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New refinements of the Wirtinger's inequality

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Abstract: In this work, two enhanced versions of Wirtinger's inequality are developed. These improvements arise when considering a weighted sum of multiple Wirtinger's inequalities. Depending on the context, one of the proposed refinements may be applicable than the other. Finally, a simple application of such refinements is presented.

Keywords: Wirtinger inequality, integral inequalities, weighted inequalities

MSC: 26D10, 26D15.

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1. Introduction

Suppose that $u(x)$ is a real function with a period of 2π. if $u(x) \in H^1(0, 2\pi)$, the classical form of the Wirtinger's inequality is written as follows [1, Chapter 2]:

$$\int_0^{2\pi} u(x)^2\,dx \le \int_0^{2\pi} u'(x)^2\,dx \quad \text{if } \left\{\int_0^{2\pi} u(x)\,dx = 0\right\}, \tag{1}$$

Two important generalizations of Inequality (1) existing in the literature are as follows [1, Chapter 2]:

$$\int_a^b u(x)^2\,dx \le \left(\frac{b-a}{2\pi}\right)^2 \int_a^b u'(x)^2\,dx \quad \text{if } \left\{u(a) = u(b) \wedge \int_a^b u(x)\,dx = 0\right\}, \tag{2}$$

$$\int_a^b u(x)^2\,dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b u'(x)^2\,dx \quad \text{if } \left\{u(a) = u(b) = 0 \vee \int_a^b u(x)\,dx = 0\right\}. \tag{3}$$

It is possible that the function $y(x)$ does not satisfy any of the conditions of Inequality (3). By substituting $u(x) = y(x) - \overline{y}$, Inequality (3) can be written in the following generalized form:

$$\int_a^b \left(y(x) - \overline{y}\right)^2\,dx \le c^2 \int_a^b y'(x)^2\,dx, \tag{4}$$

where $\overline{y} = \frac{1}{b-a}\int_a^b y(x)\,dx$ and $c = \left(\frac{b-a}{\pi}\right)$. Inequality (4) holds for every function $y(x)$ if only $y \in H^1(a,b)$. The refinement presented in this paper is based on Inequality (4).

The Wirtinger inequality has widespread applications in the stability analysis of delay differential equations. References [2–10] provide various applications of the Wirtinger's inequality. References [11–15] also present several generalizations and refinements of the Wirtinger inequality. It is important to note that a weighted sum of multiple Wirtinger inequalities can have various applications, ranging from stability analysis of systems to estimating a lower bound for the period of an orbit. The Lyapunov–Krasovskii functional, used in the stability analysis of systems, leads to the weighted sum of multiple Wirtinger inequalities [2]. In reference [16] (p. 33), it is proven that by summing multiple Wirtinger inequalities, one can estimate a lower bound for the period of the system.

If we consider Inequality (4) as a generalized form of the Wirtinger's inequality, then it is obvious that the weighted sum of n Wirtinger's inequalities for a set of n functions $\{y_i(x)\}_{i=1}^n$ can be written as follows:

$$\sum_{i=1}^n \frac{m_i}{M} \int_a^b \left(y_i(x) - \overline{y}_i\right)^2\,dx \le c^2 \sum_{i=1}^n \frac{m_i}{M} \int_a^b y_i'(x)^2\,dx, \tag{5}$$

where $\{m_i\}_{i=1}^n > 0$ and $M = \sum\limits_{i=1}^n m_i$. In this paper, it is proved that two refinements of Inequality (5) exist, as follows:

$$\sum_{i=1}^n \frac{m_i}{M} \int_a^b \left(y_i(x) - \overline{y}_i\right)^2 dx + \left\{ \begin{array}{c} q_1^2 \\ q_2^2 \end{array} \right\} \le c^2 \sum_{i=1}^n \frac{m_i}{M} \int_a^b y_i'(x)^2 \, dx, \tag{6}$$

where q_1^2 and q_2^2 depend on $\{y_i(x)\}_{i=1}^n$, $\{y_i'(x)\}_{i=1}^n$, and $\{m_i\}_{i=1}^n$. Finally, in §3, A simple example of applying these refinements is presented.

2. Main results

The primary outcome of this paper is presented in the next theorem.

Assumption: Assume $y_1, \cdots, y_n \in H^1(a,b)$ are real-valued functions on (a,b).

Theorem 1. *Consider a set of n Wirtinger inequalities applied to functions $\{y_i(x)\}_{i=1}^n$ as follows:*

$$\int_a^b \left(y_i(x) - \overline{y}_i\right)^2 dx \le c^2 \int_a^b y_i'(x)^2 \, dx, \qquad i = 1, \ldots, n, \tag{7}$$

Where $\overline{y}_i = \frac{1}{b-a} \int_a^b y_i(x) \, dx$ and $c = \left(\frac{b-a}{\pi}\right)$.

Then, there are two refined forms of the weighted sum of n Wirtinger inequalities, presented as follows:

$$\sum_{i=1}^n \frac{m_i}{M} \int_a^b \left(y_i(x) - \overline{y}_i\right)^2 dx + \left\{ \begin{array}{c} q_1^2 \\ q_2^2 \end{array} \right\} \le c^2 \sum_{i=1}^n \frac{m_i}{M} \int_a^b y_i'(x)^2 \, dx, \tag{8}$$

where $\{m_i\}_{i=1}^n > 0$, $M = \sum\limits_{i=1}^n m_i$ and

$$q_1^2 = c^2 \int_a^b y_c'(x)^2 \, dx - \int_a^b \left(y_c(x) - \overline{y_c}\right)^2 dx, \tag{9}$$

$$q_2^2 = \sum_{i=1}^n \frac{m_i}{M} \left(c^2 \int_a^b r_i'(x)^2 \, dx - \int_a^b \left(r_i(x) - \overline{r}_i\right)^2 dx \right), \tag{10}$$

$$y_c(x) = \frac{\sum\limits_{i=1}^n y_i(x) m_i}{M}, \tag{11}$$

$$y_c'(x) = \frac{\sum\limits_{i=1}^n y_i'(x) m_i}{M}, \tag{12}$$

$$r_i(x) = y_i(x) - y_c(x), \tag{13}$$

$$r_i'(x) = y_i'(x) - y_c'(x). \tag{14}$$

Please note that $\overline{r}_i = \frac{1}{b-a} \int_a^b r_i(x) \, dx$.

Proof. By substituting $\overline{y}_i = \frac{1}{b-a} \int_a^b y_i(x) \, dx$ into Inequality (7), left side of Inequality (7) is rewritten as follows:

$$\begin{aligned} \int_a^b \left(y_i(x) - \overline{y}_i\right)^2 dx &= \int_a^b y_i(x)^2 \, dx - 2\overline{y}_i \int_a^b y_i(x) \, dx + \overline{y}_i^2 \int_a^b dx \\ &= \int_a^b y_i(x)^2 \, dx - 2(b-a)\overline{y}_i^2 + (b-a)\overline{y}_i^2 \\ &= \int_a^b y_i(x)^2 \, dx - (b-a)\overline{y}_i^2 \\ &= \int_a^b y_i(x)^2 \, dx - \frac{1}{b-a} \left(\int_a^b y_i(x) \, dx \right)^2 . \end{aligned} \tag{15}$$

Therefore, the weighted sum of n Wirtinger inequalities can be rewritten as follows:

$$\sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b y_i(x)^2 \, dx - \frac{1}{b-a} \left(\int_a^b y_i(x) \, dx \right)^2 \right) \leq c^2 \sum_{i=1}^{n} \frac{m_i}{M} \int_a^b y_i'(x)^2 \, dx. \tag{16}$$

The proof of the Theorem consists of three steps: the left-hand side of Inequality (16) is derived in the first step, the right-hand side in the second, and finally, in the third step, the two expressions are subtracted.

Step 1: Derivation of the left-hand side

We assign S_1 to represent the left-hand side of Inequality (16), as follows:

$$S_1 = H_1 - H_2, \tag{17}$$

where

$$H_1 = \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b y_i(x)^2 \, dx \right), \tag{18}$$

$$H_2 = \frac{1}{b-a} \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b y_i(x) \, dx \right)^2. \tag{19}$$

First, we obtain the value of H_1. Substituting $y_i(x)$ from Eq. (13) into Eq. (18) yields:

$$\begin{aligned}
H_1 &= \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b (y_c(x) + r_i(x))^2 \, dx \right) \\
&= \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b \left(y_c(x)^2 + 2y_c(x)r_i(x) + r_i(x)^2 \right) dx \right) \\
&= \frac{1}{M} \sum_{i=1}^{n} m_i \int_a^b y_c(x)^2 \, dx + \frac{2}{M} \sum_{i=1}^{n} m_i \int_a^b y_c(x) r_i(x) \, dx + \frac{1}{M} \sum_{i=1}^{n} m_i \int_a^b r_i(x)^2 \, dx \\
&= \int_a^b y_c(x)^2 \, dx + \frac{2}{M} \int_a^b y_c(x) \left(\sum_{i=1}^{n} m_i r_i(x) \right) dx + \frac{1}{M} \sum_{i=1}^{n} m_i \int_a^b r_i(x)^2 \, dx.
\end{aligned} \tag{20}$$

From Eqs. (11) and (13), we have $\sum_{i=1}^{n} m_i r_i(x) = 0$. Therefore,

$$H_1 = \int_a^b y_c(x)^2 \, dx + \frac{1}{M} \sum_{i=1}^{n} m_i \int_a^b r_i(x)^2 \, dx. \tag{21}$$

Now, we obtain the value of H_2. Substituting $y_i(x)$ from Eq. (13) into Eq. (19) gives:

$$\begin{aligned}
H_2 &= \frac{1}{b-a} \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b (y_c(x) + r_i(x)) \, dx \right)^2 \\
&= \frac{1}{b-a} \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b y_c(x) \, dx \right)^2 + \frac{2}{b-a} \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b y_c(x) \, dx \right) \left(\int_a^b r_i(x) \, dx \right) \\
&\quad + \frac{1}{b-a} \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b r_i(x) \, dx \right)^2 \\
&= \frac{1}{b-a} \left(\int_a^b y_c(x) \, dx \right)^2 + \frac{2}{M(b-a)} \left(\int_a^b y_c(x) \, dx \right) \sum_{i=1}^{n} m_i \int_a^b r_i(x) \, dx \\
&\quad + \frac{1}{b-a} \sum_{i=1}^{n} \frac{m_i}{M} \left(\int_a^b r_i(x) \, dx \right)^2 \\
&= \frac{1}{b-a} \left(\int_a^b y_c(x) \, dx \right)^2 + \frac{2}{M(b-a)} \left(\int_a^b y_c(x) \, dx \right) \int_a^b \left(\sum_{i=1}^{n} m_i \right) r_i(x) \, dx
\end{aligned}$$

$$+\frac{1}{b-a}\sum_{i=1}^{n}\frac{m_i}{M}\left(\int_a^b r_i(x)\,dx\right)^2.$$

Similar to Eq. (20), we have $\sum\limits_{i=1}^{n} m_i r_i(x) = 0$. Therefore,

$$H_2 = \frac{1}{b-a}\left(\int_a^b y_c(x)\,dx\right)^2 + \frac{1}{b-a}\sum_{i=1}^{n}\frac{m_i}{M}\left(\int_a^b r_i(x)\,dx\right)^2. \tag{22}$$

Substituting Eq. (21) and (22) into Eq. (17) gives:

$$S_1 = \int_a^b y_c(x)^2\,dx - \frac{1}{b-a}\left(\int_a^b y_c(x)\,dx\right)^2 + \sum_{i=1}^{n}\frac{m_i}{M}\left(\int_a^b r_i(x)^2\,dx - \frac{1}{b-a}\left(\int_a^b r_i(x)\,dx\right)^2\right). \tag{23}$$

Step 2: Derivation of the right-hand side
We denote the right side of Inequality (16) by S_2, defined as follows:

$$S_2 = c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_i'(x)^2\,dx. \tag{24}$$

From Eq. (14), Eq. (24) can be rewritten as follows:

$$\begin{aligned}
S_2 &= c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_i'(x)^2\,dx\\
&= c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b \left(y_c'(x)+r_i'(x)\right)^2\,dx\\
&= c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_c'(x)^2\,dx + 2c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_c'(x)r_i'(x)\,dx + c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b r_i'(x)^2\,dx\\
&= c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_c'(x)^2\,dx + 2\frac{c^2}{M}\int_a^b\sum_{i=1}^{n} m_i y_c'(x)r_i'(x)\,dx + c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b r_i'(x)^2\,dx\\
&= c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_c'(x)^2\,dx + 2\frac{c^2}{M}\int_a^b y_c'(x)\sum_{i=1}^{n} m_i r_i'(x)\,dx + c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b r_i'(x)^2\,dx.
\end{aligned} \tag{25}$$

From Eq. (12) and (14), we have $\sum\limits_{i=1}^{n} m_i r_i'(x) = 0$. Therefore,

$$S_2 = c^2\int_a^b y_c'(x)^2\,dx + c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b r_i'(x)^2\,dx. \tag{26}$$

Step 3: Derivation of $S_2 - S_1$
From Eq. (26) and (23), we have:

$$\begin{aligned}
S_2 - S_1 = &\left(c^2\int_a^b y_c'(x)^2\,dx - \int_a^b y_c(x)^2\,dx + \frac{1}{b-a}\left(\int_a^b y_c(x)\,dx\right)^2\right)\\
&+\sum_{i=1}^{n}\frac{m_i}{M}\left(c^2\int_a^b r_i'(x)^2\,dx - \int_a^b r_i(x)^2\,dx + \frac{1}{b-a}\left(\int_a^b r_i(x)\,dx\right)^2\right).
\end{aligned} \tag{27}$$

From Eq. (15), Eq. (27) can be rewritten as follows:

$$S_2 - S_1 = \left(c^2\int_a^b y_c'(x)^2\,dx - \int_a^b \left(y_c(x) - \overline{y_c}\right)^2\,dx\right)$$

$$+\sum_{i=1}^{n}\frac{m_i}{M}\left(c^2\int_a^b r_i'(x)^2\,dx-\int_a^b\left(r_i(x)-\overline{r}_i\right)^2\,dx\right). \tag{28}$$

According to the Wirtinger inequality in Inequality (4), the value of $S_2 - S_1$ is equal to the sum of two positive terms q_1^2 and q_2^2. That is:

$$S_2-S_1=c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_i'(x)^2\,dx-\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b\left(y_i(x)-\overline{y}_i\right)^2\,dx=q_1^2+q_2^2, \tag{29}$$

where

$$q_1^2=\left(c^2\int_a^b y_c'(x)^2\,dx-\int_a^b\left(y_c(x)-\overline{y_c}\right)^2\,dx\right),$$

$$q_2^2=\sum_{i=1}^{n}\frac{m_i}{M}\left(c^2\int_a^b r_i'(x)^2\,dx-\int_a^b\left(r_i(x)-\overline{r}_i\right)^2\,dx\right).$$

Therefore, two refinements for the weighted sum of n Wirtinger inequalities in the form presented in Inequality (7) can be written as follows:

$$S_2-S_1=c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_i'(x)^2\,dx-\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b\left(y_i(x)-\overline{y}_i\right)^2\,dx\geq q_1^2, \tag{30}$$

or

$$S_2-S_1=c^2\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b y_i'(x)^2\,dx-\sum_{i=1}^{n}\frac{m_i}{M}\int_a^b\left(y_i(x)-\overline{y}_i\right)^2\,dx\geq q_2^2. \tag{31}$$

The proof is complete. ☐

3. Application of the refinements

In this section, a simple example demonstrating the application of the proven refinements is presented.

Example 1. Consider the following nonlinear system:

$$\dot{x}_1=f\left(x_1,x_2\right), \tag{32-a}$$

$$\dot{x}_2=-f\left(x_1,x_2\right)+i(t), \tag{32-b}$$

where $f : R^2 \to R$ is of class $C^1\left(R^2\right)$, $i(t)$ represents the input of the system, and the initial conditions are $x_1(0) = 0$ and $x_2(0) = 0$. Suppose that the energy of the system is defined as $E = \dot{x}_1^2 + \dot{x}_2^2$. Our aim is to find a lower bound for the time-averaged energy of the system that is independent of the system dynamics and depends only on the input $i(t)$.

First, the Wirtinger inequalities are written for $x_1(t)$ and $x_2(t)$. From Inequality (7), we have:

$$\int_0^T\left(x_1(t)-\overline{x}_1\right)^2\,dt\leq c^2\int_0^T\dot{x}_1(t)^2\,dt, \tag{33}$$

$$\int_0^T\left(x_2(t)-\overline{x}_2\right)^2\,dt\leq c^2\int_0^T\dot{x}_2(t)^2\,dt. \tag{34}$$

Therefore, a weighted sum of Inequalities (33) and (34) can be written as follows:

$$\frac{1}{2}\int_0^T\left(x_1(t)-\overline{x}_1\right)^2dt+\frac{1}{2}\int_0^T\left(x_2(t)-\overline{x}_2\right)^2dt\leq\frac{c^2}{2}\int_0^T\left(\dot{x}_1(t)^2+\dot{x}_2(t)^2\right)dt=\frac{c^2}{2}\int_0^T E\,dt. \tag{35}$$

Based on the theorem, Inequality (35) can be refined as follows:

$$q^2+\frac{1}{2}\int_0^T\left(x_1(t)-\overline{x}_1\right)^2dt+\frac{1}{2}\int_0^T\left(x_2(t)-\overline{x}_2\right)^2dt\leq\frac{c^2}{2}\int_0^T E\,dt. \tag{36}$$

From Eq. (9), we have:

$$q^2 = c^2 \int_0^T \dot{x}_c(t)^2 dt - \int_0^T (x_c(t) - \overline{x}_c)^2 \, dt. \tag{37}$$

Inequalities (33) and (34) are each multiplied by $\frac{1}{2}$ and then added together. Therefore, from Eq. (11) and (12), we have:

$$x_c(t) = \frac{x_1(t) + x_2(t)}{2}, \tag{38}$$

$$\dot{x}_c(t) = \frac{\dot{x}_1(t) + \dot{x}_2(t)}{2}, \tag{39}$$

From Eqs. (32-a) and (32-b), we have:

$$\dot{x}_1(t) + \dot{x}_2(t) = i(t), \tag{40}$$

and

$$x_1(t) + x_2(t) = \int_0^t i(s) \, ds. \tag{41}$$

Substituting Eqs. (40) and (41) into Eqs. (38) and (39) gives:

$$x_c(t) = \frac{1}{2} \int_0^t i(s) \, ds, \tag{42}$$

$$\dot{x}_c(t) = \frac{1}{2} i(t), \tag{43}$$

and

$$\overline{x}_c = \frac{1}{2T} \int_0^T \int_0^t i(s) \, ds \, dt. \tag{44}$$

From Eq. (36), we can conclude that

$$\frac{2q^2}{Tc^2} \leq \frac{1}{T} \int_0^T E \, dt = \overline{E}, \tag{45}$$

where q^2 is calculated from Eq. (37), and the functions $x_c(t)$ and $\dot{x}_c(t)$ can be obtained from Eqs. (42) and (43). Therefore, according to Inequality (45), the lower bound of the system's average energy depends solely on the input function $i(t)$.

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