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Graphical illustrations of integral inequalities involving tgs-convex functions via Riemann-Liouville fractional integrals

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Abstract: In this paper, integral midpoint type inequalities involving Riemann-Liouville fractional integrals for tgs-convex functions are proved. Two different identities are utilized to get some new integral midpoint type inequalities. One identity is used to obtain inequalities for functions whose first derivatives are tgs-convex functions and another identity is used to obtain inequalities for the functions whose second derivatives are tgs-convex. Some numerical examples along with graphical representation are also included to demonstrate the effectiveness of the results. The results demonstrate that the newly established bounds offer significant improvements and tighter estimates compared to existing inequalities in the literature.

Keywords: convex function, integral midpoint inequality, Hölder inequality, power mean inequality, Riemann-Liouville fractional integrals

MSC: 26A51, 26D15.

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1. Introduction

It has been demonstrated that inequalities are the most effective construction tools for a number of mathematical fields. Inequalities play a fundamental role in the study of integral equations, differential equations, optimization theory, and fractional calculus. Some recent mathematical inequalities can be found in [1–3].

Convexity is a phrase that has gained much focus and developed into a valuable origin of knowledge and inspiration. For enthusiastic readers, the literature on convex analysis, convex functions, and their applications can be found in [4] and tgs-convex function is available in [5]. In recent years, several authors have investigated fractional integral inequalities for different generalized convex functions, including h-convex, (α, s)-convex, and twice differentiable functions. For example, recent contributions on fractional inequalities and midpoint-type inequalities for generalized convex functions can be found in [6,7].

Fractional integrals have an essential role in the theory of inequalities and their applications. Mehreen *et al.* [8] presented the integral identities related with integral midpoint inequality for tgs-convex functions. Fahad *et al.* [9] obtain some inequalities for Caputo Fabrizio fractional integrals involving (α, s) convex functions with applications. Qaisar *et al.* obtained some integral midpoint inequalities involving fractional integrals for convex functions [10].

In addition, inequalities associated with convexity have also been extensively studied in the setting of time scales calculus, which unifies continuous and discrete analysis. Several important contributions include Ostrowski-type inequalities for functions whose derivative modulus is relatively convex on time scales, generalized Hölder and Minkowski inequalities on diamond-alpha time scales, and multidimensional reverse Hölder inequalities on time scales [11–13]. These works demonstrate the broad applicability of convexity-based inequalities in both continuous and discrete frameworks.

Because of the numerous uses of integral midpoint type inequalities [14,15] and fractional calculus [16], the objective of present work to investigate the integral midpoint type inequalities involving fractional integrals. Although tgs-convex functions can be viewed as a special case of h-convex functions, the analysis of midpoint-type fractional inequalities for tgs-convex functions leads to different estimates and sharper structures associated with the symmetry of the tgs-condition. Moreover, the present work provides graphical validations and numerical comparisons of the obtained inequalities, which, to the best of our knowledge, have not been investigated previously for this class of functions. Even in more generalized settings such as those considered in [17,18], graphical illustrations and numerical validations for tgs-convex fractional midpoint inequalities are still unavailable.

The framework of the paper is as follows: After some preliminary facts in §2, some integral midpoint type inequalities are proved in §3 and §4, for these functions whose absolute values of first or second derivatives are tgs-convex respectively. The graphical representations of obtained results are given in §5. Moreover some numerical examples are provided to make comparison of obtained results with existing results.

2. Preliminaries

Jacques Hadamard and Charles Hermite derived the integral midpoint type inequality, which is expressed as:

Definition 1 (Integral midpoint inequality)**.** Let $\zeta : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $\iota, \kappa \in I$ with $\iota < \kappa$, then the following double inequality holds:

$$\zeta\left(\frac{\iota+\kappa}{2}\right) \leq \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa} \zeta(h)dh \leq \frac{\zeta(\iota)+\zeta(\kappa)}{2}. \tag{1}$$

If ζ is concave, then (1) holds in reverse direction.

Definition 2 (h-convex function [19])**.** Let $h : J \to R$ be a non-negative function, $h \neq 0$. We say that $\zeta : I \to \mathbb{R}$ is an h-convex function, if

$$\zeta(\varrho\iota + (1-\varrho)\kappa) \leq h(\varrho)\zeta(\iota) + h(1-\varrho)\zeta(\kappa), \tag{2}$$

holds $\forall \iota, \kappa \in I, \varrho \in (0,1)$.

Definition 3 (tgs-convex function [19])**.** A function $\zeta : I \subseteq \mathbb{R} \to \mathbb{R}$ is tgs-convex, if

$$\zeta(\varrho\iota + (1-\varrho)\kappa) \leq \varrho(1-\varrho)(\zeta(\iota)+\zeta(\kappa)), \tag{3}$$

holds $\forall\ \iota, \kappa \in I, \varrho \in (0,1)$.

If we take $h(\varrho) = \varrho(1-\varrho)$ in (2), then (2) reduces to (3). Namely, definition of tgs-convex function may be regarded as special case of h-convex function, see [20].

Definition 4 (Beta function [16])**.** The classical Euler beta function, denoted by $\beta(\iota, \kappa)$, is defined by

$$\beta(\iota,\kappa) = \int_0^1 \varrho^{\iota-1}(1-\varrho)^{\kappa-1} \mathrm{d}\varrho, \tag{4}$$

for any $\iota, \kappa > 0$.

Sarikaya *et al.* [21] introduced the following integral midpoint type inequalities involving Riemann-Liouville fractional integrals for the class of classical convex functions:

Theorem 1. *Suppose that $\zeta : [\iota, \kappa] \to \mathbb{R}$ is a positive function, where $0 \leq \iota \leq \kappa$ with $\zeta \in L[\iota, \kappa]$. If ζ is a convex function on $[\iota, \kappa]$, then*

$$\zeta\left(\frac{\iota+\kappa}{2}\right) \leq \frac{\Gamma(\vartheta+1)}{2(\kappa-\iota)}[J_{\iota^+}^{\vartheta}\zeta(\kappa) + J_{\kappa^-}^{\vartheta}\zeta(\iota)] \leq \frac{\zeta(\iota)+\zeta(\kappa)}{2}, \tag{5}$$

hold, where the left and right sided Riemann-Liouville fractional integrals of the order $\vartheta \in \mathbb{R}_+ = [0, \infty)$ *are denoted by* $J^{\vartheta}_{\iota^+}\zeta$ *and* $J^{\vartheta}_{\kappa^-}\zeta$*:*

$$(J^{\vartheta}_{\iota^+}\zeta)(h) = \frac{1}{\Gamma(\vartheta)}\int_{\iota}^{h}(h-\varrho)^{\vartheta-1}\zeta(\varrho)d\varrho; \quad 0 \leq \iota < h \leq \kappa,$$
$$(J^{\vartheta}_{\kappa^-}\zeta)(h) = \frac{1}{\Gamma(\vartheta)}\int_{h}^{\kappa}(\varrho-h)^{\vartheta-1}\zeta(\varrho)d\varrho; \quad 0 \leq \iota < h \leq \kappa,$$

respectively and $\Gamma(\cdot)$ *is classical Euler Gamma function.*

Theorem 2. *Suppose that* $\zeta : [\iota, \kappa] \to \mathbb{R}$ *is a differentiable mapping on* (ι, κ) *with* $\iota < \kappa$*. Let* $|\zeta'|$ *is convex on* $[\iota, \kappa]$*, then*

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{\Gamma(\vartheta+1)}{2(\kappa-\iota)^{\vartheta}}[J^{\vartheta}_{\iota^+}\zeta(\kappa) + J^{\vartheta}_{\kappa^-}\zeta(\iota)]\right| \leq \frac{\kappa-\iota}{2(\vartheta+1)}\left(1-\frac{1}{2^{\vartheta}}\right)[\zeta'(\iota)+\zeta'(\kappa)], \tag{6}$$

holds.

Khan *et al.* [22] and Dargomir *et al.* [23] presented the following lemmas listed below:

Lemma 1. *Consider* $\zeta : I \subseteq \mathbb{R} \to \mathbb{R}$ *is differentiable on* I^o*, where* $\iota, \kappa \in I^o$ *and* $\iota < \kappa$*. If* $\zeta' \in L[\iota, \kappa]$ *and* $\vartheta > 0$*, then the equality*

$$\begin{aligned}\zeta(h) - \frac{\Gamma(\vartheta+1)}{2}\left[\frac{J^{\vartheta}_{h^-}\zeta(\iota)}{(h-\iota)^{\vartheta}} + \frac{J^{\vartheta}_{h^+}\zeta(\kappa)}{(\kappa-h)^{\vartheta}}\right] \\ = \frac{h-\iota}{2}\int_0^1 \varrho^{\vartheta}\zeta'(\varrho h + (1-\varrho)\iota)d\varrho + \frac{h-\kappa}{2}\int_0^1 \varrho^{\vartheta}\zeta'(\varrho h + (1-\varrho)\kappa)d\varrho,\end{aligned} \tag{7}$$

holds for $h \in (\iota, \kappa)$*.*

Lemma 2. *Let* $\zeta : I \subseteq \to \mathbb{R}$ *be twice differentiable function on* I^o*. Assume that* $\iota, \kappa \in I^o$ *, where* $\iota < \kappa$ *and* $\zeta'' \in L[\iota, \kappa]$*, and* $\vartheta > 0$*, then*

$$\begin{aligned}\frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{\Gamma(\vartheta+1)}{2(\kappa-\iota)^{\vartheta}}[J^{\vartheta}_{\iota^+}\zeta(\kappa) + J^{\vartheta}_{\kappa^-}\zeta(\iota)] \\ = \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\int_0^1 \varrho(1-\varrho^{\vartheta})[\zeta''(\varrho\iota+(1-\varrho)\kappa) + \zeta''((1-\varrho)\iota+\varrho\kappa)]d\varrho.\end{aligned} \tag{8}$$

Furthermore, to simplify expressions, the following notations are employed through out the next results:

$$\Re = \zeta(h) - \frac{\Gamma(\vartheta+1)}{2}\left[\frac{J^{\vartheta}_{h^-}\zeta(\iota)}{(h-\iota)^{\vartheta}} + \frac{J^{\vartheta}_{h^+}\zeta(\kappa)}{(\kappa-h)^{\vartheta}}\right],$$

and

$$\Im = \frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{\Gamma(\vartheta+1)}{2(\kappa-\iota)^{\vartheta}}[J^{\vartheta}_{\iota^+}\zeta(\kappa) + J^{\vartheta}_{\kappa^-}\zeta(\iota)].$$

Theorem 3. *Let* ζ *and* ξ *be real-valued functions on* $[\iota, \kappa]$*.*

1. Hölder Inequality for Integrals [24]: *If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, and* $|\zeta|^p, |\xi|^q$ *are integrable functions on* $[\iota, \kappa]$ *with* $q \geq 1$*, then*
$$\int_{\iota}^{\kappa}|\zeta(h)\xi(h)|\mathrm{d}h \leq \left(\int_{\iota}^{\kappa}|\zeta(h)|^p\mathrm{d}h\right)^{\frac{1}{p}}\left(\int_{\iota}^{\kappa}|\xi(h)|^q\mathrm{d}h\right)^{\frac{1}{q}}. \tag{9}$$
2. Power-mean Integral Inequality [24]: *If* $q \geq 1$ *and* $|\zeta|, |\zeta||\xi|^q$ *are integrable functions on* $[\iota, \kappa]$*, then*

$$\int_{\iota}^{\kappa}|\zeta(h)\xi(h)|\mathrm{d}h \leq \left(\int_{\iota}^{\kappa}|\zeta(h)|\mathrm{d}h\right)^{1-\frac{1}{q}}\left(\int_{\iota}^{\kappa}|\zeta(h)||\xi(h)|^q\mathrm{d}h\right)^{\frac{1}{q}}. \tag{10}$$

3. Integral midpoint type inequalities involving fractional integrals for differentiable functions

We need Lemma 1 to demonstrate the main results related to integral midpoint type inequalities involving fractional integrals.

Theorem 4. *Suppose that $\vartheta \geq 1$ and $\zeta : [\iota, \kappa] \to \mathbb{R}$ is a positive valued integrable function, where $0 \leq \iota < h < \kappa$. If ζ is a tgs-convex function on $[\iota, \kappa]$, then*

$$\Gamma(\vartheta + 1)\left[\frac{J^{\vartheta}_{h^-}\zeta(\iota)}{(h-\iota)^{\vartheta}} + \frac{J^{\vartheta}_{h^+}\zeta(\kappa)}{(\kappa-h)^{\vartheta}}\right] \leq \frac{\vartheta}{(\vartheta+1)(\vartheta+2)}[(\zeta(h)+\zeta(\iota)) + (\zeta(h)+\zeta(\kappa))], \tag{11}$$

holds.

Proof. Applying tgs-convexity of ζ

$$\zeta(\varrho h + (1-\varrho)\iota) + \zeta(\varrho h + (1-\varrho)\kappa) \leq \varrho(1-\varrho)(\zeta(h)+\zeta(\iota)) + \varrho(1-\varrho)(\zeta(h)+\zeta(\kappa)). \tag{12}$$

Multiply both sides of (12) by $\varrho^{\vartheta-1}$ and integrate w.r.t ϱ over $[0,1]$ to get

$$\begin{aligned}
&\int_0^1 \varrho^{\vartheta-1}\zeta(\varrho h + (1-\varrho)\iota)d\varrho + \int_0^1 \varrho^{\vartheta-1}\zeta(\varrho h + (1-\varrho)\kappa)d\varrho \\
&\leq (\zeta(h)+\zeta(\iota))\int_0^1 \varrho^{\vartheta}(1-\varrho)d\varrho + (\zeta(h)+\zeta(\kappa))\int_0^1 \varrho^{\vartheta}(1-\varrho)d\varrho \\
&= \frac{1}{(\vartheta+1)(\vartheta+2)}[(\zeta(h)+\zeta(\iota)) + (\zeta(h)+\zeta(\kappa))].
\end{aligned} \tag{13}$$

Now, substituting $\ell = \varrho h + (1-\varrho)\iota$ and $\jmath = \varrho h + (1-\varrho)\kappa$ in first and second terms on L.H.S of (13)

$$\int_{\iota}^{h}\left(\frac{\ell-\iota}{h-\iota}\right)^{\vartheta-1}\zeta(\ell)\frac{d\ell}{h-\iota} + \int_{\kappa}^{h}\left(\frac{\kappa-\jmath}{\kappa-h}\right)^{\vartheta-1}\zeta(\jmath)\frac{d\jmath}{h-\kappa} \leq \frac{1}{(\vartheta+1)(\vartheta+2)}[(\zeta(h)+\zeta(\iota)) + (\zeta(h)+\zeta(\kappa))].$$

Multiply both sides by ϑ

$$\frac{\vartheta}{(h-\iota)^{\vartheta}}\int_{\iota}^{h}(\ell-\iota)^{\vartheta-1}\zeta(\ell)d\ell + \frac{\vartheta}{(\kappa-h)^{\vartheta}}\int_{h}^{\kappa}(\kappa-\jmath)^{\vartheta-1}\zeta(\jmath)d\jmath \leq \frac{\vartheta}{(\vartheta+1)(\vartheta+2)}[(\zeta(h)+\zeta(\iota)) + (\zeta(h)+\zeta(\kappa))].$$

□

Theorem 5. *Consider $\zeta' \in L[\iota, \kappa]$ such that $|\zeta'|$ is tgs-convex on $[\iota, \kappa]$, then*

$$|\Re| \leq \beta(\vartheta+2,2)\left[\frac{h-\iota}{2}(|\zeta'(h)| + |\zeta'(\iota)|) + \frac{h-\kappa}{2}(|\zeta'(h)| + |\zeta'(\kappa)|)\right]. \tag{14}$$

Proof. By using the Lemma 1

$$|\Re| \leq \frac{h-\iota}{2}\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h + (1-\varrho)\iota)|d\varrho + \frac{h-\kappa}{2}\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h + (1-\varrho)\kappa)|d\varrho.$$

Since $|\zeta'|$ is tgs-convex

$$\begin{aligned}
|\Re| \leq& \frac{h-\iota}{2}\left[(|\zeta'(h)| + |\zeta'(\iota)|)\int_0^1 \varrho^{\vartheta+1}(1-\varrho)d\varrho\right] + \frac{h-\kappa}{2}\left[(|\zeta'(h)| + |\zeta'(\kappa)|)\int_0^1 \varrho^{\vartheta+1}(1-\varrho)d\varrho\right], \\
|\Re| \leq& \beta(\vartheta+2,2)\left[\frac{h-\iota}{2}(|\zeta'(h)| + |\zeta'(\iota)|) + \frac{h-\kappa}{2}(|\zeta'(h)| + |\zeta'(\kappa)|)\right].
\end{aligned}$$

□

Theorem 6. *Assume $\zeta' \in L[\iota,\kappa]$ such that $|\zeta'|^q$ $(q>1)$ is tgs-convex on $[\iota,\kappa]$, then*

$$|\Re| \leq \left(\frac{1}{(\vartheta+1)}\right)^{1-\frac{1}{q}} \left[\frac{h-\iota}{2}[|\zeta'(h)|^q\beta(\vartheta+2,2)+|\zeta'(\iota)|^q\beta(\vartheta+2,2)]^{\frac{1}{q}} + \frac{h-\kappa}{2}[|\zeta'(h)|^q\beta(\vartheta+2,2)+|\zeta'(\kappa)|^q\beta(\vartheta+2,2)]^{\frac{1}{q}}\right]. \tag{15}$$

Proof. By using Lemma 1

$$|\Re| \leq \frac{h-\iota}{2}\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h+(1-\varrho)\iota)|d\varrho + \frac{h-\kappa}{2}\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h+(1-\varrho)\kappa)|d\varrho.$$

By utilizing the power-mean integral inequality (10), we obtain

$$|\Re| \leq \frac{h-\iota}{2}\left(\int_0^1 \varrho^{\vartheta}d\varrho\right)^{1-\frac{1}{q}}\left[\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h+(1-\varrho)\iota)|^q d\varrho\right]^{\frac{1}{q}} + \frac{h-\kappa}{2}\left(\int_0^1 \varrho^{\vartheta}d\varrho\right)^{1-\frac{1}{q}}\left[\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h+(1-\varrho)\kappa)|^q d\varrho\right]^{\frac{1}{q}}.$$

Since $|\zeta'|^q$ is tgs-convex

$$|\Re| \leq \frac{h-\iota}{2}\left(\frac{1}{\vartheta+1}\right)^{1-\frac{1}{q}}\left[(|\zeta'(h)|^q+|\zeta'(\iota)|^q)\int_0^1 \varrho^{\vartheta+1}(1-\varrho)d\varrho\right]^{\frac{1}{q}} + \frac{h-\kappa}{2}\left(\frac{1}{\vartheta+1}\right)^{1-\frac{1}{q}}\left[(|\zeta'(h)|^q+|\zeta'(\kappa)|^q)\int_0^1 \varrho^{\vartheta+1}(1-\varrho)d\varrho\right]^{\frac{1}{q}}.$$

Therefore,

$$|\Re| \leq \left(\frac{1}{(\vartheta+1)}\right)^{1-\frac{1}{q}} \left[\frac{h-\iota}{2}[|\zeta'(h)|^q\beta(\vartheta+2,2)+|\zeta'(\iota)|^q\beta(\vartheta+2,2)]^{\frac{1}{q}} + \frac{h-\kappa}{2}[|\zeta'(h)|^q\beta(\vartheta+2,2)+|\zeta'(\kappa)|^q\beta(\vartheta+2,2)]^{\frac{1}{q}}\right].$$

□

Theorem 7. *Suppose $\zeta' \in L[\iota,\kappa]$ such that $|\zeta'|^q$ is tgs-convex on $[\iota,\kappa]$, then*

$$|\Re| \leq \left(\frac{1}{\vartheta p+1}\right)^{\frac{1}{p}}\left(\frac{1}{6}\right)\left[\frac{h-\iota}{2}[|\zeta'(h)|^q+|\zeta'(\iota)|^q]^{\frac{1}{q}} + \frac{h-\kappa}{2}[|\zeta'(h)|^q+|\zeta'(\kappa)|^q]^{\frac{1}{q}}\right]. \tag{16}$$

Proof. By using the Lemma 1

$$|\Re| \leq \frac{h-\iota}{2}\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h+(1-\varrho)\iota)|d\varrho + \frac{h-\kappa}{2}\int_0^1 \varrho^{\vartheta}|\zeta'(\varrho h+(1-\varrho)\kappa)|d\varrho.$$

According to Hölder inequality (9)

$$|\Re| \leq \frac{h-\iota}{2}\left(\int_0^1 \varrho^{\vartheta p}d\varrho\right)^{\frac{1}{p}}\left[\int_0^1 |\zeta'(\varrho h+(1-\varrho)\iota)|^q d\varrho\right]^{\frac{1}{q}} + \frac{h-\kappa}{2}\left(\int_0^1 \varrho^{\vartheta p}d\varrho\right)^{\frac{1}{p}}\left[\int_0^1 |\zeta'(\varrho h+(1-\varrho)\kappa)|^q d\varrho\right]^{\frac{1}{q}}.$$

Since $|\zeta'|^q$ is tgs-convex

$$|\Re| \leq \frac{h-\iota}{2}\left(\frac{1}{\vartheta p+1}\right)^{\frac{1}{p}}\left[(|\zeta'(h)|^q+|\zeta'(\iota)|^q)\int_0^1 \varrho(1-\varrho)d\varrho\right]^{\frac{1}{q}} + \frac{h-\kappa}{2}\left(\frac{1}{\vartheta p+1}\right)^{\frac{1}{p}}\left[(|\zeta'(h)|^q+|\zeta'(\kappa)|^q)\int_0^1 \varrho(1-\varrho)d\varrho\right]^{\frac{1}{q}}.$$

Therefore,

$$|\Re| \leq \left(\frac{1}{\vartheta p+1}\right)^{\frac{1}{p}}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[\frac{h-\iota}{2}[|\zeta'(h)|^q+|\zeta'(\iota)|^q]^{\frac{1}{q}}+\frac{h-\kappa}{2}[|\zeta'(h)|^q+|\zeta'(\kappa)|^q]^{\frac{1}{q}}\right].$$

□

4. Integral inequalities involving fractional integrals for twice differentiable functions

Lemma 2 is used in this section to prove the next main results.

Theorem 8. *Assume that $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ is twice differentiable function on I^o such that $|\zeta''|$ is tgs-convex function on I. Let $\iota, \kappa \in I^o$ for $\iota < \kappa$ and $|\zeta''| \in L[\iota,\kappa]$, then*

$$|\Im| \leq \frac{(\kappa-\iota)^2}{(\vartheta+1)}[\beta(3,\vartheta+2)(|\zeta''(\iota)|+|\zeta''(\kappa)|)]. \tag{17}$$

Proof. By using the Lemma 2

$$|\Im| = \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\int_0^1 \varrho(1-\varrho^\vartheta)[|\zeta''(\varrho\iota+(1-\varrho)\kappa)|+|\zeta''((1-\varrho)\iota+\varrho\kappa)|]d\varrho$$
$$\leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left[\int_0^1 \varrho(1-\varrho^\vartheta)[|\zeta''(\varrho\iota+(1-\varrho)\kappa)|]d\varrho+\int_0^1 \varrho(1-\varrho^\vartheta)[|\zeta''((1-\varrho)\iota+\varrho\kappa)|]d\varrho\right].$$

By applying the tgs-convexity of $|\zeta''|$

$$|\Im| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left[\int_0^1 \varrho(1-\varrho^\vartheta)\varrho(1-\varrho)(|\zeta''(\iota)|+|\zeta''(\kappa)|)d\varrho+\int_0^1 \varrho(1-\varrho^\vartheta)\varrho(1-\varrho)(|\zeta''(\iota)|+|\zeta''(\kappa)|)d\varrho\right]$$
$$= \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left[\int_0^1 \varrho^2(1-\varrho)(1-\varrho^\vartheta)(|\zeta''(\iota)|+|\zeta''(\kappa)|)d\varrho\int_0^1 \varrho^2(1-\varrho)(1-\varrho^\vartheta)(|\zeta''(\iota)|+|\zeta''(\kappa)|)d\varrho\right].$$

Since$\varrho^\vartheta \geq \varrho, \alpha \in (0,1]$ and $\varrho \in [0,1]$, we have $-\varrho^\vartheta \leq \varrho \Rightarrow 1-\varrho^\vartheta \leq 1-\varrho \leq (1-\varrho)^\vartheta$.

$$|\Im| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left[\int_0^1 \varrho^2(1-\varrho)^{\vartheta+1}(|\zeta''(\iota)|+|\zeta''(\kappa)|)d\varrho+\int_0^1 \varrho^2(1-\varrho)^{\vartheta+1}(|\zeta''(\iota)|+|\zeta''(\kappa)|)d\varrho\right].$$

Therefor,

$$|\Im| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}[\beta(3,\vartheta+2)(|\zeta''(\iota)|+|\zeta''(\kappa)|)+\beta(3,\vartheta+2)(|\zeta''(\iota)|+|\zeta''(\kappa)|)]$$
$$= \frac{(\kappa-\iota)^2}{(\vartheta+1)}[\beta(3,\vartheta+2)(|\zeta''(\iota)|+|\zeta''(\kappa)|)].$$

□

Theorem 9. *Assume that $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ is twice differentiable function on I^o. Let $q > 1$ such that $|\zeta''|^q$ is tgs-convex function on I. Let $\iota, \kappa \in I^o$, where $\iota < \kappa$ and $\zeta'' \in L[\iota, \kappa]$, then*

$$|\mathfrak{I}| \leq \frac{(\kappa-\iota)^2}{(\vartheta+1)}[\beta(p+1,\vartheta p+1)]^{\frac{1}{p}}\left[\frac{|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q}{6}\right]^{\frac{1}{q}}, \tag{18}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the Lemma 2

$$|\mathfrak{I}| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\int_0^1 \varrho(1-\varrho^\vartheta)[|\zeta''(\varrho\iota+(1-\varrho)\kappa)|+|\zeta''((1-\varrho)\iota+\varrho\kappa)|]d\varrho.$$

According to Hölder inequality (9)

$$\begin{aligned}|\mathfrak{I}| \leq &\frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left[\left(\int_0^1 \varrho^p(1-\varrho^\vartheta)^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1 |\zeta''(\varrho\iota+(1-\varrho)\kappa)|^q d\varrho\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1 \varrho^p(1-\varrho^\vartheta)^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1 |\zeta''((1-\varrho)\iota+\varrho\kappa)|^q d\varrho\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since $\varrho^\vartheta \geq \varrho, \vartheta \in (0,1]$ and $\varrho \in [0,1]$, we have $-\varrho^\vartheta \leq \varrho \Rightarrow 1-\varrho^\vartheta \leq 1-\varrho \leq (1-\varrho)^\vartheta$.

$$|\mathfrak{I}| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left(\int_0^1 \varrho^p(1-\varrho)^{\vartheta p} d\varrho\right)^{\frac{1}{p}}\left[\left(\int_0^1 |\zeta''(\varrho\iota+(1-\varrho)\kappa)|^q d\varrho\right)^{\frac{1}{q}}+\left(\int_0^1 |\zeta''((1-\varrho)\iota+\varrho\kappa)|^q d\varrho\right)^{\frac{1}{q}}\right].$$

Since $|\zeta''|^q$ is tgs-convex

$$\begin{aligned}|\mathfrak{I}| \leq &\frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left(\int_0^1 \varrho^p(1-\varrho)^{\vartheta p} d\varrho\right)^{\frac{1}{p}}\left[\left(\int_0^1 (|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\varrho(1-\varrho)d\varrho\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1 (|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\varrho(1-\varrho)d\varrho\right)^{\frac{1}{q}}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}|\mathfrak{I}| &\leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}[\beta(p+1,\vartheta p+1)]^{\frac{1}{p}}\left[\left(\frac{|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q}{6}\right)^{\frac{1}{q}}\right]\\ &= \frac{(\kappa-\iota)^2}{(\vartheta+1)}[\beta(p+1,\vartheta p+1)]^{\frac{1}{p}}\left[\frac{|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q}{6}\right]^{\frac{1}{q}}.\end{aligned}$$

□

Theorem 10. *Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I^o. Suppose that $q \geq 1$ such that $|\zeta''|^q$ is tgs-convex function on I. Assume that $\iota, \kappa \in I^o$, where $\iota < \kappa$ and $\zeta'' \in L[\iota, \kappa]$, then*

$$|\mathfrak{I}| \leq \frac{\vartheta(\kappa-\iota)^2}{2(\vartheta+1)(\vartheta+2)}\left(\frac{2(\vartheta+2)}{\vartheta}\right)^{\frac{1}{q}}\left(\beta(3,\vartheta+2)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\right)^{\frac{1}{q}}. \tag{19}$$

Proof. By applying Lemma 2

$$|\mathfrak{I}| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\int_0^1 \varrho(1-\varrho^\vartheta)[|\zeta''(\varrho\iota+(1-\varrho)\kappa)|+|\zeta''((1-\varrho)\iota+\varrho\kappa)|]d\varrho.$$

Applying the power-mean integral inequality (10)

$$
\begin{aligned}
|\Im| \leq & \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\Bigg[\left(\int_0^1 \varrho(1-\varrho^{\vartheta})d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho(1-\varrho^{\vartheta})|\zeta''(\varrho\iota+(1-\varrho)\kappa)|^q d\varrho\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 \varrho(1-\varrho^{\vartheta})d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho(1-\varrho^{\vartheta})|\zeta''((1-\varrho\iota)+\varrho\kappa))|^q d\varrho\right)^{\frac{1}{q}}\Bigg] \\
= & \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left(\int_0^1 \varrho(1-\varrho^{\vartheta})d\varrho\right)^{1-\frac{1}{q}}\Bigg[\left(\int_0^1 \varrho(1-\varrho^{\vartheta})|\zeta''(\varrho\iota+(1-\varrho)\kappa)|^q d\varrho\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 \varrho(1-\varrho^{\vartheta})|\zeta''((1-\varrho)\iota+\varrho\kappa)|^q d\varrho\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Simplifying

$$
\int_0^1 \varrho(1-\varrho^{\vartheta})d\varrho = \int_0^1 (\varrho-\varrho^{\vartheta+1})d\varrho = \frac{\vartheta}{2(\vartheta+2)},
$$

Since $|\zeta''|^q$ is tgs-convex

$$
\begin{aligned}
|\Im| \leq & \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left(\frac{\vartheta}{2(\vartheta+2)}\right)^{1-\frac{1}{q}}\Bigg[\left(\int_0^1 \varrho(1-\varrho^{\vartheta})((|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\varrho(1-\varrho))\right)^{\frac{1}{q}} d\varrho \\
&+\left(\int_0^1 \varrho(1-\varrho^{\vartheta})((|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\varrho(1-\varrho))\right)^{\frac{1}{q}} d\varrho\Bigg)\Bigg].
\end{aligned}
$$

Since $\varrho^{\vartheta} \geq \varrho, \vartheta \in (0,1]$ and $\varrho \in [0,1]$, we have $-\varrho^{\vartheta} \leq \varrho \Rightarrow 1-\varrho^{\vartheta} \leq 1-\varrho \leq (1-\varrho)^{\vartheta}$.

$$
\begin{aligned}
|\Im| \leq & \frac{\vartheta(\kappa-\iota)^2}{4(\vartheta+1)(\vartheta+2)}\left(\frac{2(\vartheta+2)}{\vartheta}\right)^{\frac{1}{q}}\Bigg[\left(\int_0^1 \varrho^2(1-\varrho)^{\vartheta+1}(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)d\varrho\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 \varrho^2(1-\varrho)^{\vartheta+1}(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)d\varrho\right)^{\frac{1}{q}}\Bigg] \\
= & \frac{\vartheta(\kappa-\iota)^2}{2(\vartheta+1)(\vartheta+2)}\left(\frac{2(\vartheta+2)}{\vartheta}\right)^{\frac{1}{q}}\left(\beta(3,\vartheta+2)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\right)^{\frac{1}{q}}.
\end{aligned}
$$

□

Theorem 11. *Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I^o such that $\zeta'' \in L[\iota,\kappa]$ for $\iota,\kappa \in I^o$ and $\iota < \kappa$. Let $|\zeta''|^q$ is tgs-convex on $[\iota,\kappa]$ and $q > 1$, then*

$$
|\Im| \leq \frac{(\kappa-\iota)^2}{(\vartheta+1)}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left[\left(\beta(2,\vartheta q+2)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\right)^{\frac{1}{q}}\right], \tag{20}
$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the Lemma 2

$$
|\Im| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left[\int_0^1 \varrho(1-\varrho^{\vartheta})|\zeta''(\varrho\iota+(1-\varrho)\kappa)|d\varrho+\int_0^1 \varrho(1-\varrho^{\vartheta})|\zeta''((1-\varrho)\iota+\varrho\kappa)|d\varrho\right].
$$

Applying the Hölder inequality (9)

$$
\begin{aligned}
|\Im| \leq & \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\Bigg[\left(\int_0^1 \varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1 (1-\varrho^{\vartheta})^q|\zeta''(\varrho\iota+(1-\varrho)\kappa)|^q d\varrho\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 \varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1 (1-\varrho^{\vartheta})^q|\zeta''((1-\varrho)\iota+\varrho\kappa)|^q d\varrho\right)^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

Since $|\zeta''|^q$ is tgs-convex

$$|\Im| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\Bigg[\bigg(\int_0^1 \varrho^p d\varrho\bigg)^{\frac{1}{p}}\bigg(\int_0^1 (1-\varrho^\vartheta)^q\varrho(1-\varrho)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\bigg)^{\frac{1}{q}} + \bigg(\int_0^1 \varrho^p d\varrho\bigg)^{\frac{1}{p}}\bigg(\int_0^1 (1-\varrho^\vartheta)^q\varrho(1-\varrho)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)d\varrho\bigg)^{\frac{1}{q}}\Bigg].$$

Since $\varrho^\vartheta \geq \varrho, \vartheta \in (0,1]$ and $\varrho \in [0,1]$, we have $-\varrho^\vartheta \leq \varrho \Rightarrow 1-\varrho^\vartheta \leq 1-\varrho \leq (1-\varrho)^\vartheta$.

$$\begin{aligned}|\Im| \leq& \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\bigg(\frac{1}{1+p}\bigg)^{\frac{1}{p}}\Bigg[\bigg(\int_0^1 \varrho(1-\varrho)^{\vartheta q+1}(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)d\varrho\bigg)^{\frac{1}{q}} + \bigg(\int_0^1 \varrho(1-\varrho)^{\vartheta q+1}(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)d\varrho\bigg)^{\frac{1}{q}}\Bigg] \\ =& \frac{(\kappa-\iota)^2}{(\vartheta+1)}\bigg(\frac{1}{1+p}\bigg)^{\frac{1}{p}}\Bigg[\bigg(\beta(2,\vartheta q+2)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\bigg)^{\frac{1}{q}}\Bigg].\end{aligned}$$

□

Theorem 12. *Suppose that $\zeta : I \subseteq [0,\infty) \to \mathbb{R}$ is differentiable on I^o such that $\zeta'' \in L[\iota,\kappa]$ for $\iota,\kappa \in I^o$ with $\iota < \kappa$. Let $|\zeta''|^q$ is tgs-convex on $[\iota,\kappa]$ also $q \geq 1$, then*

$$|\Im| \leq \frac{(\kappa-\iota)^2}{(\vartheta+1)}\bigg(\frac{1}{2}\bigg)^{1-\frac{1}{q}}\Bigg[\bigg(\beta(3,\vartheta q+2)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\bigg)^{\frac{1}{q}}\Bigg], \tag{21}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the Lemma 2

$$\begin{aligned}|\Im| \leq& \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\bigg[\int_0^1 \varrho(1-\varrho^\vartheta)|\zeta''(\varrho\iota+(1-\varrho)\kappa)|d\varrho + \int_0^1 \varrho(1-\varrho^\vartheta)|\zeta''((1-\varrho)\iota+\varrho\kappa)|d\varrho\bigg] \\ =& \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\bigg[\int_0^1 \varrho^{1-\frac{1}{q}}\varrho^{\frac{1}{q}}(1-\varrho^\vartheta)[|\zeta''(\varrho\iota+(1-\varrho)\kappa)|]d\varrho + \int_0^1 \varrho^{1-\frac{1}{q}}\varrho^{\frac{1}{q}}(1-\varrho^\vartheta)[|\zeta''((1-\varrho)\iota+\varrho\kappa)|]d\varrho\bigg].\end{aligned}$$

Applying the Hölder inequality (9)

$$|\Im| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\Bigg[\bigg(\int_0^1 \varrho d\varrho\bigg)^{1-\frac{1}{q}}\bigg(\int_0^1 \varrho(1-\varrho^\vartheta)^q|\zeta''(\varrho\iota+(1-\varrho)\kappa)|^q d\varrho\bigg)^{\frac{1}{q}} + \bigg(\int_0^1 \varrho d\varrho\bigg)^{1-\frac{1}{q}}\bigg(\int_0^1 \varrho(1-\varrho^\vartheta)^q|\zeta''((1-\varrho)\iota+\varrho\kappa)|^q d\varrho\bigg)^{\frac{1}{q}}\Bigg].$$

Since $|\zeta''|^q$ is the tgs-convex

$$|\Im| \leq \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\bigg(\int_0^1 \varrho d\varrho\bigg)^{1-\frac{1}{q}}\Bigg[\bigg(\int_0^1 \varrho(1-\varrho^\vartheta)^q(\varrho(1-\varrho)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q))d\varrho\bigg)^{\frac{1}{q}} + \bigg(\int_0^1 \varrho(1-\varrho^\vartheta)^q(\varrho(1-\varrho)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q))d\varrho\bigg)^{\frac{1}{q}}\Bigg].$$

Since $\varrho^{\vartheta} \geq \varrho, \vartheta \in (0,1]$ and $\varrho \in [0,1]$, we have $-\varrho^{\vartheta} \leq \varrho \Rightarrow 1-\varrho^{\vartheta} \leq 1-\varrho \leq (1-\varrho)^{\vartheta}$.

$$\begin{aligned}|\mathfrak{I}| \leq & \frac{(\kappa-\iota)^2}{2(\vartheta+1)}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \varrho^2(1-\varrho)^{\vartheta q+1}(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)d\varrho\right)^{\frac{1}{q}}\right.\\ & \left.+\left(\int_0^1 \varrho^2(1-\varrho)^{\vartheta q+1}(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)d\varrho\right)^{\frac{1}{q}}\right]\\ = & \frac{(\kappa-\iota)^2}{(\vartheta+1)}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\beta(3,\vartheta q+2)(|\zeta''(\iota)|^q+|\zeta''(\kappa)|^q)\right)^{\frac{1}{q}}\right].\end{aligned}$$

□

5. Numerical and graphical representation of inequalities

Some examples are constructed in this section for inequalities appeared in §4.

Example 1. Let $\vartheta = 1$, in (17) of Theorem 8, we get the following relation:

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2}-\frac{1}{(\kappa-\iota)}\int_\iota^\kappa \zeta(\varrho)d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left[\frac{1}{30}(|\zeta''(\iota)|+|\zeta''(\kappa)|)\right]. \tag{22}$$

Substitute $\zeta(\varrho) = \frac{1}{\varrho}$, in (22) to get

$$\left|\frac{\frac{1}{\iota}+\frac{1}{\kappa}}{2}-\frac{1}{\kappa-\iota}\int_\iota^\kappa \frac{1}{\varrho}d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left(\frac{|\frac{2}{\iota^3}|^2+|\frac{2}{\kappa^3}|^2}{30}\right).$$

Particularly, for $\iota = 1$ and $\kappa = 3$ in (22) we get

$$\left|\frac{2}{3}-\frac{1.098612}{2}\right| \leq 2\left(\frac{4.005487}{30}\right),$$
$$0.117364 \leq 0.267032.$$

Similarly, let $\vartheta = 1$ in [23, Theorem 1] then, we obtain:

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2}-\frac{1}{\kappa-\iota}\int_\iota^\kappa \zeta(\varrho)d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left[\frac{1}{24}(|\zeta''(\iota)|+|\zeta''(\kappa)|)\right]. \tag{23}$$

Substitute $\zeta(\varrho) = \frac{1}{\varrho}$, in (23) to get

$$\left|\frac{\frac{1}{\iota}+\frac{1}{\kappa}}{2}-\frac{1}{\kappa-\iota}\int_\iota^\kappa \frac{1}{\varrho}d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left(\frac{|\frac{2}{\iota^3}|^2+|\frac{2}{\kappa^3}|^2}{24}\right).$$

Particularly, for $\iota = 1$ and $\kappa = 3$ in (23) we get

$$\left|\frac{2}{3}-\frac{1.098612}{2}\right| \leq 2\left(\frac{4.005487}{24}\right),$$

$$0.117364 \leq 0.333790.$$

Remark 1. The difference of bounds, appear in Theorem 8, is 0.267032 and difference of bounds, appeared in [23, Theorem 1], is 0.333790. Hence, our result is more efficient.

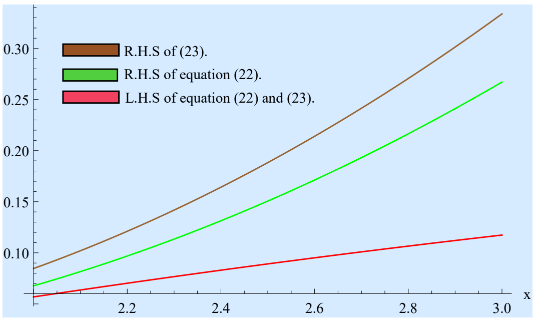

Figure 1. Graphical comparison of inequalities (22) and (23) as a function of κ where $\iota = 1$, $\vartheta = 1$, and $\zeta(\varrho) = \frac{1}{\varrho}$.

Example 2. Let $\vartheta = 1$, in (18) of Theorem 9, then we get the following relation:

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\zeta(\varrho)d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left[\left(\frac{|\zeta''(\iota)|^q + |\zeta''(\kappa)|^q}{6}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1)\right]. \tag{24}$$

Substitute $\zeta(\varrho) = \frac{1}{\varrho}$, in (24) to get

$$\left|\frac{\frac{1}{\iota}+\frac{1}{\kappa}}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\frac{1}{\varrho}d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left[\left(\frac{\left|\frac{2}{\iota^3}\right|^2 + \left|\frac{2}{\kappa^3}\right|^2}{6}\right)^{\frac{1}{2}}\beta^{\frac{1}{2}}(3,3)\right].$$

Particularly, for $\iota = 1$ and $\kappa = 3$ in (24) we get

$$\left|\frac{2}{3} - \frac{1.098612}{2}\right| \leq 2\left(\frac{4.005486}{6}\right)^{\frac{1}{2}}\left(\frac{1}{30}\right)^{\frac{1}{2}},$$

$$0.117364 \leq 0.298346.$$

Similarly, if we choose $\vartheta = 1$ in Corollary 3.6 of [14] then, we get:

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\zeta(\varrho)d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left[\left(\frac{|\zeta''(\iota)|^q + |\zeta''(\kappa)|^q}{s+1}\right)^{\frac{1}{q}}\beta^{\frac{1}{p}}(p+1,p+1)\right]. \tag{25}$$

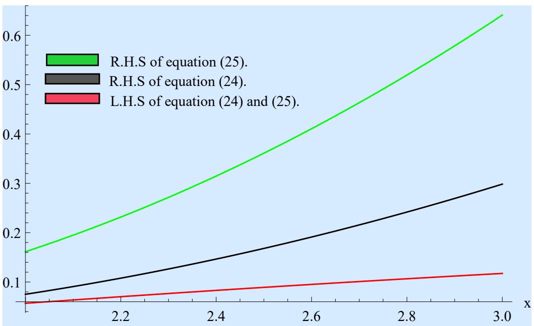

Figure 2. Graphical comparison of inequalities (24) and (25) as a function of κ where $\iota = 1$, $\vartheta = 1$, and $\zeta(\varrho) = \frac{1}{\varrho}$

Substitute $\zeta(\varrho) = \frac{1}{\varrho}$, $p = q = 2$ in (25) to get

$$\left|\frac{\frac{1}{\iota}+\frac{1}{\kappa}}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\frac{1}{\varrho}d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left[\left(\frac{\left|\frac{2}{\iota^3}\right|^2 + \left|\frac{2}{\kappa^3}\right|^2}{s+1}\right)^{\frac{1}{2}}\beta^{\frac{1}{2}}(3,3)\right].$$

Particularly, for $\iota = 1$, $\kappa = 3$ and s=0.3 in (25) we get

$$\left|\frac{2}{3} - \frac{1.098612}{2}\right| \leq 2\left(\frac{4.005486}{1.3}\right)^{\frac{1}{2}}\left(\frac{1}{30}\right)^{\frac{1}{2}},$$
$$0.117364 \leq 0.640950.$$

Remark 2. The difference of bounds, appear in Theorem 9, is 0.298346 and difference of bounds, appeared in Corollary 3.6 in [14], is 0.640950. Hence, present result is more efficient.

Example 3. Let $\vartheta = 1$, in (19) of Theorem 10, then we get the following relation:

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\zeta(\varrho)d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\frac{1}{30}(|\zeta''(\iota)|^q + |\zeta''(\kappa)|^q)\right]^{\frac{1}{q}}. \tag{26}$$

Substitute $\zeta(\varrho) = \frac{1}{\varrho}$, and $q = 2$ in (26) to get

$$\left|\frac{\frac{1}{\iota}+\frac{1}{\kappa}}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\frac{1}{\varrho}d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{2}}\left(\frac{\left|\frac{2}{\iota^3}\right|^2+\left|\frac{2}{\kappa^3}\right|^2}{30}\right)^{\frac{1}{2}}.$$

Particularly, for $\iota = 1$ and $\kappa = 3$ in (26) we get

$$\left|\frac{2}{3} - \frac{1.098612}{2}\right| \leq 2\left(\frac{1}{6}\right)^{\frac{1}{2}}\left(\frac{4.005486}{30}\right)^{\frac{1}{2}},$$
$$0.117364 \leq 0.298346.$$

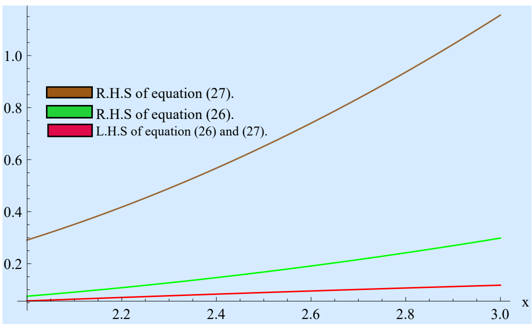

Figure 3. Graphical comparison of inequalities (26) and (27) as a function of κ where $\iota = 1$, $\vartheta = 1$, and $\zeta(\varrho) = \frac{1}{\varrho}$.

Similarly, let $\alpha = m = 1$ in [14, Theorem 3.4] then, we get:

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\zeta(\varrho)d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\frac{1}{2}(|\zeta''(\iota)|^q + |\zeta''(\kappa)|^q)\right]^{\frac{1}{q}}. \tag{27}$$

Substitute $\zeta(\varrho) = \frac{1}{\varrho}$ and $q = 2$ in (27) to get

$$\left|\frac{\frac{1}{\iota}+\frac{1}{\kappa}}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\frac{1}{\varrho}d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{2}}\left(\frac{\left|\frac{2}{\iota^3}\right|^2+\left|\frac{2}{\kappa^3}\right|^2}{2}\right)^{\frac{1}{2}}.$$

Particularly, for $\iota = 1$ and $\kappa = 3$ in (27) we get

$$\left|\frac{2}{3} - \frac{1.098612}{2}\right| \leq 2\left(\frac{1}{6}\right)^{\frac{1}{2}}\left(\frac{4.005486}{2}\right)^{\frac{1}{2}},$$

$$0.117364 \leq 1.155491.$$

Remark 3. The difference of bounds, appear in Theorem 10, is 0.298346 and difference of bounds, appeared in Theorem 3.4 in [14], is 1.155491. Hence, present result is more efficient.

Corollary 1. *For $\vartheta = 1$, the inequality (20) becomes:*

$$\left|\frac{\zeta(\iota)+\zeta(\kappa)}{2} - \frac{1}{\kappa-\iota}\int_{\iota}^{\kappa}\zeta(\varrho)d\varrho\right| \leq \frac{(\kappa-\iota)^2}{2}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left[\left(\beta(2,q+2)(|\zeta''(\iota)|^q + |\zeta''(\kappa)|^q)\right)^{\frac{1}{q}}\right]. \tag{28}$$

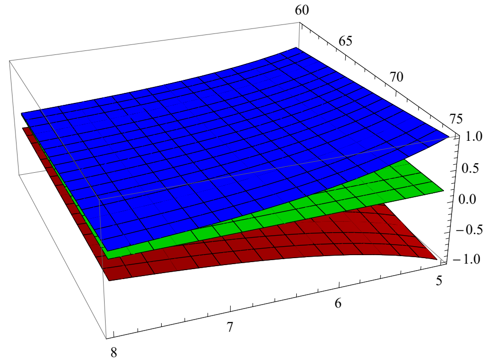

Figure 4. Graphical representation of inequality (28). The inequality's left side is shown in red colour, while its right side is shown in green colour. The right most part of inequality when we remove the absolute value from left hand side is shown in blue colour.

6. Conclusion

In this paper, the authors have generated integral midpoint type inequalities involving tgs-convex functions and Riemann-Liouville fractional integrals. The numerical and graphical representation of inequalities in §5, show that present results are more efficient as compared with the ones that already exist in [14,23]. In similar fashion, present inequalities can be discussed with different kinds of integral operators, including quantum integral operators, generalized fractional integral operators, delta or nabla integral operators in combination with tgs convexity. Moreover, some other companion inequalities including Jensen, Hardy and Fejér type can also be investigated by using the same tools, which are utilized in present work.

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