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Scaled block mean Jensen estimates and superquadratic refinements for m -convex functions via positive linear functionals

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Received: 13 June 2026; Accepted: 06 September 2026; Published: 22 September 2026

Abstract: We establish scaled block-mean Jensen-type estimates for m -convex functions in the framework of positive linear functionals. The main observation is that, after a positive functional is decomposed into two nontrivial blocks, the ordinary block identity $\mu = \alpha\mu_1 + \beta\mu_2$, must be written in the m -convex geometry as $\mu = \alpha\mu_1 + m\beta\left(\frac{\mu_2}{m}\right)$. Thus the complementary block mean is evaluated at the scaled point μ_2/m , not at μ_2 . This yields a valid two-block Jensen-type estimate under explicit domain and admissibility assumptions; a simple example shows that the unscaled analogue may fail for $m < 1$. We derive modulus-controlled corrections, approximate stability estimates, superquadratic lower bounds for the scaled block gap, and recursive versions obtained from nested block decompositions. Weighted sums, ordinary integrals, time-scale integrals, Jackson q -integrals, and positive fractional-kernel operators are obtained as direct realizations of the abstract positive-functional result.

Keywords: m -convex functions, Jensen-type inequality, positive linear functional, fractional-kernel integral, superquadratic function, time scales

MSC: 26A51, 26D15.

1. Introduction

Convexity inequalities are basic tools in analysis, optimization and applied mathematics; standard references include [1,2]. Jensen's inequality is central in this theory, and its weighted, integral and functional variants have been developed in many directions [3–5]. A convenient way to treat several of these variants simultaneously is to work with positive linear functionals, because weighted sums, expectation operators, ordinary integrals, time-scale integrals and kernel-type integrals can then be handled by the same formal argument [6–9].

For $0 < m \leq 1$, the Dragomir–Toader class of m -convex functions is governed by the asymmetric combination

$$tx + m(1-t)y, \quad 0 \leq t \leq 1,$$

which reduces to the ordinary convex combination when $m = 1$ [10]. Jensen-type inequalities for m -convex functions and related generalized convexities have attracted renewed interest, particularly in connection with fractional integral operators and generalized kernels [11,12]. Recent work also emphasizes refinements, stability estimates and fractional extensions of Jensen-type inequalities; see, for example, [13–18].

The present paper isolates a simple but important scaling phenomenon that appears when a positive functional is split into blocks in the setting of m -convexity. Suppose a positive functional is decomposed into two nontrivial blocks and let

$$\mu = \alpha\mu_1 + \beta\mu_2, \quad \alpha + \beta = 1,$$

be the ordinary identity relating the global mean to the two block means. For ordinary convexity, this is the natural two-block Jensen representation. For m -convexity, however, the same identity must be read as

$$\mu = \alpha\mu_1 + m\beta\left(\frac{\mu_2}{m}\right).$$

Consequently, the second block is evaluated at the scaled point μ_2/m . This is not a cosmetic rewriting: Example 2 shows that replacing μ_2/m by μ_2 can lead to a false inequality when $m < 1$.

The novelty of the paper is the positive-functional formulation of this scaled block mechanism. The main theorem gives the admissible two-block estimate for m -convex functions. We then derive a modulus-controlled version, an approximate stability version, and a superquadratic refinement based on the Jensen improvement of Abramovich, Jameson and Sinnamon [19]. Finally, the two-block argument is iterated along nested subweights, producing recursive estimates with accumulated correction or stability terms.

The paper is organized as follows. §2 gives definitions and standing assumptions. §3 proves the scaled two-block estimates. §4 establishes superquadratic refinements. §5 gives recursive versions. §6 and §7 record discrete, integral, time-scale, Jackson q -integral and fractional-kernel realizations. §8 gives a separated-potential illustration, and §9 concludes the paper.

2. Preliminaries and standing assumptions

Throughout the paper, $I \subseteq \mathbb{R}$ denotes an interval, $0 < m \leq 1$, and S is a nonempty set. Let L be a real linear class of functions on S containing the constant function 1.

Definition 1 (Positive linear functional). A mapping $A : L \rightarrow \mathbb{R}$ is called a positive linear functional if it is linear and satisfies

$$h(u) \geq 0 \quad (u \in S) \implies A(h) \geq 0.$$

If $w \in L$ is a nonnegative weight with $A(w) > 0$, and if $wg \in L$, we write

$$\mu_A(w; g) := \frac{A(wg)}{A(w)}.$$

When no confusion is possible, this mean is denoted by μ .

Remark 1 (Closure convention). The class L is not assumed to be an algebra. Therefore, whenever expressions such as wg , ηw , $(1 - \eta)w$, ηwg or $(1 - \eta)wg$ occur, it is explicitly assumed that these functions belong to the domain of A . This convention is used in every theorem below.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ in the sense of Hilger [20]; standard references include [21]. If $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ and

$$A(h) = \int_a^b h(t) \Delta t,$$

then the abstract results below produce continuous, discrete, hybrid and q -discrete versions by specialization; see also [22] for fractional q -calculus on time scales.

Definition 2 (m -convexity). Let $I \subseteq \mathbb{R}$ and $0 < m \leq 1$. A function $f : I \rightarrow \mathbb{R}$ is called m -convex on I if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y),$$

for all $x, y \in I$ and $t \in [0, 1]$ for which $tx + m(1 - t)y \in I$. If the inequality is strict for distinct x, y and $t \in (0, 1)$, then f is called strictly m -convex.

Definition 3 (m -admissibility). An interval I is said to be m -admissible for a pair $(x, y) \in I \times I$ if

$$tx + m(1 - t)y \in I \quad (0 \leq t \leq 1).$$

Definition 4 (Modulus-controlled m -convexity). Let $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfy $\phi(0) = 0$. A function $f : I \rightarrow \mathbb{R}$ is called m -convex with modulus ϕ if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y) - t(1 - t)\phi(|x - y|),$$

for all $x, y \in I$ such that I is m -admissible for (x, y) , and for all $t \in [0, 1]$. If ϕ is increasing and $\phi(r) > 0$ for every $r > 0$, then f is called uniformly m -convex with modulus ϕ .

The convention $\phi \equiv 0$ is allowed, so ordinary m -convexity is the qualitative case of the modulus-controlled theory. The quadratic choice $\phi(r) = \rho r^2$ gives a strongly m -convex model.

Definition 5 (Approximately m -convex functions). Let $\psi : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ satisfy $\psi(0, r) = \psi(1, r) = 0$ for $r \geq 0$. A function $f : I \rightarrow \mathbb{R}$ is called approximately m -convex with error ψ if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) + \psi(t, |x-y|),$$

for all $x, y \in I$ such that I is m -admissible for (x, y) , and for all $t \in [0, 1]$.

Definition 6 (Superquadratic function). A function $f : [0, \infty) \rightarrow \mathbb{R}$ is called superquadratic if, for every $x \geq 0$, there exists a constant $C_x \in \mathbb{R}$ such that

$$f(y) \geq f(x) + C_x(y-x) + f(|y-x|) \quad (y \geq 0).$$

A function f is called subquadratic if $-f$ is superquadratic.

The following refinement is standard; see [19,23,24].

Lemma 1 (Two-point superquadratic Jensen refinement). Let $f : [0, \infty) \rightarrow \mathbb{R}$ be superquadratic. Then, for $x, y \geq 0$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - tf((1-t)|x-y|) - (1-t)f(t|x-y|).$$

If f is subquadratic, the reverse inequality holds.

Remark 2. A nonnegative superquadratic function is convex and satisfies $f(0) = 0$ in the standard theory of superquadratic functions. The functions $f(x) = x^p$, $p \geq 2$, are typical nonnegative superquadratic examples on $[0, \infty)$. Since $m^p \leq m$ for $0 < m \leq 1$ and $p \geq 1$, these power functions are also m -convex on $[0, \infty)$.

Remark 3 (Automatic admissibility on $[0, \infty)$). If $I = [0, \infty)$ and $g(S) \subseteq [0, \infty)$, then every weighted block mean is nonnegative. Since $0 < m \leq 1$, every scaled point of the form x/m also belongs to $[0, \infty)$ whenever $x \geq 0$.

For reference, the main notation used in the two-block setting is summarized in Table 1.

Table 1. Notation for the two-block decomposition.

Symbol	Meaning
A	Positive linear functional on L
w	Nonnegative weight with $A(w) > 0$
g	Function whose weighted means are considered
η	Block selector, $0 \leq \eta \leq 1$
α	Relative mass $A(\eta w) / A(w)$
β	Relative mass $A((1-\eta)w) / A(w)$
μ	Global mean $A(wg) / A(w)$
μ_1	First block mean $A(\eta wg) / A(\eta w)$
μ_2	Second block mean $A((1-\eta)wg) / A((1-\eta)w)$
μ_2/m	Scaled complementary mean forced by m -convexity

3. Two-block scaled block-mean estimates

Let $w, g, \eta \in L$ satisfy $w \geq 0$, $0 \leq \eta \leq 1$, and assume that

$$A(w) > 0, \quad A(\eta w) > 0, \quad A((1 - \eta)w) > 0.$$

In accordance with Remark 1, assume further that all products appearing below belong to the domain of A . Define

$$\alpha := \frac{A(\eta w)}{A(w)}, \quad \beta := \frac{A((1 - \eta)w)}{A(w)}.$$

Then

$$\alpha + \beta = 1.$$

The corresponding localized means are defined by

$$\mu_1 := \frac{A(\eta wg)}{A(\eta w)}, \quad \mu_2 := \frac{A((1 - \eta)wg)}{A((1 - \eta)w)}, \quad \mu := \frac{A(wg)}{A(w)}.$$

Lemma 2 (Block-mean identity). *Under the assumptions above,*

$$\mu = \alpha\mu_1 + \beta\mu_2 = \alpha\mu_1 + m\beta\left(\frac{\mu_2}{m}\right).$$

Proof. The decompositions $w = \eta w + (1 - \eta)w$ and $wg = \eta wg + (1 - \eta)wg$, together with the linearity of A , give

$$A(wg) = A(\eta wg) + A((1 - \eta)wg), \quad A(w) = A(\eta w) + A((1 - \eta)w).$$

Division by $A(w)$ gives $\mu = \alpha\mu_1 + \beta\mu_2$. Since $m > 0$, the second summand can be rewritten as $\beta\mu_2 = m\beta(\mu_2/m)$. $\square$

Theorem 1 (Two-block m -Jensen estimate). *Let $f : I \rightarrow \mathbb{R}$ be m -convex. Assume that $\mu_1 \in I$, $\mu_2/m \in I$, and that I is m -admissible for $(\mu_1, \mu_2/m)$. Then $\mu \in I$ and*

$$f(\mu) \leq \alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right).$$

If f is strictly m -convex, equality in this m -convexity step can occur only when $\mu_1 = \mu_2/m$.

Proof. By Lemma 2,

$$\mu = \alpha\mu_1 + m(1 - \alpha)\left(\frac{\mu_2}{m}\right).$$

The admissibility assumptions ensure that this point belongs to I . Applying m -convexity with $x = \mu_1$, $y = \mu_2/m$ and $t = \alpha$ gives the desired inequality. Since both block masses are positive, $0 < \alpha < 1$, and the equality statement follows from strict m -convexity. $\square$

Corollary 1 (Classical convex limit). *If $m = 1$ in Theorem 1, then*

$$f(\mu) \leq \alpha f(\mu_1) + \beta f(\mu_2),$$

which is the standard two-block Jensen estimate for ordinary convex functions.

Theorem 2 (Modulus-controlled two-block estimate). *Let $f : I \rightarrow \mathbb{R}$ be m -convex with modulus ϕ . Assume that $\mu_1 \in I$, $\mu_2/m \in I$, and that I is m -admissible for $(\mu_1, \mu_2/m)$. Then $\mu \in I$ and*

$$f(\mu) \leq \alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) - \alpha\beta\phi\left(\left|\mu_1 - \frac{\mu_2}{m}\right|\right).$$

Proof. The proof is the proof of Theorem 1 with the correction term retained from the modulus-controlled definition. $\square$

Corollary 2 (Scaled block Jensen gap). *Under the assumptions of Theorem 2, define*

$$G_{\eta,m}(f;w,g) := \alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) - f(\mu).$$

Then

$$G_{\eta,m}(f;w,g) \geq \alpha\beta\phi\left(\left|\mu_1 - \frac{\mu_2}{m}\right|\right).$$

In particular, if f is uniformly m -convex and $0 < \alpha, \beta < 1$, then the gap is strictly positive unless $\mu_1 = \mu_2/m$.

Theorem 3 (Approximate scaled block estimate). *Let $f : I \rightarrow \mathbb{R}$ be approximately m -convex with error ψ . Assume that $\mu_1 \in I$, $\mu_2/m \in I$, and that I is m -admissible for $(\mu_1, \mu_2/m)$. Then $\mu \in I$ and*

$$f(\mu) \leq \alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) + \psi\left(\alpha, \left|\mu_1 - \frac{\mu_2}{m}\right|\right).$$

Proof. Use Lemma 2 and apply approximate m -convexity. $\square$

Corollary 3 (Uniform error form). *If*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) + \varepsilon t(1-t),$$

for some $\varepsilon \geq 0$, then

$$f(\mu) \leq \alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) + \varepsilon\alpha\beta.$$

Remark 4 (Relation with the ordinary Jensen functional). The present estimates are scaled block-mean Jensen-type estimates. When the stronger functional Jensen inequality

$$f(\mu) \leq \frac{A(w(f \circ g))}{A(w)},$$

is available for the chosen functional and function class, it can be combined with the scaled block estimates above. The additional information supplied here is the adapted block gap $G_{\eta,m}$ and the necessary occurrence of the scaled point μ_2/m .

Corollary 4 (Quadratic specialization). *If $\phi(r) = \rho r^2$ with $\rho \geq 0$, then*

$$f(\mu) \leq \alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) - \rho\alpha\beta\left(\mu_1 - \frac{\mu_2}{m}\right)^2.$$

For $m = 1$, this reduces to the usual between-block quadratic correction.

Example 1 (Quadratic model on $[0, \infty)$). Let $I = [0, \infty)$ and $f(x) = x^2$. Then f is m -convex with modulus $\phi(r) = mr^2$. Indeed, for $x, y \geq 0$ and $t \in [0, 1]$,

$$tx^2 + m(1-t)y^2 - (tx + m(1-t)y)^2 - mt(1-t)(x-y)^2 = (1-m)(1-t)(tx^2 + m(1-t)y^2) \geq 0.$$

Example 2 (Necessity of the scaled complementary mean). Let $m = 1/2$, $I = [0, \infty)$, $f(x) = x^2$, and consider two equally weighted points $x_1 = 1$ and $x_2 = 2$. Splitting the first point from the second gives

$$\alpha = \beta = \frac{1}{2}, \quad \mu_1 = 1, \quad \mu_2 = 2, \quad \mu = \frac{3}{2}.$$

The unscaled expression would give

$$\alpha f(\mu_1) + m\beta f(\mu_2) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot 4 = \frac{3}{2},$$

which is smaller than $f(\mu) = 9/4$. Hence the unscaled version is not a valid bound. The scaled estimate gives

$$\alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot 16 = \frac{9}{2}.$$

Using the quadratic modulus from Example 1, Theorem 2 improves this to

$$\frac{9}{2} - \frac{1}{4} \cdot \frac{1}{2} |1 - 4|^2 = \frac{27}{8},$$

which still bounds $f(\mu) = 9/4$.

4. Superquadratic refinements of the scaled block gap

The modulus-controlled estimate measures the m -convex separation between μ_1 and μ_2/m . Superquadraticity gives a different refinement: it measures the ordinary dispersion between the actual block means μ_1 and μ_2 . The two mechanisms are compatible because

$$\mu = \alpha\mu_1 + \beta\mu_2 = \alpha\mu_1 + m\beta\left(\frac{\mu_2}{m}\right).$$

Theorem 4 (Superquadratic scaled block-gap refinement). *Assume that $I = [0, \infty)$, $g(S) \subseteq [0, \infty)$, and $0 < m \leq 1$. Let $f : [0, \infty) \rightarrow [0, \infty)$ be both superquadratic and m -convex. Then*

$$\alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) - f(\mu) \geq \alpha f(\beta|\mu_1 - \mu_2|) + \beta f(\alpha|\mu_1 - \mu_2|) + \beta \left[mf\left(\frac{\mu_2}{m}\right) - f(\mu_2) \right].$$

Consequently,

$$\alpha f(\mu_1) + m\beta f\left(\frac{\mu_2}{m}\right) - f(\mu) \geq \alpha f(\beta|\mu_1 - \mu_2|) + \beta f(\alpha|\mu_1 - \mu_2|) \geq 0.$$

Proof. Since the means are nonnegative, Lemma 1 applied to $\mu = \alpha\mu_1 + \beta\mu_2$ gives

$$f(\mu) \leq \alpha f(\mu_1) + \beta f(\mu_2) - \alpha f(\beta|\mu_1 - \mu_2|) - \beta f(\alpha|\mu_1 - \mu_2|).$$

The endpoint case $t = 0$ in m -convexity, with $y = \mu_2/m$, gives

$$f(\mu_2) = f\left(m\frac{\mu_2}{m}\right) \leq mf\left(\frac{\mu_2}{m}\right).$$

Substitution and rearrangement yield the stated inequality. The final nonnegativity follows from the nonnegativity of f and from the endpoint estimate. $\square$

Corollary 5 (Classical superquadratic limit). *If $m = 1$ in Theorem 4, then*

$$\alpha f(\mu_1) + \beta f(\mu_2) - f(\mu) \geq \alpha f(\beta|\mu_1 - \mu_2|) + \beta f(\alpha|\mu_1 - \mu_2|),$$

which is the usual two-block superquadratic refinement of Jensen's inequality.

Corollary 6 (Power-function refinement). *Let $p \geq 2$ and $f(x) = x^p$ on $[0, \infty)$. Then*

$$\alpha\mu_1^p + m\beta\left(\frac{\mu_2}{m}\right)^p - \mu^p \geq \alpha\beta^p|\mu_1 - \mu_2|^p + \beta\alpha^p|\mu_1 - \mu_2|^p + \beta(m^{1-p} - 1)\mu_2^p.$$

Remark 5. Theorem 4 is not merely the modulus estimate with a special modulus. The superquadratic terms measure the ordinary dispersion between μ_1 and μ_2 , whereas the final bracket records the additional scaling contribution forced by m -convexity.

5. Nested recursive estimates

The two-block theorem can be iterated along a nested chain of subweights. This gives a rigorous recursive estimate without asserting the flat finite-partition monotonicity available for ordinary convexity.

Let $w_0 = w$ and

$$\mu_0 := \frac{A(wg)}{A(w)}.$$

For $k = 1, \dots, n$, let $\eta_k, \theta_k \in L$ satisfy

$$\eta_k \geq 0, \quad \theta_k \geq 0, \quad \eta_k + \theta_k = 1,$$

and define recursively

$$w_k := \eta_k w_{k-1}.$$

Assume that all products involved belong to the domain of A and that

$$A(w_k) > 0, \quad A(\theta_k w_{k-1}) > 0.$$

Define

$$\begin{aligned} \alpha_k &:= \frac{A(w_k)}{A(w_{k-1})}, & \beta_k &:= \frac{A(\theta_k w_{k-1})}{A(w_{k-1})}, \\ \mu_k &:= \frac{A(w_k g)}{A(w_k)}, & \nu_k &:= \frac{A(\theta_k w_{k-1} g)}{A(\theta_k w_{k-1})}. \end{aligned}$$

For compact notation, put

$$P_0 := 1, \quad P_k := \prod_{j=1}^k \alpha_j \quad (k \geq 1).$$

Theorem 5 (Nested qualitative estimate). *Let $f : I \rightarrow \mathbb{R}$ be m -convex. If $\mu_k \in I$, $\nu_k/m \in I$, and I is m -admissible for each pair $(\mu_k, \nu_k/m)$, then $\mu_0 \in I$ and*

$$f(\mu_0) \leq P_n f(\mu_n) + m \sum_{k=1}^n P_{k-1} \beta_k f\left(\frac{\nu_k}{m}\right).$$

Proof. Apply Theorem 1 to the split of w_{k-1} into w_k and $\theta_k w_{k-1}$, then substitute successively. At level k , the new contributions are multiplied by the accumulated factor P_{k-1} . $\square$

Theorem 6 (Nested modulus-controlled estimate). *Let $f : I \rightarrow \mathbb{R}$ be m -convex with modulus ϕ . If $\mu_k \in I$, $\nu_k/m \in I$, and I is m -admissible for each pair $(\mu_k, \nu_k/m)$, then $\mu_0 \in I$ and*

$$f(\mu_0) \leq P_n f(\mu_n) + m \sum_{k=1}^n P_{k-1} \beta_k f\left(\frac{\nu_k}{m}\right) - \sum_{k=1}^n P_{k-1} \alpha_k \beta_k \phi\left(\left|\mu_k - \frac{\nu_k}{m}\right|\right).$$

Proof. This follows by applying Theorem 2 at each nested split and summing the accumulated correction terms. $\square$

Theorem 7 (Nested approximate estimate). *Let $f : I \rightarrow \mathbb{R}$ be approximately m -convex with error ψ . If $\mu_k \in I$, $\nu_k/m \in I$, and I is m -admissible for each pair $(\mu_k, \nu_k/m)$, then $\mu_0 \in I$ and*

$$f(\mu_0) \leq P_n f(\mu_n) + m \sum_{k=1}^n P_{k-1} \beta_k f\left(\frac{\nu_k}{m}\right) + \sum_{k=1}^n P_{k-1} \psi\left(\alpha_k, \left|\mu_k - \frac{\nu_k}{m}\right|\right).$$

Corollary 7 (Quadratic recursive remainder). *If $\phi(r) = \rho r^2$ with $\rho \geq 0$, then*

$$f(\mu_0) \leq P_n f(\mu_n) + m \sum_{k=1}^n P_{k-1} \beta_k f\left(\frac{\nu_k}{m}\right) - \rho \sum_{k=1}^n P_{k-1} \alpha_k \beta_k \left(\mu_k - \frac{\nu_k}{m}\right)^2.$$

Theorem 8 (Nested superquadratic refinement). Assume the nested setting above, with $I = [0, \infty)$ and all means nonnegative. Let $f : [0, \infty) \rightarrow [0, \infty)$ be both superquadratic and m -convex. Put

$$d_k := |\mu_k - \nu_k| \quad (k = 1, \dots, n).$$

Then

$$f(\mu_0) \leq P_n f(\mu_n) + m \sum_{k=1}^n P_{k-1} \beta_k f\left(\frac{\nu_k}{m}\right) - \sum_{k=1}^n P_{k-1} \left[\alpha_k f(\beta_k d_k) + \beta_k f(\alpha_k d_k) + \beta_k \left(m f\left(\frac{\nu_k}{m}\right) - f(\nu_k) \right) \right].$$

Proof. Apply Theorem 4 to each split of w_{k-1} and substitute recursively. The terms generated at level k acquire the common factor P_{k-1} . $\square$

6. Discrete, integral, time-scale and Jackson q -integral realizations

This section records direct consequences of Theorem 2. The qualitative versions follow by taking $\phi \equiv 0$, the stability versions follow from Theorem 3, and the recursive versions follow from Theorems 6, 7 and 8. In every case, the hypotheses explicitly require the displayed arguments of f to lie in I , because for $m < 1$ the complementary block mean is evaluated at μ_2/m .

Corollary 8 (Weighted discrete sums). Let $\lambda_1, \dots, \lambda_N \geq 0$ with $\Lambda := \sum_{i=1}^N \lambda_i > 0$, and let $x_1, \dots, x_N \in \mathbb{R}$. Let $J \subset \{1, \dots, N\}$ be nonempty with nonempty complement and assume $\sum_{i \in J} \lambda_i > 0$, and $\sum_{i \notin J} \lambda_i > 0$. Define

$$\alpha := \frac{\sum_{i \in J} \lambda_i}{\Lambda}, \quad \beta := \frac{\sum_{i \notin J} \lambda_i}{\Lambda}, \quad \mu := \frac{\sum_{i=1}^N \lambda_i x_i}{\Lambda}, \quad \mu_1 := \frac{\sum_{i \in J} \lambda_i x_i}{\sum_{i \in J} \lambda_i}, \quad \mu_2 := \frac{\sum_{i \notin J} \lambda_i x_i}{\sum_{i \notin J} \lambda_i}.$$

If $f : I \rightarrow \mathbb{R}$ is m -convex with modulus ϕ , $\mu_1 \in I$, $\mu_2/m \in I$, and I is m -admissible for $(\mu_1, \mu_2/m)$, then

$$f(\mu) \leq \alpha f(\mu_1) + m \beta f\left(\frac{\mu_2}{m}\right) - \alpha \beta \phi\left(\left|\mu_1 - \frac{\mu_2}{m}\right|\right).$$

Proof. Take $S = \{1, \dots, N\}$, $A(h) = \sum_{i=1}^N h(i)$, $w(i) = \lambda_i$, $g(i) = x_i$, and let η be the characteristic function of J . $\square$

Corollary 9 (Weighted integral form). Let $p : [a, b] \rightarrow [0, \infty)$ be integrable with $\int_a^b p(x) dx > 0$, and let $g : [a, b] \rightarrow \mathbb{R}$ be measurable. Let $\eta : [a, b] \rightarrow [0, 1]$ be measurable and suppose that all displayed integrals below are finite and $\int_a^b \eta(x) p(x) dx > 0$, and $\int_a^b (1 - \eta(x)) p(x) dx > 0$. Define

$$\alpha := \frac{\int_a^b \eta(x) p(x) dx}{\int_a^b p(x) dx}, \quad \beta := \frac{\int_a^b (1 - \eta(x)) p(x) dx}{\int_a^b p(x) dx}, \quad \mu := \frac{\int_a^b p(x) g(x) dx}{\int_a^b p(x) dx}, \quad \mu_1 := \frac{\int_a^b \eta(x) p(x) g(x) dx}{\int_a^b \eta(x) p(x) dx},$$

$$\mu_2 := \frac{\int_a^b (1 - \eta(x)) p(x) g(x) dx}{\int_a^b (1 - \eta(x)) p(x) dx}.$$

If $f : I \rightarrow \mathbb{R}$ is m -convex with modulus ϕ , $\mu_1 \in I$, $\mu_2/m \in I$, and I is m -admissible for $(\mu_1, \mu_2/m)$, then the conclusion of Corollary 6.1 holds with these integral means.

Proof. Use the positive linear functional $A(h) = \int_a^b h(x) dx$ and apply Theorem 2. $\square$

Corollary 10 (Δ -integral form on time scales). Let $\mathbb{T}$ be a time scale and let $p : [a, b]_{\mathbb{T}} \rightarrow [0, \infty)$ be Δ -integrable with

$$\int_a^b p(t) \Delta t > 0.$$

Let $g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ and $\eta : [a, b]_{\mathbb{T}} \rightarrow [0, 1]$ be such that all displayed Δ -integrals are finite, and assume the two block masses are positive. Define $\alpha, \beta, \mu, \mu_1$ and μ_2 as in the weighted integral form, replacing dx by Δt . If $f : I \rightarrow \mathbb{R}$ is m -convex with modulus ϕ , $\mu_1 \in I$, $\mu_2/m \in I$, and I is m -admissible for $(\mu_1, \mu_2/m)$, then the same scaled block estimate holds.

Proof. The time-scale integral is linear and positive on nonnegative Δ -integrable functions. Apply Theorem 2 with $A(h) = \int_a^b h(t) \Delta t$. $\square$

Corollary 11 (Jackson q -integral form). Fix $t_0 > 0$ and $0 < q < 1$. Let

$$\mathbb{T}_{q,t_0} := \{t_0 q^k : k \in \mathbb{N}_0\} \cup \{0\},$$

and define

$$A_q(h) := \int_0^{t_0} h(t) d_q t = (1-q)t_0 \sum_{k=0}^{\infty} q^k h(t_0 q^k),$$

whenever the series converges. Suppose that p, g, η are defined on $\mathbb{T}_{q,t_0}$, $p \geq 0$, $0 \leq \eta \leq 1$, the total mass and the two block masses formed with the weight p are positive, and all series defining the relevant means converge absolutely. Define $\alpha_q, \beta_q, \mu_q, \mu_{1,q}$ and $\mu_{2,q}$ by replacing A with A_q and taking $w = p$. If $f : I \rightarrow \mathbb{R}$ is m -convex with modulus ϕ , $\mu_{1,q} \in I$, $\mu_{2,q}/m \in I$, and I is m -admissible for $(\mu_{1,q}, \mu_{2,q}/m)$, then

$$f(\mu_q) \leq \alpha_q f(\mu_{1,q}) + m\beta_q f\left(\frac{\mu_{2,q}}{m}\right) - \alpha_q \beta_q \phi\left(\left|\mu_{1,q} - \frac{\mu_{2,q}}{m}\right|\right).$$

Proof. The coefficients $(1-q)t_0 q^k$ are nonnegative. Hence A_q is a positive linear functional on the stated convergence class. The result follows from Theorem 2. $\square$

Remark 6 (Expectation interpretation). If A is an expectation operator on a probability space and $w \equiv 1$, then the two-block and recursive estimates become refined expectation inequalities for random variables under measurable two-way and nested decompositions of the underlying law.

7. Fractional-kernel realizations

Many fractional versions of Jensen-type inequalities are obtained by choosing a fractional integral operator with a nonnegative kernel. The positive-functional setting records the exact positivity and integrability assumptions needed for the scaled block estimate to transfer to such operators.

Let $K : [a, b] \rightarrow [0, \infty)$ be measurable and not identically zero, and let $p : [a, b] \rightarrow [0, \infty)$ be measurable. Define

$$A_K(h) := \int_a^b K(t)p(t)h(t) dt,$$

whenever the integral exists. If $0 < A_K(1) < \infty$, then A_K is a positive linear functional on the corresponding integrability class.

Corollary 12 (Kernel-fractional two-block realization). Let $K \geq 0$, $p \geq 0$, and $0 < A_K(1) < \infty$. Let $g : [a, b] \rightarrow \mathbb{R}$ and $\eta : [a, b] \rightarrow [0, 1]$ be such that all quantities below exist and are finite, and assume $A_K(\eta) > 0$, and $A_K(1-\eta) > 0$. Define

$$\alpha_K := \frac{A_K(\eta)}{A_K(1)}, \quad \beta_K := \frac{A_K(1-\eta)}{A_K(1)}, \quad \mu_K := \frac{A_K(g)}{A_K(1)}, \quad \mu_{1,K} := \frac{A_K(\eta g)}{A_K(\eta)}, \quad \mu_{2,K} := \frac{A_K((1-\eta)g)}{A_K(1-\eta)}.$$

If $f : I \rightarrow \mathbb{R}$ is m -convex with modulus ϕ , $\mu_{1,K} \in I$, $\mu_{2,K}/m \in I$, and I is m -admissible for $(\mu_{1,K}, \mu_{2,K}/m)$, then

$$f(\mu_K) \leq \alpha_K f(\mu_{1,K}) + m\beta_K f\left(\frac{\mu_{2,K}}{m}\right) - \alpha_K \beta_K \phi\left(\left|\mu_{1,K} - \frac{\mu_{2,K}}{m}\right|\right).$$

The corresponding nested fractional-kernel estimate follows from Theorem 6 with $A = A_K$.

Proof. Because $K(t)p(t) \geq 0$ almost everywhere, A_K is positive on nonnegative functions, and linearity follows from the linearity of the integral. The result is Theorem 2 with $A = A_K$ and $w \equiv 1$. $\square$

Example 3 (Riemann–Liouville-type kernel). Let $x \in (a, b]$ and $\gamma > 0$. On $[a, x]$, choose

$$K_{\gamma,x}(t) := \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}.$$

Then

$$A_{K_{\gamma,x}}(h) = \int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} p(t)h(t) dt,$$

realizes a weighted left Riemann–Liouville-type fractional integral. Corollary 7.1 gives the corresponding modulus-controlled scaled block-mean estimate. Taking $\phi \equiv 0$ gives the qualitative m -convex form, while $\phi(r) = \rho r^2$ gives the strongly m -convex fractional-kernel correction.

8. Separated-potential illustration

The following short application illustrates how the abstract estimate can be used for a separated functional. The kinetic or velocity-dependent part is kept outside the argument, while the potential term is estimated by the scaled block-mean inequality.

Proposition 1 (Averaged potential estimate). Let $[a, b]_{\mathbb{T}}$ be a compact time-scale interval, let $x : [a, b]_{\mathbb{T}} \rightarrow I$ be an admissible trajectory, let $V_0 : I \rightarrow \mathbb{R}$ be m -convex with modulus ϕ , and let $\rho : [a, b]_{\mathbb{T}} \rightarrow [0, \infty)$ satisfy

$$M := \int_a^b \rho(t) \Delta t > 0.$$

Assume that $U, V_1 : [a, b]_{\mathbb{T}} \rightarrow [0, \infty)$ satisfy $U + V_1 = 1$ and that the two block masses are positive. Define

$$\bar{x}_\rho := \frac{1}{M} \int_a^b \rho(t)x(t) \Delta t, \quad \bar{x}_U := \frac{\int_a^b U(t)\rho(t)x(t) \Delta t}{\int_a^b U(t)\rho(t) \Delta t}, \quad \bar{x}_{V_1} := \frac{\int_a^b V_1(t)\rho(t)x(t) \Delta t}{\int_a^b V_1(t)\rho(t) \Delta t}.$$

If $\bar{x}_U \in I$, $\bar{x}_{V_1}/m \in I$, and I is m -admissible for $(\bar{x}_U, \bar{x}_{V_1}/m)$, then

$$V_0(\bar{x}_\rho) \leq \alpha V_0(\bar{x}_U) + m\beta V_0\left(\frac{\bar{x}_{V_1}}{m}\right) - \alpha\beta\phi\left(\left|\bar{x}_U - \frac{\bar{x}_{V_1}}{m}\right|\right),$$

where

$$\alpha := \frac{\int_a^b U(t)\rho(t) \Delta t}{\int_a^b \rho(t) \Delta t}, \quad \beta := \frac{\int_a^b V_1(t)\rho(t) \Delta t}{\int_a^b \rho(t) \Delta t}.$$

Proof. Use $A(h) = \int_a^b h(t) \Delta t$, $w = \rho$, $g = x$ and $f = V_0$ in Theorem 2. $\square$

Remark 7. For a separated density $L(t, x, v) = K(t, v) + V_0(x)$, the proposition supplies a controlled estimate for the potential contribution generated by V_0 and leaves the remaining terms outside the block-mean argument.

9. Conclusion

We developed a scaled block-mean framework for Jensen-type estimates associated with m -convex functions and positive linear functionals. The central mechanism is the scaled block identity

$$\mu = \alpha\mu_1 + m\beta\left(\frac{\mu_2}{m}\right),$$

which is forced by the asymmetric geometry of m -convexity. This identity leads to qualitative two-block estimates, modulus-controlled corrections, approximate stability forms, superquadratic block-gap refinements, and recursive inequalities with accumulated remainder terms.

The main structural point is that the complementary block mean must be evaluated at μ_2/m rather than at μ_2 . The counterexample in §3 shows that the unscaled expression may fail for $m < 1$, while the modulus-controlled and superquadratic theorems provide explicit nonnegative lower bounds for the adapted scaled block gap. The positive-linear-functional formulation yields weighted-sum, ordinary integral, time-scale, Jackson q -integral and fractional-kernel realizations from the same abstract statement.

Possible continuations include operator-valued positive functionals, normalized positive semigroups, sharper fractional-kernel correction terms, superquadratic information-theoretic refinements and applications generated by Bregman-type functions [25].

Funding Information: The author declares that no external funding was received for this work.

Data Availability: No datasets were generated or analysed during the current study.

Conflicts of Interest: The author declares that there are no competing interests.

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