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On a Hardy-Hilbert-type integral inequality involving an adjustable function

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Abstract: The Hardy-Hilbert integral inequality has inspired a vast body of research over the past few decades, resulting in the creation of numerous new forms and generalizations of integral inequalities. In this article, we build on this line of research by introducing a new class of Hardy-Hilbert-type integral inequalities incorporating an adjustable function. This additional flexibility enables our results to bridge the gap naturally between classical cases and a variety of new ones. We provide several distinct examples to illustrate the applicability and sharpness of the derived inequalities. Additionally, we present a supplementary result that extends the main theorem, supported by concrete examples that demonstrate its validity and scope.

Keywords: Hardy-Hilbert integral inequality, Hölder integral inequality, sharp constants

MSC: 26D15, 33E20.

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1. Introduction

Integral inequalities are a fundamental tool in mathematical analysis. They provide a link between the local properties of a function, such as convexity, monotonicity and smoothness, and its global properties, which are expressed through integrals. Well-known examples include the Cauchy-Schwarz integral inequality, which establishes bounds on inner products and integrals of products of functions [1], and the Hölder integral inequality, which extends this concept to L^p spaces [2]. The Jensen integral inequality connects convexity with integration [3], demonstrating the relationship between the integral of a convex function and the function evaluated at the integral. The Grönwall integral inequality provides essential bounds for solutions of differential equations and is indispensable in stability theory [4]. The Poincaré integral inequality relates the norm of a function to the norm of its derivatives on bounded domains [4]. This plays an important role in the theory of Sobolev spaces and partial differential equations.

In the late 19th century, the German mathematician and philosopher David Hilbert discovered the celebrated Hilbert integral inequality. This inequality was later extended by Godfrey H. Hardy, as formulated below. Let $f, g : [0,\infty) \to [0,\infty)$ be Lebesgue measurable functions with $f \in L^p([0,\infty))$, i.e., $\int_0^\infty (f(x))^p dx < \infty$, and $g \in L^q([0,\infty))$. Suppose $p, q > 1$ are Hölder's conjugate exponents, i.e., $1/p + 1/q = 1$. Then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \left(\int_0^\infty (g(x))^q dx \right)^{1/q}.$$

Furthermore, the constant $\pi/\sin(\pi/p)$ is the smallest possible constant for which the inequality holds for all admissible f and g. This result is one of the cornerstones of real and harmonic analysis. Further theory and development of this inequality can be found in [5].

Due to the wide interest of this inequality, several researchers have tried to generalize this identity in many different ways. A global view of these developments can be found in the survey [6] and in the book [7]. Recent contributions include [8–23].

In this article, we focus on a new Hardy-Hilbert-type integral inequality centered around the following double integral:

$$\int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)g(y)dxdy,$$

where $f,g:[0,\infty)\to[0,\infty)$ are Lebesgue measurable functions with $f\in L^p([0,\infty))$ and $g\in L^q([0,\infty))$, and $\varphi:[-1,1]\to[0,\infty)$ is an adjustable Lebesgue measurable function.

The interest of this formulation lies in its flexibility and unifying nature: by suitable choices of the auxiliary function φ, one can recover a wide variety of known integral inequalities as special cases, as well as derive new ones with additional structural properties such as power-type singularities. Moreover, the dependence of the integrand on the normalized difference $|x-y|/(x+y)$ makes the inequality naturally invariant under positive dilations, which is an essential feature in the theory of Hardy–Hilbert-type operators.

The rest of the article is organized as follows: Some preliminaries are given in §2. In §3, we establish a Hardy-Hilbert-type integral inequality based on this integral. Examples are also given. Another integral inequality result is demonstrated in §4, illustrated by several examples. §5 ends with a conclusion.

2. Preliminaries and notations

For convenience of reference, we recall the definitions of some well-known special functions. The gamma function (or Euler's second integral) is defined by the formula

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,$$

for any $x>0$.

The beta function is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

for any $x,y>0$. Note that $\Gamma(n)=(n-1)!$ for any positive integer n.

3. Main theorem

In this section, we propose a generalization of the Hardy-Hilbert integral inequality by introducing an adjustable function in the integrand. It is stated below.

Theorem 1. *Let $f,g:[0,\infty)\to[0,\infty)$ be Lebesgue measurable functions with $f\in L^p([0,\infty))$ and $g\in L^q([0,\infty))$. Let $\varphi:[-1,1]\to[0,\infty)$ be a Lebesgue measurable function such that*

$$\Omega_p(\varphi):=\int_{-1}^1 \frac{\varphi(|x|)}{(1+x)^{1/p}(1-x)^{1/q}}dx, \tag{1}$$

is finite, where $p,q>1$ are Hölder's conjugate exponents. Then we have

$$\int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)g(y)dxdy \le \Omega_p(\varphi)\left(\int_0^\infty (f(x))^p dx\right)^{1/p}\left(\int_0^\infty (g(x))^q dx\right)^{1/q}.$$

Proof. For convenience of reference, we set

$$\mathcal{I}:=\int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)g(y)dxdy.$$

Next, we consider the change of variables $u = x + y$ and $v = (x - y)/(x + y)$ where $x, y \in (0, \infty)$. Then we have the bounds $u \in (0, \infty)$ and $v \in (-1, 1)$. Moreover, we see that $x = u(1 + v)/2$, $y = u(1 - v)/2$, and

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = -\begin{vmatrix} \frac{1+v}{2} & \frac{u}{2} \\ \frac{1-v}{2} & -\frac{u}{2} \end{vmatrix} = \frac{u}{2}.$$

So we have

$$\begin{aligned} \mathcal{I} &= \int_{-1}^{1}\int_{0}^{\infty} \varphi(|v|)\frac{1}{u} f\left(\frac{u(1+v)}{2}\right) g\left(\frac{u(1-v)}{2}\right)\frac{u}{2} du dv \\ &= \frac{1}{2}\int_{-1}^{1} \varphi(|v|)\left(\int_{0}^{\infty} f\left(\frac{u(1+v)}{2}\right) g\left(\frac{u(1-v)}{2}\right) du\right) dv, \end{aligned} \tag{2}$$

where we apply the Fubini-Tonelli integral theorem in the second equality. Now, for any fixed $v \in (-1, 1)$, the Hölder integral inequality gives

$$\begin{aligned} \mathcal{J} &:= \int_{0}^{\infty} f\left(\frac{u(1+v)}{2}\right) g\left(\frac{u(1-v)}{2}\right) du \\ &\leq \left(\int_{0}^{\infty} \left(f\left(\frac{u(1+v)}{2}\right)\right)^{p} du\right)^{1/p} \left(\int_{0}^{\infty} \left(g\left(\frac{u(1-v)}{2}\right)\right)^{q} du\right)^{1/q}. \end{aligned}$$

We consider each integral that appears in the upper bound of $\mathcal{J}$ separately.

- First, making the change of variables $x = u(1 + v)/2$, we get

$$\int_{0}^{\infty} \left(f\left(\frac{u(1+v)}{2}\right)\right)^{p} du = \frac{2}{1+v}\int_{0}^{\infty} (f(x))^{p} dx.$$

- Second, making the change of variables $x = u(1 - v)/2$, we obtain

$$\int_{0}^{\infty} \left(g\left(\frac{u(1-v)}{2}\right)\right)^{q} du = \frac{2}{1-v}\int_{0}^{\infty} (g(x))^{q} dx.$$

Hence we have

$$\begin{aligned} \mathcal{J} &\leq \left(\frac{2}{1+v}\int_{0}^{\infty} (f(x))^{p} dx\right)^{1/p} \left(\frac{2}{1-v}\int_{0}^{\infty} (g(x))^{q} dx\right)^{1/q} \\ &= \frac{2}{(1+v)^{1/p}(1-v)^{1/q}} \left(\int_{0}^{\infty} (f(x))^{p} dx\right)^{1/p} \left(\int_{0}^{\infty} (g(x))^{q} dx\right)^{1/q}. \end{aligned}$$

Substituting this into Eq. (2) gives

$$\begin{aligned} \mathcal{I} &\leq \frac{1}{2}\int_{-1}^{1} \frac{2\varphi(|v|)}{(1+v)^{1/p}(1-v)^{1/q}} \left(\int_{0}^{\infty} (f(x))^{p} dx\right)^{1/p} \left(\int_{0}^{\infty} (g(x))^{q} dx\right)^{1/q} \\ &= \left(\int_{-1}^{1} \frac{\varphi(|v|)}{(1+v)^{1/p}(1-v)^{1/q}} dv\right) \left(\int_{0}^{\infty} (f(x))^{p} dx\right)^{1/p} \left(\int_{0}^{\infty} (g(x))^{q} dx\right)^{1/q} \\ &= \Omega_{p}(\varphi) \left(\int_{0}^{\infty} (f(x))^{p} dx\right)^{1/p} \left(\int_{0}^{\infty} (g(x))^{q} dx\right)^{1/q}. \end{aligned}$$

This completes the proof. □

Remark 1. An alternative point of view is to remark that the kernel function

$$K(x,y) := \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right),$$

is homogeneous of degree -1. This observation allows one to invoke general results, such as those presented in [6]. However, for clarity of exposition and to give a fresh perspective on the constant factor involved, we choose to provide a complete, self-contained proof. The explicit definition of this constant will, in turn, motivate both classical and new integral inequalities, as described below.

Remark 2. If $\varphi \equiv 1$ in Theorem 1, then we have

$$\Omega_p(1) = \int_{-1}^{1} \frac{1}{(1+x)^{1/p}(1-x)^{(p-1)/p}} dx.$$

Making the change of variables $y = (1+x)/2$ gives $dy = (1/2)dx$ and

$$\begin{aligned}\Omega_p(1) &= \int_0^1 \frac{2}{(2y)^{1/p}(2(1-y))^{(p-1)/p}} dy = \int_0^1 y^{-1/p}(1-y)^{-(p-1)/p} dy \\ &= \int_0^1 y^{(1-1/p)-1}(1-y)^{1/p-1} dy = B\left(1-\frac{1}{p}, \frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)},\end{aligned}$$

which is well-defined since $1-1/p$ and $1/p$ are positive. Note that the last equality is from [24]. Thus Theorem 1 can be restated as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty (f(x))^p dx\right)^{1/p} \left(\int_0^\infty (g(x))^q dx\right)^{1/q},$$

and so this reduces to the classical Hardy-Hilbert integral inequality.

Let us see some applications of this result.

Example 1. By taking $\varphi(x) = x^m$ where $m \geq 0$, and $p = q = 2$, and using the substitution $z = x^2$, we see that

$$\begin{aligned}\Omega_p(\varphi) &= 2\int_0^1 x^m(1-x^2)^{-1/2} dx = 2\int_0^1 \frac{z^{m/2}(1-z)^{-1/2}}{2z^{1/2}} dz \\ &= \int_0^1 z^{(m-1)/2+1-1}(1-z)^{1/2-1} dz = B\left(\frac{m-1}{2}+1, \frac{1}{2}\right).\end{aligned}$$

Thus Theorem 1 gives

$$\int_0^\infty \int_0^\infty \frac{|x-y|^m}{(x+y)^{m+1}} f(x)g(y) dxdy \leq B\left(\frac{m-1}{2}+1, \frac{1}{2}\right) \left(\int_0^\infty (f(x))^2 dx\right)^{1/2} \left(\int_0^\infty (g(x))^2 dx\right)^{1/2}. \tag{3}$$

For further instance, given any positive integer n, using a standard geometric sum formula, we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{1}{\min\{x,y\}} \left(1-\left(\frac{|x-y|}{x+y}\right)^n\right) f(x)g(y) dxdy \\ &= 2\int_0^\infty \int_0^\infty \frac{1}{x+y} \cdot \frac{1}{1-|x-y|/(x+y)} \left(1-\left(\frac{|x-y|}{x+y}\right)^n\right) f(x)g(y) dxdy \\ &= 2\int_0^\infty \int_0^\infty \frac{1}{x+y} \left(\sum_{m=0}^{n-1} \left(\frac{|x-y|}{x+y}\right)^m\right) f(x)g(y) dxdy \\ &= 2\sum_{m=0}^{n-1} \int_0^\infty \int_0^\infty \frac{1}{x+y} \left(\frac{|x-y|}{x+y}\right)^m f(x)g(y) dxdy \\ &\leq 2\left(\sum_{m=0}^{n-1} B\left(\frac{m-1}{2}+1, \frac{1}{2}\right)\right) \left(\int_0^\infty (f(x))^2 dx\right)^{1/2} \left(\int_0^\infty (g(x))^2 dx\right)^{1/2}.\end{aligned}$$

In the next section, we demonstrate the flexibility of Theorem 1 by deriving another integral inequality using the same elements.

4. Another result

The theorem below presents a new integral inequality that depends on one adjustable function, as a direct implication of Theorem 1.

Theorem 2. *Let $f : [0,\infty) \to [0,\infty)$ be a Lebesgue measurable function with $f \in L^p([0,\infty))$, and let $\varphi : [-1,1] \to [0,\infty)$ be a Lebesgue measurable function such that*

$$\Omega_p(\varphi) := \int_{-1}^{1} \frac{\varphi(|x|)}{(1+x)^{1/p}(1-x)^{1-1/p}}dx,$$

is finite, where $p > 1$. Then we have

$$\int_0^\infty \left(\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)dx \right)^p dy \leq \Omega_p(\varphi)^p \int_0^\infty (f(x))^p dx.$$

Proof. Firstly, let us set

$$\mathcal{K} := \int_0^\infty \left(\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)dx \right)^p dy.$$

By the Fubini-Tonelli integral theorem, we can write

$$\begin{aligned} \mathcal{K} &= \int_0^\infty \left(\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)dx \right)^{p-1} \left(\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)dx \right) dy \\ &= \int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)\tilde{g}(y)dxdy, \end{aligned} \tag{4}$$

where

$$\tilde{g}(y) = \left(\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)dx \right)^{p-1}.$$

Applying Theorem 1 to f and $\tilde{g}$, we get

$$\int_0^\infty \int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)\tilde{g}(y)dxdy \leq \Omega_p(\varphi) \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \left(\int_0^\infty (\tilde{g}(x))^q dx \right)^{1/q}, \tag{5}$$

with $q = p/(p-1)$. Then we have

$$\begin{aligned} \int_0^\infty (\tilde{g}(x))^q dx &= \int_0^\infty \left(\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)dx \right)^{q(p-1)} dx \\ &= \int_0^\infty \left(\int_0^\infty \frac{1}{x+y}\varphi\left(\frac{|x-y|}{x+y}\right) f(x)dx \right)^{p} dx = \mathcal{K}. \end{aligned}$$

Combining this with Eqs. (4) and (5), we get

$$\mathcal{K} \leq \Omega_p(\varphi) \left(\int_0^\infty (f(x))^p dx \right)^{1/p} \mathcal{K}^{1/q},$$

which implies that

$$\mathcal{K}^{1/p} \leq \Omega_p(\varphi) \left(\int_0^\infty (f(x))^p dx \right)^{1/p}.$$

Lastly, by raising the inequality above to the power of p, the proof is complete. □

Example 2. By taking $\varphi(x) = x^m$ where $m \geq 0$, and $p = q = 2$, Theorem 2 gives

$$\int_0^\infty \left(\int_0^\infty \frac{1}{x+y} \left(\frac{|x-y|}{x+y} \right)^m f(x)dx \right)^p dy \leq \left(B\left(\frac{m-1}{2} + 1, \frac{1}{2} \right) \right)^p \int_0^\infty (f(x))^p dx.$$

5. Discussion and conclusion

In conclusion, we have established a new Hardy-Hilbert-type integral inequality involving an adjustable function, φ, unifying and extending several classical results. The obtained framework highlights the structural complexity of these inequalities and their dilation invariance. Future work will focus on exploring multidimensional extensions, weighted versions and potential applications to differential and functional equations.

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