \tag{1}$$

Products and powers of sequences are understood componentwise. When non-integer powers occur, the corresponding components are assumed to be positive.

The finite weighted form of the Cauchy inequality is

$$\left(\sum_{i=1}^{n} p_i x_i y_i \right)^2 \le \left(\sum_{i=1}^{n} p_i x_i^2 \right) \left(\sum_{i=1}^{n} p_i y_i^2 \right), \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{2}$$

Equivalently, in the notation (1), (2) can be written as

$$A(\mathbf{xy})^2 \le A(\mathbf{x}^2)A(\mathbf{y}^2).$$

We refer to (2) as the Cauchy inequality; see, for example, [1, Theorem 7] and [2, Chapter 1].

The second elementary tool is the weighted Chebyshev inequality. For two real n-tuples $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, define

$$\mathcal{T}_{\mathbf{p}}(\mathbf{a}, \mathbf{b}) = P_n \sum_{i=1}^{n} p_i a_i b_i - \left(\sum_{i=1}^{n} p_i a_i \right) \left(\sum_{i=1}^{n} p_i b_i \right). \tag{3}$$

The functional (3) satisfies the identity

$$\mathcal{T}_{\mathbf{p}}(\mathbf{a}, \mathbf{b}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j (a_i - a_j)(b_i - b_j), \tag{4}$$

which implies that $\mathcal{T}_{\mathbf{p}}(\mathbf{a}, \mathbf{b}) \ge 0$ if the sequences (a_i) and (b_i) are similarly ordered, and that $\mathcal{T}_{\mathbf{p}}(\mathbf{a}, \mathbf{b}) \le 0$ if they are oppositely ordered. This is the weighted Chebyshev inequality; see Chebyshev's original work [3] and the classical treatment in [1, pp. 43–44]. In terms of (1), the Chebyshev identity (4) is the sign assertion for

$$\mathcal{T}_A(\mathbf{a}, \mathbf{b}) = A(\mathbf{ab}) - A(\mathbf{a})A(\mathbf{b}) = \frac{1}{P_n^2} \mathcal{T}_{\mathbf{p}}(\mathbf{a}, \mathbf{b}). \tag{5}$$

The form (5) will be used below to interpret one-sided reflections as Chebyshev-type quantities.

We shall also use Callebaut's refinement of the Cauchy inequality. Its first published form was given for sequences by Callebaut [4]; related forms can be found in [2].

Theorem 1 (Callebaut's inequality)**.** *Let $x_i, y_i > 0$, $i = 1, \dots, n$. If either*

$$1 \le \alpha \le \beta \le 2,$$

or

$$0 \le \beta \le \alpha \le 1,$$

then

$$\begin{aligned} \left(\sum_{i=1}^{n} p_i x_i y_i \right)^2 &\le \left(\sum_{i=1}^{n} p_i x_i^{\alpha} y_i^{2-\alpha} \right) \left(\sum_{i=1}^{n} p_i x_i^{2-\alpha} y_i^{\alpha} \right) \\ &\le \left(\sum_{i=1}^{n} p_i x_i^{\beta} y_i^{2-\beta} \right) \left(\sum_{i=1}^{n} p_i x_i^{2-\beta} y_i^{\beta} \right) \\ &\le \left(\sum_{i=1}^{n} p_i x_i^2 \right) \left(\sum_{i=1}^{n} p_i y_i^2 \right). \end{aligned} \tag{6}$$

Thus the chain (6) interpolates between the square of the mixed weighted sum and the Cauchy upper bound. Besides its original sequence form, Callebaut's refinement has been used in operator and matrix settings; for example, Wada obtained refinements of the Cauchy inequality, while Moslehian, Matharu, and Aujla proved a non-commutative Callebaut inequality for weighted operator geometric means and Hadamard products [5,6].

The next result is a finite weighted form of Aczél's reverse inequality. The classical inequality goes back to Aczél [7]; an isotonic functional formulation can be found in [8, p. 125], while related forms are discussed in [2].

Theorem 2 (Aczél's inequality)**.** *Let $f_0, g_0 \in \mathbb{R}$, and let $x_i, y_i \in \mathbb{R}$, $i = 1, \ldots, n$. If*

$$P_n g_0^2 - \sum_{i=1}^{n} p_i y_i^2 > 0 \qquad or \qquad P_n f_0^2 - \sum_{i=1}^{n} p_i x_i^2 > 0,$$

then

$$\left(P_n f_0 g_0 - \sum_{i=1}^{n} p_i x_i y_i \right)^2 \geq \left(P_n f_0^2 - \sum_{i=1}^{n} p_i x_i^2 \right) \left(P_n g_0^2 - \sum_{i=1}^{n} p_i y_i^2 \right). \tag{7}$$

The weighted form (7) also has a substantial operator-theoretic literature. For instance, Moslehian established operator versions involving weighted operator geometric means, positive sesquilinear forms, unital positive linear maps on C^*-algebras, and unitarily invariant matrix norms [9].

In [10], Izumino and Pečarić proved the following weighted Ozeki inequality, which is a complement of the Cauchy inequality.

Theorem 3 (Weighted Ozeki inequality)**.** *Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be positive n-tuples satisfying*

$$0 < m_1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq M_1, \qquad 0 < m_2 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq M_2. \tag{8}$$

Let $\mathbf{p}$ be a positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$, and put $P_k = \sum_{i=1}^{k} p_i$. Then

$$\sum_{i=1}^{n} p_i x_i^2 \sum_{i=1}^{n} p_i y_i^2 - \left(\sum_{i=1}^{n} p_i x_i y_i \right)^2 \leq (M_1 M_2 - m_1 m_2)^2 \max_{1 \leq k \leq n-1} P_k (1 - P_k)$$
$$\leq \frac{1}{4} (M_1 M_2 - m_1 m_2)^2. \tag{9}$$

The assumptions (8) and the estimate (9) will be used below through a reflected version adapted to unnormalized weights.

We finish this subsection with three finite weighted forms of Beesack–Pečarić inequalities. For a set of indices $I \subset \{1, \ldots, n\}$, write I^c for its complement and

$$P_I = \sum_{i \in I} p_i, \qquad S_I(h) = \sum_{i \in I} p_i h_i.$$

Theorem 4. (See [11] and [8, p. 122].) *Let I be a non-empty proper subset of $\{1, \ldots, n\}$. Let $g_i \geq 0$, $i = 1, \ldots, n$, let $p > 1$, and assume that $S_I(g) > 0$. Define*

$$g_I^*(i) = \begin{cases} g_i, & i \in I, \\ \left(\dfrac{S_I(g^p)}{S_I(g)} \right)^{1/(p-1)}, & i \in I^c. \end{cases}$$

Then

$$\frac{\left(\sum_{i=1}^{n} p_i g_i^p \right)^{1/p}}{\sum_{i=1}^{n} p_i g_i} \geq \frac{\left(\sum_{i=1}^{n} p_i g_I^*(i)^p \right)^{1/p}}{\sum_{i=1}^{n} p_i g_I^*(i)}. \tag{10}$$

The quotient estimate (10) is the first cut-off inequality that will be applied to reflected samples.

Theorem 5. (Beckenbach's Inequality, see [11] and [8, p. 122].) *Let $f_i, g_i \geq 0$, $i = 1, \ldots, n$, let $p > 1$, and let $1/p + 1/q = 1$. Let I be a non-empty proper subset of $\{1, \ldots, n\}$. Assume that*

$$S_I(fg) > 0, \qquad S_{I^c}(g^q) > 0.$$

Define

$$\widetilde{f}_I(i) = \begin{cases} f_i, & i \in I, \\ \left(\frac{g_i S_I(f^p)}{S_I(fg)}\right)^{q/p}, & i \in I^c. \end{cases}$$

Then

$$\frac{\left(\sum_{i=1}^n p_i f_i^p\right)^{1/p}}{\sum_{i=1}^n p_i f_i g_i} \geq \frac{\left(\sum_{i=1}^n p_i \widetilde{f}_I(i)^p\right)^{1/p}}{\sum_{i=1}^n p_i \widetilde{f}_I(i) g_i}. \tag{11}$$

The inequality (11) is the corresponding two-sequence cut-off estimate.

Theorem 6. (See [11, p. 639].) *Let $f_i, g_i \geq 0$, $i = 1, \ldots, n$, let $p > 1$, and let $1/p + 1/q = 1$. Let I be a non-empty proper subset of $\{1, \ldots, n\}$. Assume that*

$$S_I(fg) > 0, \qquad S_{I^c}(g^q) > 0.$$

Then

$$\left(\sum_{i=1}^n p_i g_i^p\right)^{1/p} P_n^{1/q} - \sum_{i=1}^n p_i g_i \geq S_I(g^p)^{1/p} P_I^{1/q} - S_I(g), \tag{12}$$

and

$$\left(\sum_{i=1}^n p_i f_i^p\right)^{1/p} \left(\sum_{i=1}^n p_i g_i^q\right)^{1/q} - \sum_{i=1}^n p_i f_i g_i \geq S_I(f^p)^{1/p} S_I(g^q)^{1/q} - S_I(fg). \tag{13}$$

The two estimates (12) and (13) will be used in the Fibonacci setting after applying the reflection introduced next.

1.2. A reflection transformation for weighted sums

Associated with the weighted mean (1) is the reflection of a sequence with respect to its weighted mean:

$$R\mathbf{x} = 2A(\mathbf{x})\mathbf{1} - \mathbf{x}, \qquad (R\mathbf{x})_i = \frac{2}{P_n} \sum_{j=1}^n p_j x_j - x_i.$$

Since $A(\mathbf{1}) = 1$, we have

$$\sum_{i=1}^n p_i (R\mathbf{x})_i = \sum_{i=1}^n p_i x_i. \tag{14}$$

A stronger identity is

$$\sum_{i=1}^n p_i (R\mathbf{u})_i (R\mathbf{v})_i = \sum_{i=1}^n p_i u_i v_i. \tag{15}$$

Indeed, expanding the left-hand side gives

$$4A(\mathbf{u})A(\mathbf{v})P_n - 2A(\mathbf{u}) \sum_{i=1}^n p_i v_i - 2A(\mathbf{v}) \sum_{i=1}^n p_i u_i + \sum_{i=1}^n p_i u_i v_i,$$

which reduces to the right-hand side. This transformation and its applications to Cauchy and related inequalities were studied in [12].

The one-sided reflection is related to the Chebyshev functional. For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$\sum_{i=1}^n p_i (R\mathbf{a})_i b_i = \frac{2}{P_n} \left(\sum_{i=1}^n p_i a_i\right) \left(\sum_{i=1}^n p_i b_i\right) - \sum_{i=1}^n p_i a_i b_i = \frac{1}{P_n} \left(\sum p_i a_i\right) \left(\sum p_i b_i\right) - \frac{1}{P_n} \mathcal{T}_{\mathbf{p}}(\mathbf{a}, \mathbf{b}). \tag{16}$$

Thus the reflection identities (14) and (15) make reflected Cauchy estimates naturally produce inequalities in which the Chebyshev functional appears.

2. Main results

We now specialize the weighted sums to Fibonacci nodes. Let $(F_m)_{m\geq 0}$ denote the Fibonacci sequence,

$$F_0 = 0, \qquad F_1 = 1, \qquad F_m = F_{m-1} + F_{m-2} \quad (m \geq 2).$$

For a real-valued function f defined at the points $F_1, F_2, \ldots, F_n$, set

$$A^F_{n,\mathbf{p}}(f) = \frac{1}{P_n}\sum_{j=1}^{n} p_j f(F_j), \qquad P_n = \sum_{j=1}^{n} p_j. \tag{17}$$

Equivalently, (17) is (1) applied to $x_j = f(F_j)$.

The choice of the weights p_j can be adapted to classical summation identities for Fibonacci numbers; see, for example, [5, 8]. Some useful choices are

$$p_j = F_j^2, \qquad P_n = \sum_{j=1}^{n} F_j^2 = F_n F_{n+1}, \tag{18}$$

$$p_j = F_j, \qquad P_n = \sum_{j=1}^{n} F_j = F_{n+2} - 1, \tag{19}$$

$$p_j = F_{2j-1}, \qquad P_n = \sum_{j=1}^{n} F_{2j-1} = F_{2n}, \tag{20}$$

$$p_j = F_{2j}, \qquad P_n = \sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1, \tag{21}$$

$$p_j = jF_j, \qquad P_n = \sum_{j=1}^{n} jF_j = nF_{n+2} - F_{n+3} + 2, \tag{22}$$

$$p_j = F_{4j-2}, \qquad P_n = \sum_{j=1}^{n} F_{4j-2} = F_{2n}^2, \tag{23}$$

$$p_j = \binom{n}{j} F_j, \qquad P_n = \sum_{j=1}^{n} \binom{n}{j} F_j = F_{2n}, \tag{24}$$

$$p_j = F_j F_{j+1}, \qquad P_n = \sum_{j=1}^{n} F_j F_{j+1} = F_{n+1}^2 - \frac{1+(-1)^n}{2}. \tag{25}$$

The identities (18)- (25) are the source of the concrete normalizing constants used below.

Throughout this section we use the shorthand

$$S(h) = \sum_{i=1}^{n} p_i h(F_i), \qquad A(h) = \frac{S(h)}{P_n}.$$

If $I \subseteq \{1, \ldots, n\}$, put

$$I^c = \{1, \ldots, n\} \setminus I, \qquad P_I = \sum_{i\in I} p_i, \qquad S_I(h) = \sum_{i\in I} p_i h(F_i).$$

The reflected values are denoted by

$$(Rh)(F_i) = 2A(h) - h(F_i) = \frac{2S(h)}{P_n} - h(F_i).$$

Then

$$S(Rh) = S(h), \qquad S((Rh)(Rk)) = S(hk), \tag{26}$$

for all functions h and k defined at the Fibonacci nodes.

2.1. Chebyshev–Cauchy-type inequalities

The next result, Theorem 7, is a Cauchy estimate for a reflected mixed term. In view of (16), it is also a Chebyshev-type estimate.

Theorem 7. *For all real-valued functions f and g defined at $F_1, \ldots, F_n$,*

$$\left| 2\left(\sum_{i=1}^{n} p_i f(F_i)\right)\left(\sum_{i=1}^{n} p_i g(F_i)\right) - P_n \sum_{i=1}^{n} p_i f(F_i) g(F_i) \right| \le P_n \left(\sum_{i=1}^{n} p_i f^2(F_i)\right)^{1/2} \left(\sum_{i=1}^{n} p_i g^2(F_i)\right)^{1/2}. \tag{27}$$

Proof. Applying the Cauchy inequality to the sequences $f(F_i)$ and $(Rg)(F_i)$, and using (26), gives

$$|S(fRg)| \le \sqrt{S(f^2)S((Rg)^2)} = \sqrt{S(f^2)S(g^2)}.$$

Since

$$S(fRg) = \frac{2S(f)S(g)}{P_n} - S(fg),$$

multiplication by P_n gives (27). □

Theorem 8. *Let I be a non-empty proper subset of $\{1, 2, \ldots, n\}$. Then the following assertions hold.*

i) *If $Rg(F_i) \ge 0$ for all i and $S_I(Rg) > 0$, define*

$$\bar{g}_I(F_i) = \begin{cases} Rg(F_i), & i \in I, \\ \dfrac{S_I((Rg)^2)}{S_I(Rg)}, & i \in I^c. \end{cases}$$

Then

$$\frac{\sqrt{S(g^2)}}{S(g)} \ge \frac{\sqrt{S(\bar{g}_I^2)}}{S(\bar{g}_I)}. \tag{28}$$

i) *If $f(F_i) \ge 0$, $Rg(F_i) \ge 0$ for all i, and $S_I(fRg) > 0$, $S_{I^c}((Rg)^2) > 0$, define*

$$\bar{f}_I(F_i) = \begin{cases} f(F_i), & i \in I, \\ \dfrac{Rg(F_i) S_I(f^2)}{S_I(fRg)}, & i \in I^c. \end{cases}$$

Then

$$\frac{\sqrt{S(f^2)}}{S(fRg)} \ge \frac{\sqrt{S(\bar{f}_I^2)}}{S(\bar{f}_I Rg)}. \tag{29}$$

iii) *If $Rf(F_i) \ge 0$, $g(F_i) \ge 0$ for all i, and $S_I(Rf\,g) > 0$, $S_{I^c}(g^2) > 0$, define*

$$\widehat{f}_I(F_i) = \begin{cases} Rf(F_i), & i \in I, \\ \dfrac{g(F_i) S_I((Rf)^2)}{S_I(Rf\,g)}, & i \in I^c. \end{cases}$$

Then

$$\frac{\sqrt{S(f^2)}}{S(Rf\,g)} \ge \frac{\sqrt{S(\widehat{f}_I^2)}}{S(\widehat{f}_I g)}. \tag{30}$$

iv) *If* $Rg(F_i) \geq 0$ *for all* i, *then*

$$\sqrt{P_n S(g^2) - S(g)} \geq \sqrt{P_I S_I((Rg)^2) - S_I(Rg)}. \tag{31}$$

v) *If* $f(F_i) \geq 0$ *and* $Rg(F_i) \geq 0$ *for all* i, *then*

$$\sqrt{S(f^2)S(g^2) - S(fRg)} \geq \sqrt{S_I(f^2)S_I((Rg)^2) - S_I(fRg)}. \tag{32}$$

Proof. We apply Theorems 4, 5, and 6 with $p = q = 2$, to the finite weighted sums over the Fibonacci nodes.

For (i), Theorem 4 is applied to the non-negative sequence $Rg(F_i)$. Since $S(Rg) = S(g)$ and $S((Rg)^2) = S(g^2)$, it gives (28). The definition of $\bar{g}_I$ is exactly the corresponding cut-off sequence.

For (ii), Theorem 5 is applied to the pair (f, Rg). The cut-off sequence becomes $\bar{f}_I$, because on I^c it is

$$\frac{Rg(F_i)S_I(f^2)}{S_I(fRg)}.$$

This gives (29). For (iii), the same argument is applied to the pair (Rf, g), and the global identity $S((Rf)^2) = S(f^2)$ gives (30).

For (iv), use (12) with the sequence Rg. The local term remains $S_I((Rg)^2)$; in general it cannot be replaced by $S_I(g^2)$. Finally, (v) follows from (13) applied to (f, Rg), again using only the global equality $S((Rg)^2) = S(g^2)$. □

2.2. Callebaut and Aczél inequalities for Fibonacci-reflected moments

The estimates (28)–(32) are useful when the index set is chosen according to Fibonacci summation identities. The weighted sum $A^F_{n,\mathbf{p}}$ may be viewed as expectation with respect to the probability distribution $\mathbb{P}(J = j) = p_j/P_n$ on the Fibonacci nodes. The reflection $Rh = 2A(h) - h$ replaces each sampled value by its mirror image with respect to the weighted mean. Therefore, Callebaut's inequality controls a whole interpolation scale of reflected mixed moments, while Aczél's inequality provides a reverse estimate for reflected mixed products.

Theorem 9 (Reflected Callebaut inequality)**.** *Let* f *and* g *be functions defined at* $F_1, \ldots, F_n$ *such that* $Rf(F_i) > 0$ *and* $Rg(F_i) > 0$ *for all* i. *If either* $1 \leq \alpha \leq \beta \leq 2$ *or* $0 \leq \beta \leq \alpha \leq 1$, *then*

$$\begin{aligned} S(fg)^2 &\leq S\big((Rf)^\alpha (Rg)^{2-\alpha}\big) S\big((Rf)^{2-\alpha}(Rg)^\alpha\big) \\ &\leq S\big((Rf)^\beta (Rg)^{2-\beta}\big) S\big((Rf)^{2-\beta}(Rg)^\beta\big) \\ &\leq S(f^2)S(g^2). \end{aligned} \tag{33}$$

Proof. Apply Theorem 1 to the positive sequences $Rf(F_i)$ and $Rg(F_i)$, and use (26). □

Theorem 10 (Reflected Aczél inequality)**.** *Let* $f_0, g_0 \in \mathbb{R}$, *and let* f *and* g *be real-valued functions defined at* $F_1, \ldots, F_n$. *If*

$$P_n g_0^2 - S(g^2) > 0 \qquad \text{or} \qquad P_n f_0^2 - S(f^2) > 0,$$

then

$$\left(P_n f_0 g_0 - \frac{2S(f)S(g)}{P_n} + S(fg)\right)^2 \geq \big(P_n f_0^2 - S(f^2)\big)\big(P_n g_0^2 - S(g^2)\big). \tag{34}$$

Proof. Apply Theorem 2 to the sampled sequences $f(F_i)$ and $(Rg)(F_i)$. Since

$$S(fRg) = \frac{2S(f)S(g)}{P_n} - S(fg), \qquad S((Rg)^2) = S(g^2),$$

the result is precisely (34). □

2.3. A converse inequality

We next derive a reflected form of the weighted Ozeki inequality for the unnormalized weights used throughout this paper. The proof consists of applying Theorem 1.3 to the normalized weights $q_i = p_i/P_n$, and then returning to the original weighted sums.

Theorem 11. *Let* $\mathbf{p} = (p_1, \dots, p_n)$ *be a positive n-tuple, and put*

$$P_k = \sum_{i=1}^{k} p_i, \qquad k = 1, \dots, n.$$

Let f and g be real-valued functions defined at $F_1, \dots, F_n$*. Suppose that*

$$0 < m_1 \le f(F_1) \le f(F_2) \le \cdots \le f(F_n) \le M_1, \tag{35}$$

and

$$0 < m_2 \le Rg(F_1) \le Rg(F_2) \le \cdots \le Rg(F_n) \le M_2. \tag{36}$$

Then

$$\begin{aligned} S(f^2)S(g^2) - \left(\frac{2S(f)S(g)}{P_n} - S(fg)\right)^2 &\le (M_1M_2 - m_1m_2)^2 \max_{1\le k\le n-1} P_k(P_n - P_k) \\ &\le \frac{P_n^2}{4}(M_1M_2 - m_1m_2)^2. \end{aligned} \tag{37}$$

Proof. Put $q_i = p_i/P_n$ and $Q_k = P_k/P_n$. Then $\sum_i q_i = 1$. Apply Theorem 3 to

$$x_i = f(F_i), \qquad z_i = Rg(F_i) = \frac{2S(g)}{P_n} - g(F_i),$$

with the normalized weights q_i. Multiplication by P_n^2 gives the right-hand side in (37). The left-hand side is transformed by the reflection identities

$$\sum_{i=1}^{n} p_i z_i^2 = S(g^2), \qquad \sum_{i=1}^{n} p_i x_i z_i = \frac{2S(f)S(g)}{P_n} - S(fg).$$

Finally, $P_k(P_n - P_k) \le P_n^2/4$. □

Remark 1. The condition (35) does not require global monotonicity of the function f. It only requires that the sampled sequence $f(F_1), \dots, f(F_n)$ is non-decreasing. Similarly, (36) is a condition on the reflected sampled sequence $Rg(F_1), \dots, Rg(F_n)$. Since $Rg(F_i) = 2A(g) - g(F_i)$, this is equivalent to saying that the sampled sequence $g(F_1), \dots, g(F_n)$ is ordered in the opposite direction.

3. Applications

3.1. Odd-even Fibonacci splitting

Let $p_i = F_i$, $i = 1, \dots, 2m$, and put

$$\Pi_m = \sum_{i=1}^{2m} F_i = F_{2m+2} - 1.$$

For a function u defined at $F_1, \dots, F_{2m}$, write

$$S_m(u) = \sum_{i=1}^{2m} F_i u(F_i),$$

$$S_m^o(u)=\sum_{r=1}^{m}F_{2r-1}u(F_{2r-1}),\qquad S_m^e(u)=\sum_{r=1}^{m}F_{2r}u(F_{2r}).$$

By (20) and (21),

$$\sum_{r=1}^{m}F_{2r-1}=F_{2m},\qquad \sum_{r=1}^{m}F_{2r}=F_{2m+1}-1.$$

Corollary 1 (Odd-even Fibonacci splitting). *If $u(F_i)\geq 0$ for $i=1,\dots,2m$, then*

$$\sqrt{\Pi_m S_m(u^2)}-S_m(u)\geq\sqrt{F_{2m}S_m^o(u^2)}-S_m^o(u),\tag{38}$$

$$\sqrt{\Pi_m S_m(u^2)}-S_m(u)\geq\sqrt{(F_{2m+1}-1)S_m^e(u^2)}-S_m^e(u).\tag{39}$$

Proof. Choose $g=Ru$ in Theorem 8(iv). Then $Rg=u$, $S(g)=S(u)$, and $S(g^2)=S(u^2)$. Applying (31) to the odd index set and then to the even index set gives the two displayed inequalities. □

3.2. Fibonacci–Lucas moments

Let $(L_m)_{m\geq 0}$ denote the Lucas sequence,

$$L_0=2,\qquad L_1=1,\qquad L_m=L_{m-1}+L_{m-2}\quad(m\geq 2).$$

We use the standard identities

$$F_iL_i=F_{2i},\qquad F_iL_i^2=F_{3i}+(-1)^iF_i,\qquad L_iF_i^2=\frac{L_{3i}-(-1)^iL_i}{5},$$

which are classical Fibonacci–Lucas identities; see [13,14].

In the rest of this subsection the weights are $p_i=F_i$, $i=1,\dots,2m$, and

$$\xi(F_i)=F_i,\qquad \lambda(F_i)=L_i.$$

The closed forms used below are

$$C_m=\sum_{i=1}^{2m}F_i^3=\frac{F_{6m+2}+5-6F_{2m-1}}{10},\tag{40}$$

$$D_m=\sum_{i=1}^{2m}F_iL_i^2=\frac{F_{6m+2}+2F_{2m-1}-3}{2},\tag{41}$$

$$Q_m=\sum_{i=1}^{2m}F_i^2L_i=\frac{L_{6m+2}-2L_{2m-1}-5}{10}.\tag{42}$$

The identities (40)–(42) follow from the classical Fibonacci–Lucas identities displayed above.

Corollary 2 (Callebaut interpolation for Fibonacci–Lucas moments). *Let $1\leq\alpha\leq\beta\leq 2$ or $0\leq\beta\leq\alpha\leq 1$. Then*

$$\begin{aligned}Q_m^2&\leq\left(\sum_{i=1}^{2m}F_i^{\alpha+1}L_i^{2-\alpha}\right)\left(\sum_{i=1}^{2m}F_i^{3-\alpha}L_i^{\alpha}\right)\\&\leq\left(\sum_{i=1}^{2m}F_i^{\beta+1}L_i^{2-\beta}\right)\left(\sum_{i=1}^{2m}F_i^{3-\beta}L_i^{\beta}\right)\\&\leq C_mD_m.\end{aligned}\tag{43}$$

Proof. Apply Theorem 9 to $f=R\xi$ and $g=R\lambda$. Then $Rf=\xi$ and $Rg=\lambda$, so the middle factors in (33) become the displayed Fibonacci–Lucas sums. The left endpoint is $S_m(\xi\lambda)^2=Q_m^2$, while the right endpoint is $S_m(\xi^2)S_m(\lambda^2)=C_mD_m$. This proves (43). □

Corollary 3 (Aczél estimate for Fibonacci–Lucas moments). *Let $a, b \in \mathbb{R}$. If*

$$a^2 > \frac{C_m}{\Pi_m} \qquad \text{or} \qquad b^2 > \frac{D_m}{\Pi_m},$$

then

$$(\Pi_m ab - Q_m)^2 \geq (\Pi_m a^2 - C_m)(\Pi_m b^2 - D_m). \tag{44}$$

Proof. Apply Theorem 10 to $f = \xi$ and $g = R\lambda$. Since $Rg = \lambda$,

$$\frac{2S_m(\xi)S_m(g)}{\Pi_m} - S_m(\xi g) = S_m(\xi\lambda) = Q_m,$$

and $S_m(g^2) = S_m(\lambda^2) = D_m$. Also $S_m(\xi^2) = C_m$, and hence (44) follows. □

Corollary 4 (Ozeki converse for Fibonacci–Lucas moments). *With the notation above,*

$$\begin{aligned} C_m D_m - Q_m^2 &\leq F_{2m}(F_{2m+1} - 1)(F_{4m} - 1)^2 \\ &= \frac{(F_{2m+2} - 1)^2 - (F_{2m-1} - 1)^2}{4}(F_{4m} - 1)^2 \\ &\leq \frac{(F_{2m+2} - 1)^2}{4}(F_{4m} - 1)^2. \end{aligned} \tag{45}$$

Proof. Apply Theorem 11 with weights $p_i = F_i$, with $f = \xi$, and with $g = R\lambda$. Then

$$f(F_i) = F_i, \qquad Rg(F_i) = \lambda(F_i) = L_i,$$

so the required sampled sequences are increasing. We may take $m_1 = m_2 = 1$, $M_1 = F_{2m}$, and $M_2 = L_{2m}$. Since $F_{2m}L_{2m} = F_{4m}$, the Ozeki factor is $(F_{4m} - 1)^2$. Moreover,

$$S_m(f^2) = C_m, \qquad S_m(g^2) = S_m(\lambda^2) = D_m, \qquad S_m(fRg) = S_m(\xi\lambda) = Q_m.$$

For Fibonacci weights,

$$P_k = \sum_{i=1}^{k} F_i = F_{k+2} - 1, \qquad P_{2m} = F_{2m+2} - 1.$$

Hence the maximum in Theorem 11 is

$$\max_{1 \leq k \leq 2m-1} (F_{k+2} - 1)(F_{2m+2} - F_{k+2}).$$

This is the maximization of the parabola $x(P_{2m} - x)$ on the strictly increasing set $x = F_{k+2} - 1$. Since

$$F_{2m} - 1 \leq \frac{F_{2m+2} - 1}{2} \leq F_{2m+1} - 1,$$

it is enough to compare the neighboring values $k = 2m - 2$ and $k = 2m - 1$. Their difference is

$$(F_{2m+1} - 1)F_{2m} - (F_{2m} - 1)F_{2m+1} = F_{2m-1} > 0.$$

Therefore the maximum is attained at $k = 2m - 1$ and equals

$$(F_{2m+1} - 1)F_{2m}.$$

This proves the first line of (45). The identity

$$F_{2m}(F_{2m+1} - 1) = \frac{(F_{2m+2} - 1)^2 - (F_{2m-1} - 1)^2}{4}$$

gives the second line, and the last line is immediate. □

Remark 2. The same scheme gives purely Fibonacci estimates by taking $u(F_i) = F_i$ in Corollary 1. Then

$$\sum_{i=1}^{2m} F_i^2 = F_{2m}F_{2m+1}, \qquad \sum_{i=1}^{2m} F_i^3 = \frac{F_{6m+2} + 5 - 6F_{2m-1}}{10},$$

and the odd and even refinements follow from (38) and (39).

4. Concluding remarks

The purpose of the preceding sections is to set up a flexible model in which classical weighted inequalities can be combined with explicit Fibonacci summation identities. In particular, each of the normalizations listed in (18)–(25) may be inserted into the weighted scheme used in Theorem 7, Theorem 8, and the reflected Callebaut, Aczél, and Ozeki forms. The last two sections show this procedure for the Fibonacci weights $p_i = F_i$, the odd-even decomposition of indices, and Fibonacci-Lucas moments; the same argument applies to the other choices of weights from the displayed list.

The construction is not limited to taking the weight and the sampled value from the same Fibonacci expression. For example, one may take

$$p_j = \binom{n}{j}, \qquad f(F_j) = F_j,$$

or

$$p_j = F_j, \qquad f(F_j) = F_{j+1},$$

and then apply the same weighted inequalities to the corresponding sums. More generally, any inequality which admits a positive weighted finite-sum form can be used in this framework. The choice of the weights determines the normalizing constant, while the choice of the sampled functions determines the Fibonacci or Lucas moments that occur in the final estimate. Thus Corollaries 1, 2, 3, and 4 should be viewed as representative examples rather than as an exhaustive list of possible applications.

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