The Pečarić jornal of
Mathematical Inequalities

PSR Press

Article

On m-convex functions of the second type

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Received: 05 February 2026; Accepted: 28 April 2026; Published: 17 June 2026

Abstract: In this study, we give a new m-convex function that is called an m-convex of the second type and its some properties. Moreover, some integral inequalities are examined for each m-convex function of the second type.

Keywords: m-convex function, integral inequalities

MSC: 26D15, 26D10.

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1. Introduction and preliminaries

The concept of a convex function is equivalent to the set of all points forming the convex function's graph being convex [1]. Thus, problems involving convex functions are associated with convex sets. As a result of a thorough understanding of convex geometry, many mathematicians are better able to address problems from a geometric perspective.

Convex functions can occasionally be found in proof methods given in various math- ematics topics. For example, Kittaneh used a simple property of convex functions to provide a better lower bound on the numerical radius of bounded linear operators. Con- sequently, the theory of convex functions remains a popular topic. Various classes of convex functions are studied by many mathematicians. One of these is the class of m- convex functions given by Toader, whose definition is given below [2].

A function $f : [0,b] \to \mathbb{R}$ is called an m-convex function, where $m \in (0,1]$ and $b > 0$, if for all $x, y \in [0,b]$ and $t \in [0,1]$ we have

$$f\left(tx + m(1-t)y\right) \leq tf(x) + m(1-t)f(y).$$

If $m = 1$, then it is a convex function. If $f(tx) \leq tf(x)$, then it is a star-shaped function.

The following relations are true for a non-negative continuous function which vanishes at the origin [3] Convex functions $\subset$ Star - shaped functions and also m-Convex functions $\subset$ Star - shaped. Integral inequalities of m-convex functions have also been studied by some mathe- maticians [4–7]. Furthermore, inequalities satisfied by functions defined on linear operators can also be found in the literature [8,9].

On the other hand, Takahashi introduced the concept of convexity at a metric distance where the linear vector space conditions are not satisfied [10]. Fixed point theorems on these convex abstract metric spaces have become a subject of considerable study [11–14]. Moreover, the definition of m-convexity in b-metric spaces is defined by taking inspiration from m-convex functions in [15]. However, this definition corresponds to a very special case of convex metric spaces [16].

In this article, we adapt the definition given by Sertbaş et al. to functions [16]. First, we give some properties of these functions. Then, we calculate some basic integral in- equalities.

2. m-convex functions of the second type

Definition 1. $f : [0,b] \to \mathbb{R}$, $b > 0$ is called to be an m- convex of the second type for some $m \in (0,1]$ if

$$f\left(tx + (1-t)my\right) \leq tf(x) + (1-mt)f(y),$$

for any $x, y \in [0,b]$ and $t \in [0,1]$.

https://pisrt.org/psr-press/journals/pjmi
Pečarić J. Math. Ineq. **2026**, 2, 113-117; doi:10.30538/pjmi2026.0025

It is obvious that a non negative m-convex function and non negative continuous convex function which vanishes at the origin are an m-convex of the second type.

Theorem 1. *If a function $f : [0,b] \to \mathbb{R}$, $b > 0$ is a second type m-convex with $m \in (0,1)$, then the following statements are satisfied;*

i) *f is a non negative function,*

ii) *The sequence $\{f(m^n x)\}$, $n \in \mathbb{N}$ converges for all $x \in [0,b]$ and*

$$\lim_{n\to+\infty} f(m^n x) \le \min\{f(0), f(x)\},$$

is satisfied.

iii) *f is continuous at zero if and only if $f(0) \le f(x)$ for each $x \in [0,b]$.*

Proof. In this case, for any $x, y \in [0,b]$

$$f(y) = f(y + 0mx) \le f(y) + (1-m)f(x),$$

and then $f(x) \ge 0$ for each $x \in [0,b]$. On the other hand, for any $x \in [0,b]$

$$f(mx) = f(0 + (1-0)mx) \le f(x),$$

is true. Therefore,

$$0 \le f(m^n x) \le f(m^{n-1}x) \le \cdots \le f(mx) \le f(x),$$

is holds for $n \in \mathbb{N}$. It means that the sequence $\{f(m^n x)\}$ is monotone deceasing and bounded and so it converges. Also, for $n \in \mathbb{N}$

$$f(m^n x) = f(m^n x + (1-m^n)0) \le m^n f(x) + (1-m^{n+1})f(0),$$

is correct. From these results

$$\lim_{n\to+\infty} f(m^n x) \le \min\{f(0), f(x)\},$$

is obtained. Moreover, if f is continuous at zero, then from the last limit the inequality $f(0) \le f(x)$ is get for each $x \in [0,b]$.

On the contrary, suppose that the inequality $f(0) \le f(x)$ is true for each $x \in [0,b]$. In this case, for any $x \in [0,b]$ there exist an element $t \in [0,1]$ such that $x = tb$ and

$$0 \le f(x) - f(0) = f(tb) - f(0) \le tf(b) + (1-mt)f(0) - f(0) = t[f(b) - mf(0)], \tag{1}$$

is correct and so f continuous at zero. □

Corollary 1. *If a function $f : [0,b] \to \mathbb{R}$, $b > 0$ is an m-convex of the second type with $m \in (0,1)$ and continuous at zero, then*

$$0 \le \underline{\lim}_{x\to 0} \frac{f(x) - f(0)}{x} \le \overline{\lim}_{x\to 0} \frac{f(x) - f(0)}{x} \le \frac{f(b) - mf(0)}{b}.$$

Also, if $f(x)$ is vanish at origin and continuous on $[0,b]$, then it is differentiable at $x = 0$.

Proof. According to previous theorem and the inequality (1) the first assertion can be proved.

If $f(0) = 0$, then it is a star shaped function. Therefore, it is obtained the second assertion by using Theorem 6 in [3]. □

Theorem 2. *Assume that a function $f : [0,b] \to \mathbb{R}$, $b > 0$ is an m-convex of the second type with $m \in (0,1)$, then it is bounded on $[0,b]$ and following statements are hold*

i) *f is a bounded function on $[0,b]$,*

ii) *For any $x \in [0,b]$, $\overline{\lim}_{y\to x} f(my) \le f(x)$.*

Proof. For any $x \in [0,b]$ there exist a unique element $t \in [0,1]$ such that $x = tb$ and

$$0 \leq f(x) = f(tb) \leq tf(b) + (1-mt)f(0) \leq f(b) + f(0),$$

this means that f is a bounded function on $[0,b]$.

On the other hand, δ is a positive or negative number and small enough such that $x + n\delta \in [0,b]$, $n \in \mathbb{N}$. In this case,

$$f(m(x+\delta)) = f\left(\frac{m(x+n\delta)}{n} + \left(1-\frac{1}{n}\right)mx\right) \leq \frac{1}{n} f(m(x+n\delta)) + \left(1-\frac{m}{n}\right)f(x).$$

If $\delta \to 0$, then it must be $n \to +\infty$. Since f is a bounded function,

$$\overline{\lim}_{y\to x} f(my) \leq f(x),$$

is get. This idea is in [17]. □

3. Some inequalities

Theorem 3. *Let $f : [0,b] \to \mathbb{R}$, $b > 0$ be an m-convex of the second type and Lebesgue integrable on $[ma,c]$, $0 \leq a < c \leq b$. Then*

$$f\left(m\frac{a+c}{2}\right) \leq \frac{3-m}{2(c-a)}\int_a^c f(x)\,dx \leq (3-m)\left(\frac{c-ma}{c-a}\right)\frac{f(c)+(2-m)f(a)}{4}.$$

Proof. Because f is an m-convex of the second type a, for all $x, y \in [a,b]$ we have

$$f\left(m\frac{x+y}{2}\right) \leq \frac{f(x)+(2-m)f(y)}{2}.$$

If $x = ta + (1-t)c$ and $y = tc + (1-t)a$ are chosen, then we have

$$f\left(m\frac{a+c}{2}\right) \leq \frac{(f(ta+(1-t)c) + (2-m)f(tc+(1-t)a))}{2}.$$

We obtain by integrating the last inequality

$$f\left(m\frac{a+c}{2}\right) \leq \frac{3-m}{2(c-a)}\int_a^c f(x)\,dx.$$

On the other hand,

$$\int_a^c f(y)\,dy \leq \int_{ma}^c f(y)\,dy \leq (c-ma)\frac{f(c)+(2-m)f(a)}{2}.$$

This is completed the proof of theorem. □

Theorem 4. *Suppose that $f : [0,b] \to \mathbb{R}$, $b > 0$ is a function and f' is an m-convex of the second type on $[0,mb]$, then for $0 \leq a < b$*

$$\frac{f(mb)+(2-m)f(ma)}{3-m} - \frac{1}{(b-a)}\int_a^b f(mx)\,dx \leq \frac{m(b-a)}{3-m}\,\frac{(2-m)f'(b)+f'(a)}{4}.$$

Proof. Because f' is bounded on $[0,mb]$ and the points of discontinuity of a derivative function are only of the second type, f' must be continuous on $[0,mb]$. In this case, we use the idea given in [18], and so

$$\int_0^1 (1-(3-m)t)f'(tma+(1-t)mb)\,dt = \frac{(2-m)f(ma)+f(mb)}{m(b-a)} - \frac{3-m}{m(b-a)^2}\int_a^b f(mx)\,dx,$$

is get. Because f' is an m-convex of the second type on $[0, mb]$,

$$\begin{aligned}\int_0^1 (1-(3-m)t)f'(tma+(1-t)mb)\,dt &\leq \int_0^1 (1-2t)f'(tma+(1-t)mb)\\ \leq \int_0^1 |1-2t|f'(tma+(1-t)mb)\,dt &\leq \int_0^1 |1-2t|\big(tf'(a)+(1-mt)f'(b)\big)\,dt\\ = \frac{(2-m)f'(b)+f'(a)}{4}.\end{aligned}$$

From this inequality,

$$\frac{f(mb)+(2-m)f(ma)}{3-m} - \frac{1}{(b-a)}\int_a^b f(mx)\,dx \leq \frac{m(b-a)}{3-m}\frac{(2-m)f'(b)+f'(a)}{4}.$$

□

Corollary 2. *Under the previous theorem, the following inequality is obtained*

$$f(ma) - \frac{1}{(b-a)}\int_a^b f(mx)\,dx < (b-a)\frac{(2-m)f'(b)+f'(a)}{8},$$

Proof. Since f' is a non negative continuous function on $[0, mb]$, $f(x) = f(0) + \int_0^x f'(\tau)d\tau$ is a monotone increasing function. Also, $\frac{m}{3-m} < \frac{1}{2}$ is true for $m < 1$. In this case, the desired result from the previous theorem is obtained. □

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