The Pečarić journal of
Mathematical Inequalities

PSR Press

Article

New insights into Hermite-Hadamard type inequalities via generalized Riemann-Liouville fractional integrals

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Received: 22 November 2025; Accepted: 19 February 2026; Published: 17 June 2026

Abstract: This study aims to extend the classical Hermite-Hadamard-type inequalities by employing recently introduced (k,l)-type fractional integrals, which are formulated within the framework of the Riemann-Liouville approach. These integrals are characterized by two exponential parameters, k and l, defined via the (k,l)-gamma function. In particular, we established new inequalities involving the arithmetic, geometric, and harmonic (k,l)-Riemann-Liouville fractional integrals. Notably, when $k=l$, these integrals reduce to k-Riemann-Liouville fractional integrals. Additionally, several foundational identities related to the general (k,l)-Riemann-Liouville fractional integrals are presented. Subsequently, various related inequalities are established using the convexity properties of differentiable functions. These results contribute to the field of fractional calculus and its role in mathematical analysis.

Keywords: Convex functions, Hermite-Hadamard inequalities, (k,l)-gamma function, (k,l)-Riemann-Liouville integrals

MSC: 26D15, 26D10, 26A33, 52A41.

1. Introduction and Preliminaries

Convex functions represent a foundational class in mathematical analysis and have been widely studied since the classical work of Hardy, Littlewood, and Pólya [1]. One of the most famous results involving convex functions is the Hermite-Hadamard inequality, which provides bounds of the mean value of a convex function over a closed interval (see [2, p. 137] and [3]). Formally, if $\mathcal{U}$ is convex function defined on the interval of real numbers $[u_1, v_1]$ with $u_1, v_1 \in \mathbb{R}$ and $u_1 < v_1$, then

$$\mathcal{U}\left(\frac{u_1+v_1}{2}\right) \leq \frac{1}{v_1-u_1}\int_{u_1}^{v_1}\mathcal{U}(x)\,dx \leq \frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}. \tag{1}$$

Hermite-Hadamard inequalities are often used to characterize and explore the fundamental properties of convex function. As a result, Hermite-Hadamard inequalities, which are widely used to derive various results and are fundamental in the theory of convex functions. Various forms of convexities and their generalizations have been explored in multiple scientific disciplines [4,5]. These generalizations have enriched the study of convexity and its applications in inequality theory.

In parallel, fractional calculus, the study of integrals and derivatives of arbitrary order has gained substantial interest. Fractional operators are inherently nonlocal and memory-preserving, making them powerful tools for modeling real-world phenomena. Their use in inequality theory has also become prominent, particularly in the development of fractional versions of the Hermite-Hadamard inequality and related results [6–8].

To further generalize these results, fractional integrals involving two exponential parameters, denoted as (k,l)-Riemann-Liouville fractional integrals have been recently introduced [9]. This paper focuses on establishing Hermite-Hadamard inequalities via the (k,l)-Riemann-Liouville fractional integrals. We present several novel results involving the arithmetic, geometric, and harmonic versions of these integrals. Also for $k = l$, these operators reduce the k-Riemann-Liouville fractional integrals.

We also include foundational properties and identities for the (k,l)-integrals and derive various inequalities under convexity assumptions. These contributions not only extend classical results, but also offer new insights into the intersection of convex analysis and fractional calculus.

Fractional calculus has emerged as a powerful mathematical framework, offering diverse applications across various theoretical and practical fields. It has found applications in various scientific domains. In applied mathematics, fractional operators have played an essential role in enhancing and broadening the scope of integral inequalities. Notably, the k-Riemann-Liouville fractional integral has emerged as a core instrument and a deriving force for investigation in various areas such as inequalities, differential equations, and related integral problems; see [10–13].

Definition 1. Let $\mathcal{U} : [u_1, v_1] \to \mathbb{R}$ and $\Upsilon > 0$. The k-Riemann-Liouville fractional integrals of order Υ are defined as follows [14],

$$I^{\Upsilon}_{u_1^+,k}\mathcal{U}(x) = \frac{1}{k\Gamma_k(\Upsilon)} \int_{u_1}^{x} (x-t)^{\frac{\Upsilon}{k}-1} \mathcal{U}(t)\, dt, \qquad x > u_1, \tag{2}$$

$$I^{\Upsilon}_{v_1^-,k}\mathcal{U}(x) = \frac{1}{k\Gamma_k(\Upsilon)} \int_{x}^{v_1} (t-x)^{\frac{\Upsilon}{k}-1} \mathcal{U}(t)\, dt, \qquad x < v_1. \tag{3}$$

Kuldeep defined the (k,l)-gamma function, which generalizes the k-gamma function [15].

Definition 2. [16] If $\mathrm{Re}(\Upsilon) > 0$ and let Υ be any non-positive integer complex number then (k,l)-gamma function defined as:

$$\Gamma_{(k,l)}(\Upsilon) = \int_0^{\infty} t^{\Upsilon-1} e^{-\frac{t^k}{l}}\, dt, \tag{4}$$

for all $k, l, \Upsilon > 0$ and $q \in \mathbb{N}$, the (k,l)-gamma function possesses the following properties [15] and [17] hold:

$$\Gamma_{(k,l)}(q+\Upsilon) = \frac{l\Upsilon}{q}\Gamma_{(k,l)}(\Upsilon),$$

$$\Gamma_{(k,l)}(nq+\Upsilon) = l^n \left(\frac{\Upsilon}{q}\right)\left(\frac{\Upsilon}{q}+1\right)\cdots\left(\frac{\Upsilon}{q}+(n-1)\right)\Gamma_{(k,l)}(\Upsilon),$$

$$\Gamma_{(k,l)}(\Upsilon) = \left(\frac{l}{q}\right)^{\frac{\Upsilon}{k}} \Gamma_k(\Upsilon) = \left(\frac{l^{\frac{\Upsilon}{q}}}{q}\right)\Gamma\left(\frac{\Upsilon}{q}\right).$$

Remark 1. If $k = l$, this yields the k-gamma function Υ_k, while taking $k = l = 1$ leads to the standard gamma function Γ.

An advanced form of Riemann-Liouville fractional integrals, known as the general (k,l)-Riemann-Liouville fractional integral, serves as an extension of the classical k-fractional integrals and is defined as follows.

Definition 3. Let $[u_1, v_1] \subseteq [0, +\infty]$, and let $\mathcal{U} \in L_1[u_1, v_1]$. Then general (k,l)-Riemann-Liouville fractional integrals with order $\Upsilon > 0$ are defined as

$$I^{\Upsilon}_{u_1^+,\Psi_0(k,l)}\mathcal{U}(x) = \frac{1}{k\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)} \int_{u_1}^{x} (x-t)^{\frac{\Upsilon}{\Psi_0(k,l)}-1} \mathcal{U}(t)\, dt, \qquad x > u_1, \tag{5}$$

$$I^{\Upsilon}_{v_1^-,\Psi_0(k,l)}\mathcal{U}(x) = \frac{1}{k\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)} \int_{x}^{v_1} (t-x)^{\frac{\Upsilon}{\Psi_0(k,l)}-1} \mathcal{U}(t)\, dt, \qquad x < v_1, \tag{6}$$

where $\Gamma_{(k,l)}$ is the (k,l)-gamma function and $\Psi_0 :]0,+\infty[\times]0,+\infty[\to]0,+\infty[$ is a function satisfying $\Psi_0(k,k) = k$.

By choosing different form of the function $\Psi_0(k,l)$, various special cases of generalized (k,l)-Riemann-Liouville fractional integrals can be derived.

(1) By choosing $\Psi_0(k,l) = \sqrt{kl}$, we derive the geometric form of the (k,l)-Riemann-Liouville fractional integrals $G^{\Upsilon}_{u_1^+,\Psi_0(k,l)}\mathcal{U}(x)$ and $G^{\Upsilon}_{v_1^-,\Psi_0(k,l)}\mathcal{U}(x)$.
(2) By choosing $\Psi_0(k,l) = \frac{k+l}{2}$, we derive the arithmetic form of (k,l)-Riemann-Liouville fractional integrals $A^{\Upsilon}_{u_1^+,\Psi_0(k,l)}\mathcal{U}(x)$ and $A^{\Upsilon}_{v_1^-,\Psi_0(k,l)}\mathcal{U}(x)$.
(3) By choosing $\Psi_0(k,l) = \frac{k^2}{l}$, we derive the harmonic form of (k,l)-Riemann-Liouville fractional integrals $H^{\Upsilon}_{u_1^+,\Psi_0(k,l)}\mathcal{U}(x)$ and $H^{\Upsilon}_{v_1^-,\Psi_0(k,l)}\mathcal{U}(x)$.

Remark 2. By setting $l = k$, the fractional integral (k,l)-Riemann-Liouville defined in Eqs. (5) and (6) reduced to the standard fractional integrals k-Riemann-Liouville presented in Eqs. (2) and (3) (see [18]). Moreover, if $l = k = 1$, this yields the classical Riemann-Liouville fractional integrals outlined in (see [19]).

This manuscript is outlined such as: in §2, we derive the famous Hermite-Hadamard inequality for the newly defined integrals. In §3, we present some relevant midpoint-type inequalities, and in §4, we derive some trapezoidal-type inequalities. Furthermore, some novel inequalities are obtained by making different choices of $\Psi_0(k,l)$ and is concluded with conclusion describing the future research directions as well.

2. Hermite-Hadamard inequality

The classical Hermite-Hadamard inequality provides an important estimate for the integral average of a convex function over an interval. This inequality can be generalized in the context of fractional calculus. In particular, by employing the (k,l)-Riemann-Liouville fractional integrals, we obtain new versions of Hermite-Hadamard inequalities.

Theorem 1. *Let $\mathcal{U} : [u_1, v_1] \to \mathbb{R}$ be a convex function defined in the interval $[u_1, v_1]$, and suppose that $\mathcal{U} \in L_1[u_1, v_1]$, then*

$$\mathcal{U}\left(\frac{u_1+v_1}{2}\right) \leq \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)\right]$$
$$\leq \frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}. \tag{7}$$

Proof. Assuming $\mathcal{U}$ is convex, and $\xi, \omega \in [u_1, v_1]$ with $\zeta = \frac{1}{2}$,

$$\mathcal{U}\left(\frac{\xi+\omega}{2}\right) \leq \frac{\mathcal{U}(\xi)+\mathcal{U}(\omega)}{2}. \tag{8}$$

Putting $\omega = \frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}$ and $\xi = \frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}$, then Eq. (8) become

$$2\mathcal{U}\left(\frac{u_1+v_1}{2}\right) \leq \mathcal{U}\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right) + \mathcal{U}\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right). \tag{9}$$

Multiplying Eq. (9) by $\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}$, and by performing integrating with respect to ζ over the interval $[0,1]$, we obtain

$$2\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\int_0^1 \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}d\zeta \leq \int_0^1 \mathcal{U}\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}d\zeta + \int_0^1 \mathcal{U}\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}d\zeta,$$

or

$$\frac{2\Psi_0(k,l)}{\Upsilon}\mathcal{U}\left(\frac{u_1+v_1}{2}\right) \leq M_1 + M_2, \tag{10}$$

where

$$M_1 = \int_0^1 \mathcal{U}\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}d\zeta, \tag{11}$$

and

$$M_2 = \int_0^1 \mathcal{U}\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right) \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1} \, d\zeta. \tag{12}$$

After putting $\xi = \frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}$ in (11), and $\omega = \frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}$ in (12), we get

$$M_1 = \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}}}{(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \int_{\frac{u_1+v_1}{2}}^{v_1} (v_1-\xi)^{\frac{\Upsilon}{\Psi_0(k,l)}-1} \mathcal{U}(\xi)\, d\xi,$$

and

$$M_2 = \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}}}{(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \int_{u_1}^{\frac{u_1+v_1}{2}} (\omega-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}-1} \mathcal{U}(\omega)\, d\omega.$$

So, we get

$$2\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\frac{\Psi_0(k,l)}{\Upsilon} \le \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}}}{(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[\int_{\frac{u_1+v_1}{2}}^{v_1} (v_1-\xi)^{\frac{\Upsilon}{\Psi_0(k,l)}-1} \mathcal{U}(\xi)\, d\xi \right.$$
$$\left. + \int_{u_1}^{\frac{u_1+v_1}{2}} (\omega-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}-1} \mathcal{U}(\omega)\, d\omega\right].$$

After simplifications, we get

$$\mathcal{U}\left(\frac{u_1+v_1}{2}\right) \le \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)} \mathcal{U}(u_1)\right]. \tag{13}$$

Having established one part of the theorem. We now move on to establish the second part, noting that $\mathcal{U}$ is convex function, then, for $\zeta \in [0,1]$, we have

$$\mathcal{U}\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right) \le \frac{\zeta}{2}\mathcal{U}(u_1) + \frac{2-\zeta}{2}\mathcal{U}(v_1), \tag{14}$$

and

$$\mathcal{U}\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right) \le \frac{\zeta}{2}\mathcal{U}(v_1) + \frac{2-\zeta}{2}\mathcal{U}(u_1). \tag{15}$$

By adding (14) and (15), we get

$$\mathcal{U}\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right) + \mathcal{U}\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right) \le \mathcal{U}(u_1) + \mathcal{U}(v_1). \tag{16}$$

Similarly, multiplying by $\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}$ on both sides of Eq. (16), and then integrating with respect to ζ over $[0,1]$, we get the second part of the inequality as

$$\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)} \mathcal{U}(u_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)} \mathcal{U}(v_1)\right] \le \frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}. \tag{17}$$

By combining (13) and (17), we get the required result. □

Corollary 1. *If $k = l$ in Theorem 1, then we recover Theorem 7 in [20].*

Corollary 2. *If $k = l = 1$ in Theorem 1, then we recover Theorem 4 as presented in [21].*

Corollary 3. *If $k = l = \Upsilon = 1$ in Theorem 1, then the classical Hermite-Hadamard inequality (1) is recovered.*

Corollary 4. *By choosing $\Psi_0(k,l)=\sqrt{kl}$ in Theorem 1, we derive the following inequality involving the geometric form of (k,l)-Riemann-Liouville fractional integrals:*

$$\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\leq\frac{2^{\frac{\Upsilon}{\sqrt{kl}}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1-u_1)^{\frac{\Upsilon}{\sqrt{kl}}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}}\mathcal{U}(u_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}}\mathcal{U}(v_1)\right]$$
$$\leq\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}.$$

Corollary 5. *By choosing $\Psi_0(k,l)=\frac{k+l}{2}$ in Theorem 1, we derive the following inequality involving the arithmetic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\leq\frac{2^{\frac{2\Upsilon}{k+l}}k\Upsilon\Gamma_{(k,l)}\left(\frac{2k\Upsilon}{k+l}\right)}{(k+l)(v_1-u_1)^{\frac{2\Upsilon}{k+l}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}}\mathcal{U}(u_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}}\mathcal{U}(v_1)\right]$$
$$\leq\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}.$$

Corollary 6. *By choosing $\Psi_0(k,l)=\frac{k^2}{l}$ in Theorem 1, we derive the following inequality involving the harmonic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\leq\frac{2^{\frac{l\Upsilon}{k^2}-1}l\Upsilon\Gamma_{(k,l)}\left(\frac{l\Upsilon}{k}\right)}{k(v_1-u_1)^{\frac{l\Upsilon}{k^2}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k^2}{l}}\mathcal{U}(u_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k^2}{l}}\mathcal{U}(v_1)\right]$$
$$\leq\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}.$$

3. Midpoint type inequalities

The midpoint-type inequalities offer a way to estimate the value of a convex function at the midpoint of an interval through the use of fractional integrals. In the context of (k,l)-fractional integral, these inequalities extend the classical Hermite-Hadamard inequality by introducing bounds.

Lemma 1. *Let $\mathcal{U}:[u_1,v_1]\to\mathbb{R}$ be a differentiable function on open interval (u_1,v_1). Suppose that the derivative $\mathcal{U}'$ belongs to the function space $L_1[u_1,v_1]$, then*

$$\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)$$
$$=\frac{v_1-u_1}{4}\left[\int_0^1\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta-\int_0^1\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta\right]. \quad (18)$$

Proof. Let

$$M_1=\int_0^1\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta,$$

and

$$M_2=\int_0^1\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta.$$

Integrating M_1 by parts, we get

$$M_1=\frac{2\Upsilon}{(v_1-u_1)\Psi_0(k,l)}\int_0^1\mathcal{U}\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}\,d\zeta-\frac{2}{v_1-u_1}\mathcal{U}\left(\frac{u_1+v_1}{2}\right).$$

Substituting $\xi = \frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}$, we get

$$M_1 = \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}\Upsilon}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}} \int_{\frac{u_1+v_1}{2}}^{v_1} (v_1-\xi)^{\frac{\Upsilon}{\Psi_0(k,l)}-1}\mathcal{U}(\xi)\,d\xi - \frac{2}{v_1-u_1}\mathcal{U}\left(\frac{u_1+v_1}{2}\right).$$

After some calculations, we get

$$M_1 = \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)\right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right)\left(\frac{2}{v_1-u_1}\right). \tag{19}$$

Similarly, adopting the same procedure for M_2, we get

$$M_2 = \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right] + \mathcal{U}\left(\frac{u_1+v_1}{2}\right)\left(\frac{2}{v_1-u_1}\right). \tag{20}$$

By using (19) and (20), it follows that

$$\begin{aligned} M_1 - M_2 = {} & \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right] \\ & - \left(\frac{4}{v_1-u_1}\right)\mathcal{U}\left(\frac{u_1+v_1}{2}\right). \end{aligned}$$

By multiplying both sides of above equation by $\frac{v_1-u_1}{4}$, we derive the desired identity. □

Using above lemma, we establish the following (k,l)–fractional Hadamard-type inequality.

Theorem 2. *Let $\mathcal{U} : [u_1, v_1] \to \mathbb{R}$ be a differentiable function on the interval (u_1, v_1). If the absolute value of the derivative, $|\mathcal{U}'|$ is convex on $[u_1, v_1]$, then*

$$\begin{aligned} & \left| \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right| \\ & \quad \leq \frac{v_1-u_1}{4}\left(\frac{\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)}\right)\left[|\mathcal{U}'(u_1)| + |\mathcal{U}'(v_1)|\right]. \end{aligned} \tag{21}$$

Proof. By using the properties of absolute value function and the Lemma 1, we get the following

$$\begin{aligned} & \left| \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right| \\ & \leq \frac{v_1-u_1}{4}\left[\int_0^1 \left|\mathcal{U}'\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right)\right| \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta + \int_0^1 \left|\mathcal{U}'\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right)\right| \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta\right]. \end{aligned}$$

Using the convexity of $|\mathcal{U}'|$, we have

$$\begin{aligned} & \left| \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}\Psi_0(k,l)} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right| \\ & \leq \frac{v_1-u_1}{4}\left[\int_0^1 \left(\frac{\zeta}{2}|\mathcal{U}'(u_1)| + \frac{2-\zeta}{2}|\mathcal{U}'(v_1)|\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta + \int_0^1 \left(\frac{\zeta}{2}|\mathcal{U}'(v_1)| + \frac{2-\zeta}{2}|\mathcal{U}'(u_1)|\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta\right] \\ & = \frac{v_1-u_1}{4}\left[|\mathcal{U}'(u_1)|\int_0^1 \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta + |\mathcal{U}'(v_1)|\int_0^1 \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta\right] \end{aligned}$$

$$= \frac{v_1 - u_1}{4} \left(\frac{\Psi_0(k,l)}{\Psi_0(k,l) + \Upsilon} \right) \left[|\mathcal{U}'(u_1)| + |\mathcal{U}'(v_1)| \right].$$

□

Corollary 7. *If $k = l$ in Theorem 2, we obtain a special case of Theorem 3.1 presented in [22].*

Corollary 8. *If $k = l = 1$ in Theorem 2, a particular case of Theorem 5 from [21] is recovered.*

Corollary 9. *If $k = l = \Upsilon = 1$ in Theorem 2, then Theorem 2.2 is presented in [23].*

Corollary 10. *By choosing $\Psi_0(k,l) = \sqrt{kl}$ in Theorem 2, we derive the following inequality involving the geometric form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left| \frac{2^{\frac{\Upsilon}{\sqrt{kl}}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1 - u_1)^{\frac{\Upsilon}{\sqrt{kl}}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\le \frac{v_1 - u_1}{4} \left(\frac{\sqrt{kl}}{\Upsilon + \sqrt{kl}} \right) \left[|\mathcal{U}'(u_1)| + |\mathcal{U}'(v_1)| \right].$$

Corollary 11. *By choosing $\Psi_0(k,l) = \frac{k+l}{2}$ in Theorem 2, we derive the following inequality involving the arithmetic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left| \frac{2^{\frac{2\Upsilon}{k+l}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{2k\Upsilon}{k+l}\right)}{\frac{k+l}{2}(v_1 - u_1)^{\frac{2\Upsilon}{k+l}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\le \frac{v_1 - u_1}{4} \left(\frac{\frac{k+l}{2}}{\Upsilon + \frac{k+l}{2}} \right) \left[|\mathcal{U}'(u_1)| + |\mathcal{U}'(v_1)| \right].$$

Corollary 12. *By choosing $\Psi_0(k,l) = \frac{k^2}{l}$ in Theorem 2, we derive the following inequality involving the harmonic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left| \frac{2^{\frac{l\Upsilon}{k^2}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{l\Upsilon}{k}\right)}{\frac{k^2}{l}(v_1 - u_1)^{\frac{l\Upsilon}{k^2}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k^2}{l}} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k^2}{l}} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\le \frac{v_1 - u_1}{4} \left(\frac{\frac{k^2}{l}}{\Upsilon + \frac{k^2}{l}} \right) \left[|\mathcal{U}'(u_1)| + |\mathcal{U}'(v_1)| \right].$$

Similarly, using the Lemma 1 we get the following result.

Theorem 3. *Let $\mathcal{U} : [u_1, v_1] \to \mathbb{R}$ be a differentiable function on an interval (u_1, v_1). If $|\mathcal{U}'|^r$ is convex on $[u_1, v_1]$, then*

$$\left| \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1 - u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\le \frac{v_1 - u_1}{4} \left(\frac{\Psi_0(k,l)}{\Upsilon s + \Psi_0(k,l)} \right)^{\frac{1}{s}} \left[\left(\frac{|\mathcal{U}'(u_1)|^r + 3|\mathcal{U}'(v_1)|^r}{4} \right)^{\frac{1}{r}} + \left(\frac{3|\mathcal{U}'(u_1)|^r + |\mathcal{U}'(v_1)|^r}{4} \right)^{\frac{1}{r}} \right],$$

where $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. By using the properties of absolute value function and the Lemma 1, we can write (21)

$$\frac{4}{v_1-u_1}\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq\int_0^1\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta+\int_0^1\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta.$$

By applying Hölder inequality, we get

$$\frac{4}{v_1-u_1}\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq\left(\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}.$$

Since $|\mathcal{U}'|^r$ is convex, we have

$$\frac{4}{v_1-u_1}\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq\left(\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(u_1)|^r+\frac{2-\zeta}{2}|\mathcal{U}'(v_1)|^r\right)d\zeta\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(v_1)|^r+\frac{2-\zeta}{2}|\mathcal{U}'(u_1)|^r\right)d\zeta\right)^{\frac{1}{r}}$$
$$=\left(\frac{\Psi_0(k,l)}{\Upsilon s+\Psi_0(k,l)}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right].$$

Multiplying $\frac{v_1-u_1}{4}$ on both sides, we derive the desired inequality. □

Corollary 13. *If $k=l$ in Theorem 3, we obtain Theorem 3.2 from [22].*

Corollary 14. *If $k=l=1$ in Theorem 3, we obtain Theorem 6 from [24].*

Corollary 15. *If $k=l=\Upsilon=1$ in Theorem 3, we obtain the following inequality:*

$$\left|\frac{1}{v_1-u_1}\int_{u_1}^{v_1}\mathcal{U}(\xi)\,d\xi-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\leq\frac{v_1-u_1}{4}\left(\frac{1}{s+1}\right)^{\frac{1}{s}}$$
$$\times\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right].$$

Corollary 16. *By choosing $\Psi_0(k,l)=\sqrt{kl}$ in Theorem 3, we derive the following inequality involving the geometric form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{2^{\frac{\Upsilon}{\sqrt{kl}}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1-u_1)^{\frac{\Upsilon}{\sqrt{kl}}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$

$$\leq \frac{v_1-u_1}{4}\left(\frac{\sqrt{kl}}{\Upsilon s+\sqrt{kl}}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right],$$

where $r>1$ *and* $\frac{1}{r}+\frac{1}{s}=1$.

Corollary 17. *By choosing* $\Psi_0(k,l)=\frac{k+l}{2}$ *in Theorem 3, we derive the following inequality involving the arithmetic form of* (k,l)*-Riemann-Liouville fractional integrals:*

$$\left|\frac{2^{\frac{2\Upsilon}{k+l}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{2k\Upsilon}{k+l}\right)}{\left(\frac{k+l}{2}\right)(v_1-u_1)^{\frac{2\Upsilon}{k+l}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\frac{k+l}{2}}{\Upsilon s+\frac{k+l}{2}}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right],$$

where $r>1$ *and* $\frac{1}{r}+\frac{1}{s}=1$.

Corollary 18. *By choosing* $\Psi_0(k,l)=\frac{k^2}{l}$ *in Theorem 3, we derive the following inequality involving the harmonic form of* (k,l)*-Riemann-Liouville fractional integrals:*

$$\left|\frac{2^{\frac{l\Upsilon}{k^2}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{l\Upsilon}{k}\right)}{\frac{k^2}{l}(v_1-u_1)^{\frac{l\Upsilon}{k^2}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k^2}{l}}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k^2}{l}}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\frac{k^2}{l}}{\Upsilon s+\frac{k^2}{l}}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right],$$

where $r>1$ *and* $\frac{1}{r}+\frac{1}{s}=1$.

Theorem 4. *Assume that* $\mathcal{U}:[u_1,v_1]\to\mathbb{R}$ *be a differentiable function on the open interval* (u_1,v_1)*. If* $|\mathcal{U}'|^r$ *is convex on* $[u_1,v_1]$*, then*

$$\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq \frac{v_1-u_1}{4}\left[\frac{2\Psi_0(k,l)}{s(\Upsilon s+\Psi_0(k,l))}+\frac{|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{r}\right],$$

where $r>1$ *and* $\frac{1}{r}+\frac{1}{s}=1$.

Proof. By using the properties of absolute value function and the Lemma 1, we can write (21)

$$\frac{4}{v_1-u_1}\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq \int_0^1\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta+\int_0^1\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\,d\zeta.$$

By applying Young's inequality as

$$xy\leq \frac{1}{s}x^s+\frac{1}{r}y^r, \tag{22}$$

we get

$$\begin{aligned}&\frac{4}{v_1-u_1}\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\\&\leq\frac{1}{s}\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta+\frac{1}{r}\int_0^1\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|^r d\zeta\\&\quad+\frac{1}{s}\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta+\frac{1}{r}\int_0^1\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|^r d\zeta.\end{aligned}\tag{23}$$

Since $|\mathcal{U}'|^r$ is convex, we have

$$\begin{aligned}&\frac{4}{v_1-u_1}\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\\&\leq\frac{1}{s}\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta+\frac{1}{r}\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(u_1)|^r+\frac{2-\zeta}{2}|\mathcal{U}'(v_1)|^r\right)d\zeta\\&\quad+\frac{1}{s}\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta+\frac{1}{r}\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(v_1)|^r+\frac{2-\zeta}{2}|\mathcal{U}'(u_1)|^r\right)d\zeta\\&=\frac{2}{s}\int_0^1\zeta^{\frac{\Upsilon s}{\Psi_0(k,l)}}\,d\zeta+\frac{1}{r}\int_0^1\left[|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r\right]d\zeta\\&=\frac{2}{s}\left(\frac{\Psi_0(k,l)}{\Upsilon s+\Psi_0(k,l)}\right)+\frac{|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{r}.\end{aligned}\tag{24}$$

Multiplying both sides of (24) by $(v_1-u_1)/4$, we derive the desired inequality. □

Corollary 19. *If $k=l$ in Theorem 4, we get the following inequality involving k-Riemann-Liouville integrals:*

$$\begin{aligned}&\left|\frac{2^{\frac{\Upsilon}{k}-1}\Upsilon\Gamma_k(\Upsilon)}{(v_1-u_1)^{\frac{\Upsilon}{k}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,k}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,k}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\\&\leq\frac{v_1-u_1}{4}\left[\frac{2k}{s(\Upsilon s+k)}+\frac{|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{r}\right].\end{aligned}\tag{25}$$

Corollary 20. *If $k=l=1$ in Theorem 4, we deduce the following inequality involving Riemann-Liouville integrals:*

$$\begin{aligned}&\left|\frac{2^{\Upsilon-1}\Upsilon\Gamma(\Upsilon)}{(v_1-u_1)^{\Upsilon}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\\&\leq\frac{v_1-u_1}{4}\left[\frac{2}{s(\Upsilon s+1)}+\frac{|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{r}\right].\end{aligned}\tag{26}$$

Corollary 21. *If $k=l=\Upsilon=1$ in Theorem 4, then we get the following inequality*

$$\left|\frac{1}{v_1-u_1}\int_{u_1}^{v_1}\mathcal{U}(x)\,dx-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\leq\frac{v_1-u_1}{4}\left[\frac{2}{s(s+1)}+\frac{|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{r}\right].\tag{27}$$

Corollary 22. *By choosing $\Psi_0(k,l)=\sqrt{kl}$ in Theorem 4, we derive the following inequality involving the geometric form of (k,l)-Riemann-Liouville fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\frac{\Upsilon}{\sqrt{kl}}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1-u_1)^{\frac{\Upsilon}{\sqrt{kl}}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\\&\leq\frac{v_1-u_1}{4}\left[\frac{2\sqrt{kl}}{s(\Upsilon s+\sqrt{kl})}+\frac{|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{r}\right],\end{aligned}\tag{28}$$

where $r > 1$ *and* $\frac{1}{r} + \frac{1}{s} = 1$.

Corollary 23. *By choosing* $\Psi_0(k,l) = \frac{k+l}{2}$ *in Theorem 4, we derive the following inequality involving the arithmetic form of* (k,l)*-Riemann-Liouville fractional integrals:*

$$\left| \frac{2^{\frac{2\Upsilon}{k+l}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{2k\Upsilon}{k+l}\right)}{\left(\frac{k+l}{2}\right)(v_1-u_1)^{\frac{2\Upsilon}{k+l}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\leq \frac{v_1-u_1}{4} \left[\frac{2\left(\frac{k+l}{2}\right)}{s\left(\Upsilon s + \frac{k+l}{2}\right)} + \frac{|\mathcal{U}'(u_1)|^r + |\mathcal{U}'(v_1)|^r}{r} \right], \tag{29}$$

where $r > 1$ *and* $\frac{1}{r} + \frac{1}{s} = 1$.

Corollary 24. *By choosing* $\Psi_0(k,l) = \frac{k^2}{l}$ *in Theorem 4, we derive the following inequality involving the harmonic form of* (k,l)*-Riemann-Liouville fractional integrals:*

$$\left| \frac{2^{\frac{\Upsilon l}{k^2}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{\Upsilon l}{k}\right)}{\left(\frac{k^2}{l}\right)(v_1-u_1)^{\frac{\Upsilon l}{k^2}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k^2}{l}} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k^2}{l}} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\leq \frac{v_1-u_1}{4} \left[\frac{2\left(\frac{k^2}{l}\right)}{s\left(\Upsilon s + \frac{k^2}{l}\right)} + \frac{|\mathcal{U}'(u_1)|^r + |\mathcal{U}'(v_1)|^r}{r} \right], \tag{30}$$

where $r > 1$ *and* $\frac{1}{r} + \frac{1}{s} = 1$.

Theorem 5. *Let* $\mathcal{U} : [u_1, v_1] \to \mathbb{R}$ *be a differentiable function on an open interval* (u_1, v_1)*. If* $|\mathcal{U}'|^r$ *is convex on the interval* $[u_1, v_1]$*, then*

$$\left| \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\leq \frac{v_1-u_1}{4} \left(\frac{\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)}\right) \left(\frac{1}{2(\Upsilon+2\Psi_0(k,l))}\right)^{\frac{1}{r}} \times \left\{ \left((\Upsilon+\Psi_0(k,l))|\mathcal{U}'(u_1)|^r + (\Upsilon+3\Psi_0(k,l))|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}} \right.$$
$$\left. + \left((\Upsilon+3\Psi_0(k,l))|\mathcal{U}'(u_1)|^r + (\Upsilon+\Psi_0(k,l))|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}} \right\}, \tag{31}$$

where $r > 1$.

Proof. By using the properties of absolute value function and Lemma 1, we can write

$$\frac{4}{v_1-u_1} \left| \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$
$$\leq \int_0^1 \left| \mathcal{U}'\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right) \right| \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}} d\zeta + \int_0^1 \left| \mathcal{U}'\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right) \right| \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}} d\zeta. \tag{32}$$

By applying the power mean inequality, we get

$$\frac{4}{v_1-u_1} \left| \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)} \mathcal{U}(u_1) \right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right) \right|$$

$$\leq \left(\int_0^1 \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}} d\zeta\right)^{1-\frac{1}{r}} \left(\int_0^1 \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}} \left|\mathcal{U}'\left(\frac{\zeta u_1}{2} + \frac{(2-\zeta)v_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}$$
$$+ \left(\int_0^1 \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}} d\zeta\right)^{1-\frac{1}{r}} \left(\int_0^1 \zeta^{\frac{\Upsilon}{\Psi_0(k,l)}} \left|\mathcal{U}'\left(\frac{\zeta v_1}{2} + \frac{(2-\zeta)u_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}. \tag{33}$$

By using the convexity of $|\mathcal{U}'|^r$, we have

$$\frac{4}{v_1-u_1}\left|\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq \left(\frac{\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)}\right)^{1-\frac{1}{r}}\left[\frac{|\mathcal{U}'(u_1)|^r}{2}\left(\frac{\Psi_0(k,l)}{\Upsilon+2\Psi_0(k,l)}\right) + \frac{|\mathcal{U}'(v_1)|^r}{2}\left(\frac{2\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)} - \frac{\Psi_0(k,l)}{\Upsilon+2\Psi_0(k,l)}\right)\right]^{\frac{1}{r}}$$
$$+ \left(\frac{\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)}\right)^{1-\frac{1}{r}}\left[\frac{|\mathcal{U}'(v_1)|^r}{2}\left(\frac{\Psi_0(k,l)}{\Upsilon+2\Psi_0(k,l)}\right) + \frac{|\mathcal{U}'(u_1)|^r}{2}\left(\frac{2\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)} - \frac{\Psi_0(k,l)}{\Upsilon+2\Psi_0(k,l)}\right)\right]^{\frac{1}{r}}$$
$$= \left(\frac{\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)}\right)\left(\frac{1}{2(\Upsilon+2\Psi_0(k,l))}\right)^{\frac{1}{r}}\left[(\Upsilon+\Psi_0(k,l))|\mathcal{U}'(u_1)|^r + (\Upsilon+3\Psi_0(k,l))|\mathcal{U}'(v_1)|^r\right]^{\frac{1}{r}}$$
$$+ \left(\frac{\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)}\right)\left(\frac{1}{2(\Upsilon+2\Psi_0(k,l))}\right)^{\frac{1}{r}}\left[(\Upsilon+\Psi_0(k,l))|\mathcal{U}'(v_1)|^r + (\Upsilon+3\Psi_0(k,l))|\mathcal{U}'(u_1)|^r\right]^{\frac{1}{r}}. \tag{34}$$

After simplification, we get the required inequality. □

Corollary 25. *If $k = l$ in Theorem 5, then we derive Theorem 8 in [20].*

Corollary 26. *If $k = l = 1$ in Theorem 5, we derive Theorem 5 from [21].*

Corollary 27. *If $k = l = \Upsilon = 1$ in Theorem 5, we derive Theorem 2.2 from [25].*

Corollary 28. *By choosing $\Psi_0(k,l) = \sqrt{kl}$ in Theorem 5, we derive the following inequality involving the geometric form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{2^{\frac{\Upsilon}{\sqrt{kl}}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1-u_1)^{\frac{\Upsilon}{\sqrt{kl}}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}}\mathcal{U}(u_1)\right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\sqrt{kl}}{\Upsilon+\sqrt{kl}}\right)\left(\frac{1}{2(\Upsilon+2\sqrt{kl})}\right)^{\frac{1}{r}}$$
$$\times\left\{\left[(\Upsilon+\sqrt{kl})|\mathcal{U}'(u_1)|^r + (\Upsilon+3\sqrt{kl})|\mathcal{U}'(v_1)|^r\right]^{\frac{1}{r}} + \left[(\Upsilon+3\sqrt{kl})|\mathcal{U}'(u_1)|^r + (\Upsilon+\sqrt{kl})|\mathcal{U}'(v_1)|^r\right]^{\frac{1}{r}}\right\}. \tag{35}$$

Corollary 29. *By choosing $\Psi_0(k,l) = \frac{k+l}{2}$ in Theorem 5, we derive the following inequality involving the arithmetic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{2^{\frac{2\Upsilon}{k+l}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{2k\Upsilon}{k+l}\right)}{\left(\frac{k+l}{2}\right)(v_1-u_1)^{\frac{2\Upsilon}{k+l}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}}\mathcal{U}(u_1)\right] - \mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\frac{k+l}{2}}{\Upsilon+\frac{k+l}{2}}\right)\left(\frac{1}{2(\Upsilon+k+l)}\right)^{\frac{1}{r}} \times\left\{\left[\left(\Upsilon+\frac{k+l}{2}\right)|\mathcal{U}'(u_1)|^r + \left(\Upsilon+\frac{3(k+l)}{2}\right)|\mathcal{U}'(v_1)|^r\right]^{\frac{1}{r}}\right.$$
$$\left. + \left[\left(\Upsilon+\frac{3(k+l)}{2}\right)|\mathcal{U}'(u_1)|^r + \left(\Upsilon+\frac{k+l}{2}\right)|\mathcal{U}'(v_1)|^r\right]^{\frac{1}{r}}\right\}. \tag{36}$$

Corollary 30. *By choosing $\Psi_0(k,l)=\frac{k^2}{l}$ in Theorem 5, we derive the following inequality involving the harmonic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\frac{\Upsilon l}{k^2}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{\Upsilon l}{k}\right)}{\left(\frac{k^2}{l}\right)(v_1-u_1)^{\frac{\Upsilon l}{k^2}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k^2}{l}}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k^2}{l}}\mathcal{U}(u_1)\right]-\mathcal{U}\left(\frac{u_1+v_1}{2}\right)\right|\\&\leq\frac{v_1-u_1}{4}\left(\frac{\frac{k^2}{l}}{\Upsilon+\frac{k^2}{l}}\right)\left(\frac{1}{2\left(\Upsilon+2\frac{k^2}{l}\right)}\right)^{\frac{1}{r}}\\&\quad\times\left\{\left[\left(\Upsilon+\frac{k^2}{l}\right)|\mathcal{U}'(u_1)|^r+\left(\Upsilon+3\frac{k^2}{l}\right)|\mathcal{U}'(v_1)|^r\right]^{\frac{1}{r}}+\left[\left(\Upsilon+3\frac{k^2}{l}\right)|\mathcal{U}'(u_1)|^r+\left(\Upsilon+\frac{k^2}{l}\right)|\mathcal{U}'(v_1)|^r\right]^{\frac{1}{r}}\right\}.\end{aligned}\tag{37}$$

4. Trapezoidal-type inequalities

In this section, we aim to establish new trapezoidal-type inequalities involving the (k,l)-Riemann-Liouville fractional integrals. These inequalities serve as fractional counterparts of the classical trapezoidal inequalities and offer refined bounds under convexity assumptions.

To derive our main results, we begin by presenting a useful lemma that plays a central role in the analysis. The lemma provides a foundation for bounding the difference between the fractional integral average and the classical trapezoidal mean of a function. It will be instrumental in formulating and proving the fractional generalizations presented in the subsequent theorems.

Lemma 2. *Suppose $\mathcal{U}:[u_1,v_1]\to\mathbb{R}$ be a differentiable function on an open interval (u_1,v_1), and its derivative $\mathcal{U}'\in L_1[u_1,v_1]$, then*

$$\begin{aligned}&\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\\&=\frac{v_1-u_1}{4}\left[\int_0^1\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta-\int_0^1\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta\right].\end{aligned}\tag{38}$$

Proof. Let

$$M_1=\int_0^1\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta,\tag{39}$$

and

$$M_2=\int_0^1\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta.\tag{40}$$

Integrating M_1 by parts, we get

$$M_1=\frac{2\mathcal{U}(v_1)}{v_1-u_1}-\frac{2\Upsilon}{\Psi_0(k,l)(v_1-u_1)}\int_0^1\mathcal{U}\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}-1}d\zeta.\tag{41}$$

Substituting $\xi=\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}$, we get

$$M_1=\frac{2\mathcal{U}(v_1)}{v_1-u_1}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}\Upsilon}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}}\int_{\frac{u_1+v_1}{2}}^{v_1}(v_1-\xi)^{\frac{\Upsilon}{\Psi_0(k,l)}-1}\mathcal{U}(\xi)\,d\xi.\tag{42}$$

After some calculations, we get

$$M_1=\frac{2\mathcal{U}(v_1)}{v_1-u_1}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)\right].\tag{43}$$

Similarly, adopting the same procedure for M_2, we get

$$M_2=\frac{2\mathcal{U}(u_1)}{v_1-u_1}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]. \tag{44}$$

By using (43) and (44), it follows that

$$M_1+M_2=\frac{2[\mathcal{U}(u_1)+\mathcal{U}(v_1)]}{v_1-u_1}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}+1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}+1}}\times\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]. \tag{45}$$

Hence, multiplying the above equation by $(v_1-u_1)/4$, we get the identity. □

Based on Lemma 2, the following inequality involving (k,l)-fractional integral inequality is obtained.

Theorem 6. *Let $\mathcal{U}:[u_1,v_1]\to\mathbb{R}$ be a differentiable function on an open interval (u_1,v_1). If $|\mathcal{U}'|$ is convex on $[u_1,v_1]$, then*

$$\begin{aligned}&\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|\\&\le\frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Psi_0(k,l)+\Upsilon}\right)\left[|\mathcal{U}'(u_1)|+|\mathcal{U}'(v_1)|\right].\end{aligned} \tag{46}$$

Proof. By using Lemma 2 and the properties of absolute value function, we have

$$\begin{aligned}&\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|\\&\le\frac{v_1-u_1}{4}\left[\int_0^1\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta+\int_0^1\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta\right].\end{aligned} \tag{47}$$

Using the convexity of $|\mathcal{U}'|$, we have

$$\begin{aligned}&\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|\\&\le\frac{v_1-u_1}{4}\left[\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(u_1)|+\frac{2-\zeta}{2}|\mathcal{U}'(v_1)|\right)\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta\right]\\&\quad+\frac{v_1-u_1}{4}\left[\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(v_1)|+\frac{2-\zeta}{2}|\mathcal{U}'(u_1)|\right)\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta\right]\\&=\frac{v_1-u_1}{4}\left[|\mathcal{U}'(u_1)|\int_0^1\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta+|\mathcal{U}'(v_1)|\int_0^1\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta\right]\\&=\frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Upsilon+\Psi_0(k,l)}\right)\left[|\mathcal{U}'(u_1)|+|\mathcal{U}'(v_1)|\right].\end{aligned} \tag{48}$$

□

Corollary 31. *If $k=l$ in Theorem 6, we obtain Remark 5.4 in [26].*

Corollary 32. *If $k=l=1$ in Theorem 6, we obtain Corollary 5.4 in [26].*

Corollary 33. *If $k = l = \Upsilon = 1$, then we get Theorem 2.2 in [23].*

Corollary 34. *By choosing $\Psi_0(k,l) = \sqrt{kl}$ in Theorem 6, we derive the following inequality involving the geometric form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{\Upsilon}{\sqrt{kl}}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1-u_1)^{\frac{\Upsilon}{\sqrt{kl}}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Upsilon+\sqrt{kl}}\right)\left[|\mathcal{U}'(u_1)|+|\mathcal{U}'(v_1)|\right]. \tag{49}$$

Corollary 35. *By choosing $\Psi_0(k,l) = \frac{k+l}{2}$ in Theorem 6, we derive the following inequality involving the arithmetic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{2\Upsilon}{k+l}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{2k\Upsilon}{k+l}\right)}{\left(\frac{k+l}{2}\right)(v_1-u_1)^{\frac{2\Upsilon}{k+l}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Upsilon+\frac{k+l}{2}}\right)\left[|\mathcal{U}'(u_1)|+|\mathcal{U}'(v_1)|\right]. \tag{50}$$

Corollary 36. *By choosing $\Psi_0(k,l) = \frac{k^2}{l}$ in Theorem 6, we derive the following inequality involving the harmonic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{\Upsilon l}{k^2}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{\Upsilon l}{k}\right)}{\left(\frac{k^2}{l}\right)(v_1-u_1)^{\frac{\Upsilon l}{k^2}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k^2}{l}}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k^2}{l}}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Upsilon+\frac{k^2}{l}}\right)\left[|\mathcal{U}'(u_1)|+|\mathcal{U}'(v_1)|\right]. \tag{51}$$

Similarly, using Lemma 2, we get the following result.

Theorem 7. *Consider a function $\mathcal{U} : [u_1, v_1] \to \mathbb{R}$ that is differentiable on (u_1, v_1). If the function $|\mathcal{U}'|^r$ is convex over the closed interval $[u_1, v_1]$, then*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Upsilon s+\Psi_0(k,l)}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}} + \left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right], \tag{52}$$

where $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. By using the properties of absolute value function and Lemma 2, we can write

$$\frac{4}{v_1-u_1}\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$\leq \int_0^1 \left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta + \int_0^1 \left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta. \tag{53}$$

By applying Hölder's inequality, we get

$$\frac{4}{v_1-u_1}\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{Y}{\Psi_0(k,l)}-1}kY\Gamma_{(k,l)}\left(\frac{kY}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{Y}{\Psi_0(k,l)}}}\left[I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$\leq\left(\int_0^1\left(1-\zeta^{\frac{Ys}{\Psi_0(k,l)}}\right)d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1\left(1-\zeta^{\frac{Ys}{\Psi_0(k,l)}}\right)d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}. \tag{54}$$

Since $|\mathcal{U}'|^r$ is convex, we have

$$\frac{4}{v_1-u_1}\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{Y}{\Psi_0(k,l)}-1}kY\Gamma_{(k,l)}\left(\frac{kY}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{Y}{\Psi_0(k,l)}}}\left[I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$\leq\left(\int_0^1\left(1-\zeta^{\frac{Ys}{\Psi_0(k,l)}}\right)d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(u_1)|^r+\frac{2-\zeta}{2}|\mathcal{U}'(v_1)|^r\right)d\zeta\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1\left(1-\zeta^{\frac{Ys}{\Psi_0(k,l)}}\right)d\zeta\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{\zeta}{2}|\mathcal{U}'(v_1)|^r+\frac{2-\zeta}{2}|\mathcal{U}'(u_1)|^r\right)d\zeta\right)^{\frac{1}{r}}$$
$$=\left(\frac{Y}{Ys+\Psi_0(k,l)}\right)^{\frac{1}{s}}\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{Y}{Ys+\Psi_0(k,l)}\right)^{\frac{1}{s}}\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}$$
$$=\left(\frac{Y}{Ys+\Psi_0(k,l)}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right]. \tag{55}$$

Multiplying both sides by $(v_1-u_1)/4$, we derive the desired inequality. □

Corollary 37. *If $k=l$ in Theorem 7, we get Corollary 5.7 in [26].*

Corollary 38. *If $k=l=1$ in Theorem 7, we get Corollary 5.6 from [26].*

Corollary 39. *If $k=l=Y=1$ in Theorem 7, we get Remark 5.5 in [26].*

Corollary 40. *By choosing $\Psi_0(k,l)=\sqrt{kl}$ in Theorem 7, we derive the geometric form of (k,l)-Riemann-Liouville fractional integrals $G^{Y}_{u_1^+,\Psi_0(k,l)}\mathcal{U}(x)$ and $G^{Y}_{v_1^-,\Psi_0(k,l)}\mathcal{U}(x)$.*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{Y}{\sqrt{kl}}-1}kY\Gamma_{(k,l)}\left(\frac{kY}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1-u_1)^{\frac{Y}{\sqrt{kl}}}}\left[I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}}\mathcal{U}(v_1)+I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}}\mathcal{U}(u_1)\right]\right|$$
$$\leq\frac{v_1-u_1}{4}\left(\frac{Y}{Ys+\sqrt{kl}}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right], \tag{56}$$

where $r>1$ and $\frac{1}{r}+\frac{1}{s}=1$.

Corollary 41. *By choosing $\Psi_0(k,l)=\frac{k+l}{2}$ in Theorem 7, we derive the following inequality involving the arithmetic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{2Y}{k+l}-1}kY\Gamma_{(k,l)}\left(\frac{2kY}{k+l}\right)}{\left(\frac{k+l}{2}\right)(v_1-u_1)^{\frac{2Y}{k+l}}}\left[I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}}\mathcal{U}(v_1)+I^{Y}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}}\mathcal{U}(u_1)\right]\right|$$

$$\leq \frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Upsilon s+\frac{k+l}{2}}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right], \tag{57}$$

where $r>1$ *and* $\frac{1}{r}+\frac{1}{s}=1$.

Corollary 42. *By choosing* $\Psi_0(k,l)=\frac{k^2}{l}$ *in Theorem 7, we derive the following inequality involving the harmonic form of* (k,l)*-Riemann-Liouville fractional integrals:*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon l}{k^2}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{\Upsilon l}{k}\right)}{\left(\frac{k^2}{l}\right)(v_1-u_1)^{\frac{\Upsilon l}{k^2}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k^2}{l}}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k^2}{l}}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{4}\left(\frac{\Upsilon}{\Upsilon s+\frac{k^2}{l}}\right)^{\frac{1}{s}}\left[\left(\frac{|\mathcal{U}'(u_1)|^r+3|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}+\left(\frac{3|\mathcal{U}'(u_1)|^r+|\mathcal{U}'(v_1)|^r}{4}\right)^{\frac{1}{r}}\right], \tag{58}$$

where $r>1$ *and* $\frac{1}{r}+\frac{1}{s}=1$.

Theorem 8. *Let* $\mathcal{U}:[u_1,v_1]\to\mathbb{R}$ *be a function that is differentiable on the interval* (u_1,v_1)*. Suppose that* $|\mathcal{U}'|^r$ *is convex on closed interval* $[u_1,v_1]$*, for some fixed* $r>1$*. Then*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{2^{2+\frac{1}{r}}}\left(\frac{\Upsilon}{\Upsilon+\Psi_0(k,l)}\right)\left\{\left(\frac{\Upsilon+1}{2(\Upsilon+2)}|\mathcal{U}'(u_1)|^r+\frac{3\Upsilon+7}{2(\Upsilon+2)}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\frac{3\Upsilon+7}{2(\Upsilon+2\Psi_0(k,l))}|\mathcal{U}'(u_1)|^r+\frac{\Upsilon+\Psi_0(k,l)}{2(\Upsilon+2\Psi_0(k,l))}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right\}. \tag{59}$$

Proof. By using the properties of absolute value function and Lemma 2, we can write (46) as

$$\frac{4}{v_1-u_1}\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$\leq\int_0^1\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta+\int_0^1\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta. \tag{60}$$

Utilizing the power mean inequality, we get

$$\frac{4}{v_1-u_1}\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$\leq\left(\int_0^1\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)\left|\mathcal{U}'\left(\frac{\zeta u_1}{2}+\frac{(2-\zeta)v_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)d\zeta\right)^{1-\frac{1}{r}}\left(\int_0^1\left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right)\left|\mathcal{U}'\left(\frac{\zeta v_1}{2}+\frac{(2-\zeta)u_1}{2}\right)\right|^r d\zeta\right)^{\frac{1}{r}}. \tag{61}$$

Using the convexity of $|\mathcal{U}'|^r$, we have

$$\frac{4}{v_1-u_1}\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2}-\frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1}k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}}\left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1)+I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$

$$\leq \left(\int_0^1 \left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right) d\zeta\right)^{1-\frac{1}{r}} \left(\int_0^1 \left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right) \left[\frac{\zeta}{2}|\mathcal{U}'(u_1)|^r + \frac{2-\zeta}{2}|\mathcal{U}'(v_1)|^r\right] d\zeta\right)^{\frac{1}{r}}$$
$$+ \left(\int_0^1 \left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right) d\zeta\right)^{1-\frac{1}{r}} \left(\int_0^1 \left(1-\zeta^{\frac{\Upsilon}{\Psi_0(k,l)}}\right) \left[\frac{\zeta}{2}|\mathcal{U}'(v_1)|^r + \frac{2-\zeta}{2}|\mathcal{U}'(u_1)|^r\right] d\zeta\right)^{\frac{1}{r}}. \quad (62)$$

Therefore,

$$\frac{4}{v_1-u_1}\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{\Upsilon}{\Psi_0(k,l)}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\Psi_0(k,l)}\right)}{\Psi_0(k,l)(v_1-u_1)^{\frac{\Upsilon}{\Psi_0(k,l)}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\Psi_0(k,l)}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\Psi_0(k,l)}\mathcal{U}(u_1)\right]\right|$$
$$= \left(\frac{\Upsilon}{\Upsilon+\Psi_0(k,l)}\right)^{1-\frac{1}{r}} \left\{ \left[\frac{|\mathcal{U}'(u_1)|^r}{2}\left(\frac{\Upsilon}{2(\Upsilon+2\Psi_0(k,l))}\right) + \frac{|\mathcal{U}'(v_1)|^r}{2}\left(\frac{3}{2} - \frac{2\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)} + \frac{\Psi_0(k,l)}{\Upsilon+2\Psi_0(k,l)}\right)\right]^{\frac{1}{r}}\right.$$
$$\left. + \left[\frac{|\mathcal{U}'(v_1)|^r}{2}\left(\frac{\Upsilon}{2(\Upsilon+2\Psi_0(k,l))}\right) + \frac{|\mathcal{U}'(u_1)|^r}{2}\left(\frac{3}{2} - \frac{2\Psi_0(k,l)}{\Upsilon+\Psi_0(k,l)} + \frac{\Psi_0(k,l)}{\Upsilon+2\Psi_0(k,l)}\right)\right]^{\frac{1}{r}} \right\}$$
$$= \frac{v_1-u_1}{2^{2+\frac{1}{r}}}\left(\frac{\Upsilon}{\Upsilon+\Psi_0(k,l)}\right)\left\{\left(\frac{\Upsilon+1}{2(\Upsilon+2)}|\mathcal{U}'(u_1)|^r + \frac{3\Upsilon+7}{2(\Upsilon+2)}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{3\Upsilon+7}{2(\Upsilon+2\Psi_0(k,l))}|\mathcal{U}'(u_1)|^r + \frac{\Upsilon+\Psi_0(k,l)}{2(\Upsilon+2\Psi_0(k,l))}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right\}. \quad (63)$$

After simplifications, we get the required inequality. □

Corollary 43. *If $k = l$ in Theorem 8, we get Corollary 5.10 from [26].*

Corollary 44. *If $k = l = 1$ in Theorem 8, we get Corollary 5.9 from [26].*

Corollary 45. *If $k = l = \Upsilon = 1$ in Theorem 8, we get Remark 5.6 in [26].*

Corollary 46. *By choosing $\Psi_0(k,l) = \sqrt{kl}$ in Theorem 8, we derive the following inequality involving the geometric form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{\Upsilon}{\sqrt{kl}}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{k\Upsilon}{\sqrt{kl}}\right)}{\sqrt{kl}(v_1-u_1)^{\frac{\Upsilon}{\sqrt{kl}}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\sqrt{kl}}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\sqrt{kl}}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{2^{2+\frac{1}{r}}}\left(\frac{\Upsilon}{\Upsilon+\sqrt{kl}}\right)\left\{\left(\frac{1+\Upsilon}{2(2+\Upsilon)}|\mathcal{U}'(u_1)|^r + \frac{7+3\Upsilon}{2(2+\Upsilon)}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{3\Upsilon+7}{2(\Upsilon+2\sqrt{kl})}|\mathcal{U}'(u_1)|^r + \frac{\Upsilon+\sqrt{kl}}{2(\Upsilon+2\sqrt{kl})}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right\}. \quad (64)$$

Corollary 47. *By choosing $\Psi_0(k,l) = \frac{k+l}{2}$ in Theorem 8, we derive the following inequality involving the arithmetic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left|\frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{2\Upsilon}{k+l}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{2k\Upsilon}{k+l}\right)}{\frac{k+l}{2}(v_1-u_1)^{\frac{2\Upsilon}{k+l}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+,\frac{k+l}{2}}\mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-,\frac{k+l}{2}}\mathcal{U}(u_1)\right]\right|$$
$$\leq \frac{v_1-u_1}{2^{2+\frac{1}{r}}}\left(\frac{\Upsilon}{\Upsilon+\frac{k+l}{2}}\right)\left\{\left(\frac{1+\Upsilon}{2(2+\Upsilon)}|\mathcal{U}'(u_1)|^r + \frac{7+3\Upsilon}{2(2+\Upsilon)}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{7+3\Upsilon}{2\left(2+\Upsilon\cdot\frac{k+l}{2}\right)}|\mathcal{U}'(u_1)|^r + \frac{\Upsilon+\frac{k+l}{2}}{2\left(\Upsilon+2\cdot\frac{k+l}{2}\right)}|\mathcal{U}'(v_1)|^r\right)^{\frac{1}{r}}\right\}. \quad (65)$$

Corollary 48. *By choosing $\Psi_0(k,l) = \frac{k^2}{l}$ in Theorem 8, we derive the following inequality involving the harmonic form of (k,l)-Riemann-Liouville fractional integrals:*

$$\left| \frac{\mathcal{U}(u_1)+\mathcal{U}(v_1)}{2} - \frac{2^{\frac{\Upsilon l}{k^2}-1} k\Upsilon\Gamma_{(k,l)}\left(\frac{\Upsilon l}{k}\right)}{\frac{k^2}{l}(v_1-u_1)^{\frac{\Upsilon l}{k^2}}} \left[I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^+, \frac{k^2}{l}} \mathcal{U}(v_1) + I^{\Upsilon}_{\left(\frac{u_1+v_1}{2}\right)^-, \frac{k^2}{l}} \mathcal{U}(u_1) \right] \right|$$
$$\leq \frac{v_1-u_1}{2^{2+\frac{1}{r}}} \left(\frac{\Upsilon}{\Upsilon+\frac{k^2}{l}} \right) \left\{ \left(\frac{1+\Upsilon}{2(2+\Upsilon)} |\mathcal{U}'(u_1)|^r + \frac{7+3\Upsilon}{2(2+\Upsilon)} |\mathcal{U}'(v_1)|^r \right)^{\frac{1}{r}} \right.$$
$$\left. + \left(\frac{7+3\Upsilon}{2\left(2+\frac{\Upsilon k^2}{l}\right)} |\mathcal{U}'(u_1)|^r + \frac{\Upsilon+\frac{k^2}{l}}{2\left(\Upsilon+2\frac{k^2}{l}\right)} |\mathcal{U}'(v_1)|^r \right)^{\frac{1}{r}} \right\}. \tag{66}$$

5. Conclusion

In this study, we explored a variety of Hermite-Hadamard-type inequalities involving the (k,l)-Riemann-Liouville fractional integrals by utilizing the convexity properties of differentiable functions. Several key theorems were established, each extending the classical Hermite-Hadamard inequality into the fractional setting defined by the parameters k and l. Notably, our results include arithmetic, geometric, and harmonic forms of fractional inequalities, and they encompass several previously known results as special cases. These findings contribute to a broader understanding of how fractional operators interact with convexity, and they open new directions for further generalizations in the theory of inequalities.

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