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Study of a new class of integral inequalities

Christophe Chesneau

Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France;
christophe.chesneau@gmail.com

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Abstract: This article introduces and analyzes a new class of integral inequalities relating the integrals of two functions over different intervals. Using classical tools such as the Hermite-Hadamard, Steffensen and Young integral inequalities, we derive several refined bounds under monotonicity and convexity assumptions.

Keywords: integral inequalities, Hermite-Hadamard integral inequality, Steffensen integral inequality, Young integral inequality

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1. Introduction

Integrals are among the most fundamental and versatile tools in mathematics. They can be used to quantify a wide range of phenomena, including area, volume, accumulated change and average values. However, in many practical and theoretical contexts, exact evaluation of integrals is either impossible or extremely difficult. Consequently, various approximation techniques have been developed to estimate their values with the desired level of accuracy.

One particularly effective class of approximation method relies on integral inequalities, which aim to provide lower and/or upper bounds for integrals that are both tractable and often sharp. A rich variety of such inequalities has been established in the literature over the years. See, for example, [1–6].

In recent decades, there has been a significant increase in research aimed at refining, extending and generalizing these inequalities to broader settings and more complex classes of functions, see [7–22].

In this article, we contribute to the existing literature by investigating a particular class of integral inequalities of the following general form:

$$\sqrt{\left(\int_I f(t)dt\right)\left(\int_J g(t)dt\right)}+\sqrt{\left(\int_K f(t)dt\right)\left(\int_L g(t)dt\right)}\leq \Omega,$$

where f and g denote two real-valued functions, I, J, K and L are given intervals, and Ω represents an upper bound. To the best of our knowledge, integral inequalities of this type have not been widely studied in existing literature, despite their potential applications in mathematical modelling, numerical integration and error estimation. Motivated by this observation, we derive several new results under various structural assumptions on the functions f and g, including monotonicity and convexity conditions.

Our approach makes use of several classical integral inequalities as auxiliary tools, notably the Hermite-Hadamard, Steffensen and Young integral inequalities. For the sake of completeness and clarity, we present all proofs in full detail.

The remainder of the article is organized as follows: §2 presents the main results. Supplementary propositions and theorems are provided in §3. Finally, §4 concludes the article.

2. Main results

2.1. Direct approach

A direct approach to our integral inequality is presented in the theorem below, providing a fundamental benchmark result.

Theorem 1. *Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ such that $b > a$, and $f, g : [a, b] \to [0, +\infty)$ be integrable functions. Then, for any $x \in [a, b]$, the following inequality holds:*

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq 2\sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}.$$

Proof. Since f and g are non-negative, we have

$$\int_a^x f(t)dt \leq \int_a^b f(t)dt, \quad \int_x^b g(t)dt \leq \int_a^b g(t)dt, \quad \int_x^b f(t)dt \leq \int_a^b f(t)dt,$$

and

$$\int_a^x g(t)dt \leq \int_a^b g(t)dt.$$

Therefore, we have

$$\begin{aligned}
&\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \\
&\leq \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)} + \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)} \\
&= 2\sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}.
\end{aligned}$$

This completes the proof. □

This theorem yields the following question: *Can we improve the constant factor* 2*?* The answer is provided in the first result of the next subsection.

2.2. Technical approach

We start by establishing a technical lemma, which is of independent interest and may prove useful in related contexts.

Lemma 1. *For any $p, q, r, s \geq 0$, we have*

$$\sqrt{pq} + \sqrt{rs} \leq \sqrt{(p+r)(q+s)}.$$

Proof. We propose two proofs for this lemma.

Proof 1. The following equivalences hold:

$$\begin{aligned}
&\sqrt{pq} + \sqrt{rs} \leq \sqrt{(p+r)(q+s)} \quad \Leftrightarrow \quad \left(\sqrt{pq} + \sqrt{rs}\right)^2 \leq (p+r)(q+s) \\
&\Leftrightarrow \quad pq + rs + 2\sqrt{pq}\sqrt{rs} \leq pq + sp + rq + rs \quad \Leftrightarrow \quad 2\sqrt{pq}\sqrt{rs} \leq sp + rq \\
&\Leftrightarrow \quad 2\sqrt{sp}\sqrt{rq} \leq sp + rq \quad \Leftrightarrow \quad sp + rq - 2\sqrt{sp}\sqrt{rq} \geq 0 \\
&\Leftrightarrow \quad (\sqrt{sp} - \sqrt{rq})^2 \geq 0.
\end{aligned}$$

Since the last inequality clearly holds, the desired inequality follows, which completes the proof.

Proof 2. Let $n \in \mathbb{N}\backslash\{0\}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in [0, +\infty)$. Then the Cauchy-Schwarz inequality implies that

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)}.$$

Applying this inequality to $n = 2$, $a_1 = \sqrt{p}$, $a_2 = \sqrt{r}$, $b_1 = \sqrt{q}$ and $b_2 = \sqrt{s}$ yields the desired result, which completes the proof. □

Remark 1. The appeal of Proof 2 in Lemma 1 lies in the fact that the equality case of the Cauchy–Schwarz inequality is well understood: it holds if and only if there exists a constant λ such that $a_1 = \lambda b_1$ and $a_2 = \lambda b_2$. This condition implies that $a_1 b_2 = a_2 b_1$, and hence, with the notation of the lemma, $sp = rq$.

The theorem below states our main result, which refines Theorem 1 by employing Lemma 1.

Theorem 2. *Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ such that $b > a$, and $f, g : [a, b] \to [0, +\infty)$ be integrable functions. Then, for any $x \in [a, b]$, the following inequality holds:*

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}.$$

Proof. Lemma 1 ensures that, for any $p, q, r, s \geq 0$,

$$\sqrt{pq} + \sqrt{rs} \leq \sqrt{(p+r)(q+s)}.$$

Applying this inequality to the non-negative integrals

$$p = \int_a^x f(t)dt, \quad q = \int_x^b g(t)dt, \quad r = \int_x^b f(t)dt, \quad s = \int_a^x g(t)dt,$$

and using the Chasles integral relation, we get

$$\begin{aligned}
&\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \\
&\leq \sqrt{\left(\int_a^x f(t)dt + \int_x^b f(t)dt\right)\left(\int_a^x g(t)dt + \int_x^b g(t)dt\right)} \\
&= \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}.
\end{aligned}$$

This concludes the proof. □

Remark 2. With reference to Remark 1, the equality case in Theorem 2 holds if, for any $x \in [a, b]$,

$$\left(\int_a^x f(t)dt\right)\left(\int_a^x g(t)dt\right) = \left(\int_x^b f(t)dt\right)\left(\int_x^b g(t)dt\right).$$

This implies that $f = g = 0$ when evaluated at $x = a$ and $x = b$.

The constant factor 2 in Theorem 1 is thus reduced to 1 in Theorem 2 through the appropriate application of Lemma 1. It seems difficult to improve the constant 1. As a simple example, by considering $a = 0$, $b = 1$, $f = g = 1$, and taking into account that $\sup_{x \in [0,1]} x(1-x) = 1/4$, then we have

$$\begin{aligned}
\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} &= 2\sqrt{x(1-x)} \\
&\leq 2\sqrt{\frac{1}{4}} = 1 = \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}.
\end{aligned}$$

Theorem 2 therefore provides a sharper result and can serve as a foundation for deriving new classes of integral inequalities using intermediate analytical tools. This observation is further supported in the next section, where several propositions are established to illustrate the approach.

3. Other results

3.1. Some propositions

The proposition below presents an integral inequality based on a boundedness condition for the main functions.

Proposition 1. *Let $a, b \in \mathbb{R}$ such that $b > a$, and $f, g : [a, b] \to [0, +\infty)$ be integrable functions satisfying, for any $x \in [a, b]$,*

$$f(x) + g(x) \leq 2.$$

Then, for any $x \in [a, b]$, the following inequality holds:

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq b - a.$$

Proof. Applying Theorem 2, we get

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}. \tag{1}$$

Using the inequalities, for any $x \in [a, b]$, $f(x) \geq 0$ and $0 \leq g(x) \leq 2 - f(x)$, we have

$$\begin{aligned}\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right) &\leq \left(\int_a^b f(t)dt\right)\left(\int_a^b (2 - f(t))\, dt\right) \\ &= \left(\int_a^b f(t)dt\right)\left(2(b-a) - \int_a^b f(t)dt\right) \\ &= A\left(2(b-a) - A\right),\end{aligned} \tag{2}$$

where

$$A = \int_a^b f(t)dt.$$

Since, for any $x \in [a, b]$, $f(x) \geq 0$ and $f(x) \leq 2 - g(x) \leq 2$, we have $A \in [0, 2(b-a)]$. Let us introduce the function

$$h(y) = y\left(2(b-a) - y\right)$$

with $y \in [0, 2(b-a)]$. Then we have $h'(y) = 2\left((b-a) - y\right)$ so that $h'(y_\star) = 0$ if, and only if, $y_\star = b - a$, and h achieves a maximum at this value. Therefore, we have

$$A\left(2(b-a) - A\right) = h(A) \leq h(y_\star) = (b-a)\left(2(b-a) - (b-a)\right) = (b-a)^2. \tag{3}$$

Combining Eqs. (1), (2) and (3), we get

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{(b-a)^2} = b - a.$$

This ends the proof. □

Notice that the obtained upper bound is independent of f and g; it is simply $b - a$.

The proposition below presents another integral inequality that is subject to a monotonicity assumption. The Chebyshev integral inequality is a key of the proof.

Proposition 2. *Let $a, b \in \mathbb{R}$ such that $b > a$, and $f, g : [a, b] \to [0, +\infty)$ be monotonic functions of the same monotonicity. Then, for any $x \in [a, b]$, the following inequality holds:*

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{(b-a)\int_a^b f(t)g(t)dt}.$$

Proof. Applying Theorem 2, we get

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}. \quad (4)$$

Since f and g are monotonic functions of the same monotonicity, the Chebyshev integral inequality gives

$$\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right) \leq (b-a)\int_a^b f(t)g(t)dt. \quad (5)$$

Combining Eqs. (4) and (5), we get

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{(b-a)\int_a^b f(t)g(t)dt}.$$

This completes the proof. □

The obtained upper bound thus depends on the integral $\int_a^b f(t)g(t)dt$, which in some circumstances can be simpler to evaluate than the separate integrals of f and g.

The proposition below presents another integral inequality that is subject to a convex assumption. The Hermite-Hadamard integral inequality is crucial to the proof.

Proposition 3. *Let $a, b \in \mathbb{R}$ such that $b > a$, and $f, g : [a, b] \to [0, +\infty)$ be convex functions. Then, for any $x \in [a, b]$, the following inequality holds:*

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \frac{b-a}{2}\sqrt{(f(a)+f(b))(g(a)+g(b))}.$$

Proof. Applying Theorem 2, we get

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}. \quad (6)$$

Since f and g are convex, the (right-hand side of the) Hermite-Hadamard integral inequality gives

$$\int_a^b f(t)dt \leq (b-a)\frac{f(a)+f(b)}{2},$$

and

$$\int_a^b g(t)dt \leq (b-a)\frac{g(a)+g(b)}{2}.$$

This, together with the non-negativity of the integrals involved, yields

$$\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right) \leq (b-a)^2\frac{(f(a)+f(b))(g(a)+g(b))}{4}. \quad (7)$$

Combining Eqs. (6) and (7), we get

$$\sqrt{\left(\int_a^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_a^x g(t)dt\right)} \leq \sqrt{(b-a)^2 \frac{(f(a)+f(b))\,(g(a)+g(b))}{4}}$$
$$= \frac{b-a}{2}\sqrt{(f(a)+f(b))\,(g(a)+g(b))}.$$

This completes the proof. □

Therefore, the obtained upper bound depends on the endpoints of f and g, which are easier to determine than the individual integrals of f and g.

The proposition below presents an original integral inequality that is subject to a monotonicity assumption. The Steffensen integral inequality plays a key role in the proof.

Proposition 4. *Let $a, b \in \mathbb{R}$ such that $b > a$, $f, g : [a, b] \to [0, +\infty)$ be integrable non-increasing functions, and $h : [a, b] \to [0, 1]$ be a function. We set*

$$\lambda = \int_a^b h(x)dx$$

Then, for any $x \in [b-\lambda, b]$, the following inequality holds:

$$\sqrt{\left(\int_{b-\lambda}^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_{b-\lambda}^x g(t)dt\right)} \leq \sqrt{\left(\int_a^b f(t)h(t)dt\right)\left(\int_a^b g(t)h(t)dt\right)}.$$

Proof. Applying Theorem 2 with $b-\lambda$ instead of a, we get

$$\sqrt{\left(\int_{b-\lambda}^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_{b-\lambda}^x g(t)dt\right)} \leq \sqrt{\left(\int_{b-\lambda}^b f(t)dt\right)\left(\int_{b-\lambda}^b g(t)dt\right)}. \quad (8)$$

Since f and g are integrable non-increasing, the (left-hand side of the) Steffensen integral inequality gives

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)h(t)dt,$$

and

$$\int_{b-\lambda}^b g(t)dt \leq \int_a^b g(t)h(t)dt.$$

This, together with the non-negativity of the integrals involved, yields

$$\left(\int_{b-\lambda}^b f(t)dt\right)\left(\int_{b-\lambda}^b g(t)dt\right) \leq \left(\int_a^b f(t)h(t)dt\right)\left(\int_a^b g(t)h(t)dt\right). \quad (9)$$

Combining Eqs. (8) and (9), we get

$$\sqrt{\left(\int_{b-\lambda}^x f(t)dt\right)\left(\int_x^b g(t)dt\right)} + \sqrt{\left(\int_x^b f(t)dt\right)\left(\int_{b-\lambda}^x g(t)dt\right)} \leq \sqrt{\left(\int_a^b f(t)h(t)dt\right)\left(\int_a^b g(t)h(t)dt\right)}.$$

This completes the proof. □

This inequality is quite innovative, with the use of an integral in the integral limits and a manageable upper bound.

We conclude our analysis with two supplementary theorems.

3.2. Supplementary theorems

The theorem below is a functional variant of Theorem 2. It incorporates two adjustable functions, h and k.

Theorem 3. *Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ such that $b > a$, $f, g : [a,b] \to [0,+\infty)$ be integrable functions and $h, k : [a,b] \to [0,1]$ be functions. Then the following inequality holds:*

$$\sqrt{\left(\int_a^b f(t)h(t)dt\right)\left(\int_a^b g(t)(1-k(t))dt\right)} + \sqrt{\left(\int_a^b f(t)\left(1-h(t)\right)dt\right)\left(\int_a^b g(t)k(t)dt\right)} \leq \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}.$$

Proof. Lemma 1 ensures that, for any $p, q, r, s \geq 0$,

$$\sqrt{pq} + \sqrt{rs} \leq \sqrt{(p+r)(q+s)}.$$

Applying this inequality to the non-negative integrals

$$p = \int_a^b f(t)h(t)dt, \quad q = \int_a^b g(t)\left(1-k(t)\right)dt, \quad r = \int_a^b f(t)\left(1-h(t)\right)dt,$$

and

$$s = \int_a^b g(t)k(t)dt,$$

and using the linearity of the integral, we get

$$\begin{aligned}
&\sqrt{\left(\int_a^b f(t)h(t)dt\right)\left(\int_a^b g(t)(1-k(t))dt\right)} + \sqrt{\left(\int_a^b f(t)\left(1-h(t)\right)dt\right)\left(\int_a^b g(t)k(t)dt\right)} \\
&\leq \sqrt{\left(\int_a^b f(t)h(t)dt + \int_a^b f(t)\left(1-h(t)\right)dt\right)\left(\int_a^b g(t)k(t)dt + \int_a^b g(t)\left(1-k(t)\right)dt\right)} \\
&= \sqrt{\left(\int_a^b f(t)\left(h(t)+1-h(t)\right)dt\right)\left(\int_a^b g(t)\left(k(t)+1-k(t)\right)dt\right)} \\
&= \sqrt{\left(\int_a^b f(t)dt\right)\left(\int_a^b g(t)dt\right)}.
\end{aligned}$$

This concludes the proof. □

Once again, we highlight the simplicity of the obtained upper bound.

The theorem below combines Lemma 1 with the Young integral inequality to produce a new result.

Theorem 4. *Let $a, b \geq 0$, $f, g : [0,+\infty) \to [0,+\infty)$ be continuous and strictly increasing functions such that $f(0) = 0$ and $g(0) = 0$, and f^{-1} and g^{-1} be their inverse functions, respectively. Then the following inequality holds:*

$$\sqrt{\left(\int_0^a f(t)dt\right)\left(\int_0^b g^{-1}(t)dt\right)} + \sqrt{\left(\int_0^b f^{-1}(t)dt\right)\left(\int_0^a g(t)dt\right)} \leq \sqrt{\left(bf^{-1}(b) + f(a)\left(a - f^{-1}(b)\right)\right)\left(bg^{-1}(b) + g(a)\left(a - g^{-1}(b)\right)\right)}.$$

Proof. Lemma 1 ensures that, for any $p, q, r, s \geq 0$,

$$\sqrt{pq} + \sqrt{rs} \leq \sqrt{(p+r)(q+s)}.$$

Applying this inequality to the non-negative integrals

$$p = \int_0^a f(t)dt, \quad q = \int_0^b g^{-1}(t)dt, \quad r = \int_0^b f^{-1}(t)dt, \quad s = \int_0^a g(t)dt,$$

we obtain

$$\sqrt{\left(\int_0^a f(t)dt\right)\left(\int_0^b g^{-1}(t)dt\right)}+\sqrt{\left(\int_0^b f^{-1}(t)dt\right)\left(\int_0^a g(t)dt\right)} \leq \sqrt{\left(\int_0^a f(t)dt+\int_0^b f^{-1}(t)dt\right)\left(\int_0^a g(t)dt+\int_0^b g^{-1}(t)dt\right)}. \tag{10}$$

Using the (right-hand side of the) Young integral inequality as presented in [19, Theorem 1], we get

$$\int_0^a f(t)dt+\int_0^b f^{-1}(t)dt \leq bf^{-1}(b)+f(a)\left(a-f^{-1}(b)\right),$$

and

$$\int_0^a g(t)dt+\int_0^b g^{-1}(t)dt \leq bg^{-1}(b)+g(a)\left(a-g^{-1}(b)\right).$$

This, together with the non-negativity of the integrals involved, yields

$$\left(\int_0^a f(t)dt+\int_0^b f^{-1}(t)dt\right)\left(\int_0^a g(t)dt+\int_0^b g^{-1}(t)dt\right) \leq \left(bf^{-1}(b)+f(a)\left(a-f^{-1}(b)\right)\right)\left(bg^{-1}(b)+g(a)\left(a-g^{-1}(b)\right)\right). \tag{11}$$

Combining Eqs. (10) and (11), we obtain

$$\sqrt{\left(\int_0^a f(t)dt\right)\left(\int_0^b g^{-1}(t)dt\right)}+\sqrt{\left(\int_0^b f^{-1}(t)dt\right)\left(\int_0^a g(t)dt\right)} \leq \sqrt{(bf^{-1}(b)+f(a)(a-f^{-1}(b)))(bg^{-1}(b)+g(a)(a-g^{-1}(b)))}.$$

This concludes the proof. □

Therefore, the obtained upper bound depends on the endpoints of f, f^{-1}, g and g^{-1} in a sharp way.

4. Conclusion

In this article, we present new results for a class of integral inequalities involving products of integrals over distinct intervals, subject to monotonicity and convexity assumptions on the involved functions. Our approach relies on classical tools, such as the Hermite-Hadamard, Steffensen and Young integral inequalities, to provide sharper bounds and a framework for deriving further inequalities. Future work could involve extending the approach to more general function classes, multidimensional integrals, or applications in numerical analysis and mathematical modelling.

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