The Pečarić journal of
Mathematical Inequalities

PSR Press

Article

Two extensions of a Steffensen-type inequality

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Received: 05 February 2026; Accepted: 28 April 2026; Published: 07 May 2026

Abstract: We extend the Steffensen-type inequality, proved in the recent paper under concavity and certain smoothness assumptions, to a weighted measure $d\mu = wdx$ and a relaxation on smoothness. We also give two numerical examples where the original inequality cannot be applied, while our assumptions are satisfied.

Keywords: Steffensen inequality, Stieltjes integral, finite measure

MSC: 26D15.

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1. Introduction and preliminaries

Integral inequalities are extremely useful tools in analysis, utilized in computing error bounds, stability estimates, and also used in the optimization theory. Among classical results (such as Cauchy–Schwarz, Jensen, Ostrowski inequalities), the Steffensen inequality is distinct as it controls the product integral $\int_a^b f(x)g(x)dx$ via integrals of f over subintervals whose lengths are determined by $\int_a^b g(x)dx$. In its original form [1], if $f \geq 0$ is non-increasing and $0 \leq g(x) \leq 1$, then

$$\int_{b-\lambda}^{b} f(x)dx \leq \int_a^b f(x)g(x)dx \leq \int_a^{a+\lambda} f(x)dx, \qquad \lambda = \int_a^b g(x)dx,$$

and countless extensions have been developed since [2].

Over the last decade, several directions of generalization have been made. First, measure-theoretic and weighted formulations restate Steffensen-type bounds on general finite measure spaces or with nontrivial densities w; see the survey monograph [2] and the book [3]. Second, functional-analytic refinements connect Steffensen-type inequalities to sharp L^1-L^∞ bounds for weighted integrals and convex functionals, present in Rabier's work [4,5]. Third, extension to the theory to time scales and to q-calculus [6–9]. Finally, recent papers by Chesneau introduced Steffensen-type inequalities for concave functions, together with multi-function right–hand side variants [10,11].

In this article, we are highly motivated by the integration-by-parts approach in [11], where the following theorem is proved.

Theorem 1. *Let $a, b \in \mathbb{R}$ with $b > a$, $f\colon [a,b] \to [0,+\infty)$ be a twice differentiable concave function and $g\colon [a,b] \to [0,1]$ be a differentiable integrable function. Let*

$$k = \int_a^b g(x)dx.$$

Then the following inequalities hold:

$$f(b-k) - f(b)(1-g(b)) - f(a)g(a) \geq \int_a^b f(x)g'(x) \geq f(b)g(b) + f(a)(1-g(a)) - f(a+k). \tag{1}$$

We are also motivated by the weighted measure approach in [2,3]. For those reasons, we develop a Steffensen–type inequality for the Lebesgue–Stieltjes (simply: Stieltjes) integral and for the finite measure setting.

We now set up our notation.

Let $a < b$ and let $w : [a,b] \to (0,\infty)$ be integrable. Define the finite Borel measure

$$d\mu(x) = w(x)dx, \qquad W := \mu([a,b]) = \int_a^b w(x)dx.$$

Let $\Phi(x) := \mu([a,x]) = \int_a^x w(t)dt$ and let $Q : [0,W] \to [a,b]$ be the (left-continuous) μ-quantile map

$$Q(m) := \inf\{x \in [a,b] : \ \Phi(x) \geq m\}.$$

We say that $f : [a,b] \to \mathbb{R}$ is *μ-absolutely continuous* if there exists $D_\mu f \in L^1(\mu)$ such that

$$f(x) = f(a) + \int_{[a,x]} D_\mu f(t) d\mu(t) \quad \text{for all } x \in [a,b].$$

If $w > 0$ a.e. and f is (classically) absolutely continuous, then $D_\mu f = f'/w$ μ-a.e. We will use the notation $f \in AC$ and $g \in BV$ for functions that are absolutely continuous and for functions with bounded variation, respectively. We will often include the domain of functions by writing $g \in BV([a,b])$. Besides we also use the right-continuity convention for the Stjelties integral (and for functions with bounded variation), so that when we deal with the half-open interval $(a,b]$, we interpret $g(a)$ as $g(a_+)$ and $g(b)$ as $g(b_-)$, and we write

$$\int_a^b f dg = \int_{(a,b]} f dg.$$

The goal of our results is to relax the smoothness used for the concave case in [11] to an AC and a BV setting, and replace Lebesgue measure by a general weight, which is useful in numerical integration with graded meshes and in kernel-weighted estimation [3,11].

2. Main results

2.1. Generalization to AC/BV setting

Now, we generalize the inequality (1) to the AC/BV setting.

Theorem 2. *Let $a < b$. Assume that:*

- *$f : [a,b] \to \mathbb{R}$ is absolutely continuous, $f' \in L^1([a,b])$ is (essentially) decreasing on $[a,b]$ and* $\operatorname{ess\,inf} f' > -\infty$
- *$g \in BV([a,b])$ and $0 \leq g \leq 1$ a.e. on $[a,b]$.*

Set

$$k := \int_a^b g(x)dx \in [0, b-a].$$

Then the following double inequality holds:

$$f(b-k) - f(b)\big(1-g(b)\big) - f(a)g(a) \geq \int_{(a,b]} f dg \geq f(b)g(b) + f(a)\big(1-g(a)\big) - f(a+k). \tag{2}$$

Proof. Since $f \in AC$ and $g \in BV([a,b])$, the integration by parts for the Stieltjes integral gives

$$\int_{(a,b]} f dg = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx. \tag{3}$$

Thus it remains to bound the Lebesgue integral $\int_a^b f'(x)g(x)dx$. For that, we claim that if $\varphi \in L^1([a,b])$ is (essentially) decreasing with $\operatorname{ess\,inf} \varphi > -\infty$ and $h : [a,b] \to [0,1]$ is integrable with

$$\ell := \int_a^b h(x)dx,$$

then

$$\int_{b-\ell}^{b} \varphi(x)dx \le \int_a^b \varphi(x)h(x)dx \le \int_a^{a+\ell} \varphi(x)dx, \tag{4}$$

which can be seen as a Steffensen inequality without sign restriction.

To remove the sign restriction, pick a constant $C \in \mathbb{R}$ such that $\varphi + C \ge 0$ a.e. on $[a,b]$ (for instance, take $C \ge -\operatorname{ess\,inf}_{[a,b]} \varphi$). Now, apply the classical Steffensen inequality to $\varphi + C$ and h to obtain

$$\int_{b-\ell}^{b} (\varphi + C)dx \le \int_a^b (\varphi + C)hdx \le \int_a^{a+\ell} (\varphi + C)dx.$$

Since

$$\int_{b-\ell}^{b} Cdx = \ell C = \int_a^b Chdx = \int_a^{a+\ell} Cdx,$$

the constant C cancels, yielding (4).

We now apply (4) to $\varphi = f'$ and $h = g$. Since f' is (essentially) decreasing and $0 \le g \le 1$, letting $\ell = k = \int_a^b g(x)dx$ in (4) gives

$$\int_{b-k}^{b} f'(x)dx \le \int_a^b f'(x)g(x)dx \le \int_a^{a+k} f'(x)dx. \tag{5}$$

Because f is absolutely continuous, we may rewrite the endpoint integrals as

$$\int_{b-k}^{b} f'(x)dx = f(b) - f(b-k), \qquad \int_a^{a+k} f'(x)dx = f(a+k) - f(a).$$

Substituting into (5) yields

$$f(b) - f(b-k) \le \int_a^b f'(x)g(x)dx \le f(a+k) - f(a). \tag{6}$$

Finally, using the lower bound in (6) in (3) gives the upper bound:

$$\int_{(a,b]} fdg \le f(b)g(b) - f(a)g(a) - \big(f(b) - f(b-k)\big) = f(b-k) - f(b)\big(1 - g(b)\big) - f(a)g(a).$$

Similarly, using the upper bound in (6) in (3) gives the lower bound:

$$\int_{(a,b]} fdg \ge f(b)g(b) - f(a)g(a) - \big(f(a+k) - f(a)\big) = f(b)g(b) + f(a)\big(1 - g(a)\big) - f(a+k).$$

This yields (2). □

2.2. Generalization to finite measures

Theorem 3. *Let $a < b$ and let $w : [a,b] \to (0,\infty)$ be integrable. Define the finite measure and the total mass quantity*

$$d\mu(x) = w(x)dx, \qquad W := \mu([a,b]) = \int_a^b w(x)dx.$$

Assume that:

- *$f : [a,b] \to [0,\infty)$ absolutely continuous and $D_\mu f(x) := \frac{f'(x)}{w(x)}$ is (essentially) decreasing and essentially bounded on $[a,b]$,*
- *$g \in BV([a,b])$ and $0 \le g \le 1$ a.e. on $[a,b]$.*

Set

$$k := \int_a^b g(x)d\mu(x) = \int_a^b g(x)w(x)dx \in [0, W].$$

Let $x_-, x_+ \in [a,b]$ be defined by the conditions

$$\int_a^{x_-} w(x)dx = k, \qquad \int_{x_+}^b w(x)dx = k,$$

(equivalently, $\mu([a,x_-]) = k$ and $\mu([x_+,b]) = k$). Then the following inequalities hold:

$$f(x_+) - f(b)\big(1-g(b)\big) - f(a)g(a) \geq \int_a^b f dg \geq f(b)g(b) + f(a)\big(1-g(a)\big) - f(x_-). \tag{7}$$

Proof. We will utilize the Stieltjes integral approach once more. Since $f \in AC$ and $g \in BV([a,b])$, the integration by parts gives

$$\int_{(a,b]} f dg = f(b)g(b) - f(a)g(a) - \int_{[a,b)} g df. \tag{8}$$

The measure df is absolutely continuous with respect to the Lebesgue measure, hence $df(x) = f'(x)dx$. Using w, we can now represent df in terms of $d\mu$:

$$df(x) = \frac{f'(x)}{w(x)} w(x)dx = D_\mu f(x) d\mu(x).$$

Substitute this into (8) to obtain

$$\int_{(a,b]} f dg = f(b)g(b) - f(a)g(a) - \int_a^b g(x) D_\mu f(x) d\mu(x). \tag{9}$$

We now introduce the change of variables. Define the map

$$\Phi(x) := \int_a^x w(t)dt, \qquad x \in [a,b],$$

so that $\Phi(a) = 0$, $\Phi(b) = W := \int_a^b w$, and $d(\Phi(x)) = w(x)dx = d\mu(x)$. Assuming $w > 0$ a.e., Φ is strictly increasing and has an (a.e.) inverse $Q : [0,W] \to [a,b]$.

Define

$$\widetilde{g}(u) := g(Q(u)), \qquad \widetilde{h}(u) := D_\mu f(Q(u)) = \frac{f'(Q(u))}{w(Q(u))}, \qquad u \in [0,W].$$

Then $0 \leq \widetilde{g} \leq 1$ and, by change of variables $u = \Phi(x)$,

$$k = \int_a^b g d\mu = \int_0^W \widetilde{g}(u)du.$$

Moreover, since $D_\mu f$ is decreasing in x and Q is increasing, $\widetilde{h}$ is decreasing in u.

Now apply the classical Steffensen inequality in $[0,W]$ to the decreasing function $\widetilde{h}$ and the function $\widetilde{g} \in [0,1]$:

$$\int_{W-k}^W \widetilde{h}(u)du \leq \int_0^W \widetilde{h}(u)\widetilde{g}(u)du \leq \int_0^k \widetilde{h}(u)du. \tag{10}$$

Note that in case $\widetilde{h}$ attains negative values, we can repeat the same argument as we did in AC/BV setting to remove the sign restriction.

We now convert each term back to x. First,

$$\int_0^W \widetilde{h}(u)\widetilde{g}(u)du = \int_a^b D_\mu f(x) g(x) d\mu(x) = \int_a^b g D_\mu f d\mu.$$

Next, define $x_- \in [a,b]$ and $x_+ \in [a,b]$ by

$$\int_a^{x_-} w(x)dx = k, \qquad \int_{x_+}^b w(x)dx = k,$$

or equivalently, $\Phi(x_-) = k$ and $\Phi(x_+) = W - k$. Then

$$\int_0^k \widetilde{h}(u)du = \int_a^{x_-} D_\mu f d\mu = \int_a^{x_-} f'(x)dx = f(x_-) - f(a),$$

and similarly

$$\int_{W-k}^W \widetilde{h}(u)du = \int_{x_+}^b D_\mu f d\mu = \int_{x_+}^b f'(x)dx = f(b) - f(x_+).$$

Thus (10) becomes

$$f(b) - f(x_+) \le \int_a^b g D_\mu f d\mu \le f(x_-) - f(a). \tag{11}$$

We now use the upper bound in (11) in (9) to obtain

$$\int_{(a,b]} f dg \ge f(b)g(b) - f(a)g(a) - (f(x_-) - f(a)) = f(b)g(b) + f(a)(1 - g(a)) - f(x_-).$$

Similarly, we use the lower bound in (11) for

$$\int_{(a,b]} f dg \le f(b)g(b) - f(a)g(a) - (f(b) - f(x_+)) = f(x_+) - f(b)(1 - g(b)) - f(a)g(a).$$

This gives (7). □

Remark 1. If we additionally assume that g is differentiable (or just absolutely continuous), we can write

$$\begin{aligned}\int_a^b f dg = \int_a^b f(x)g'(x)dx &= \int_a^b f(x)\frac{g'(x)}{w(x)}w(x)dx \\ &= \int_a^b f(x)\frac{g'(x)}{w(x)}d\mu(x),\end{aligned}$$

and the final form can be utilized in (7).

3. Numerical examples

In this section we provide two numerical examples to our results. These examples are chosen in such a way they meet the assumptions of our theorems but they do not meet the conditions of [11], or where the inequality cannot be utilized.

Example 1. Let $[a, b] = [0, 1]$ and define

$$f(x) = \sqrt{x}, \qquad g(x) = \mathbf{1}_{[1/2,1]}(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

Then $f \ge 0$ and f is absolutely continuous on $[0, 1]$. Moreover,

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad x > 0,$$

is (essentially) decreasing and belongs to $L^1([0, 1])$. The function g satisfies $0 \le g \le 1$ and $g \in BV([0, 1])$, but g is not differentiable (it has a jump at $x = \frac{1}{2}$). Also, f is not twice differentiable on $[0, 1]$ (due to the singularity of f' at 0). Hence the assumptions of inequality 1 are not met.

We compute

$$k = \int_0^1 g(x)dx = \int_{1/2}^1 1dx = \frac{1}{2}.$$

Since g has a single jump of size $\Delta g(\frac{1}{2}) = 1$ at $x = \frac{1}{2}$ and is constant elsewhere, the Stieltjes integral reduces to

$$\int_{(0,1]} f dg = f\left(\tfrac{1}{2}\right) \Delta g\left(\tfrac{1}{2}\right) = \sqrt{\tfrac{1}{2}}.$$

The bounds in Theorem 2 become (here $g(1) = 1$ and $g(0) = 0$)

$$\text{Upper bound} = f(1-k) - f(1)(1-g(1)) - f(0)g(0) = f\left(\tfrac{1}{2}\right) = \sqrt{\tfrac{1}{2}},$$

and

$$\text{Lower bound} = f(1)g(1) + f(0)(1-g(0)) - f(0+k) = 1 - \sqrt{\tfrac{1}{2}}.$$

Therefore

$$\sqrt{\tfrac{1}{2}} \geq \sqrt{\tfrac{1}{2}} = \int_{(0,1]} f dg \geq 1 - \sqrt{\tfrac{1}{2}},$$

and numerical values are

$$0.70710678 \ \geq \ 0.70710678 \ \geq \ 0.29289322.$$

Example 2. Let $[a,b] = [0,1]$ and consider the non-Lebesgue finite measure $d\mu = w dx$ with

$$w(x) = 2x+1, \qquad d\mu(x) = (2x+1)dx.$$

Then

$$W = \mu([0,1]) = \int_0^1 (2x+1)dx = 2.$$

Let

$$f(x) = 1 - x^2 \quad (\text{concave}, \mathcal{C}^2,\ f \geq 0), \qquad g(x) = x \quad (0 \leq g \leq 1,\ g' \equiv 1).$$

We have

$$D_\mu f(x) = \frac{f'(x)}{w(x)} = \frac{-2x}{2x+1}.$$

Note that $D_\mu f$ is decreasing on $[0,1]$ since

$$\frac{d}{dx}\left(\frac{-2x}{2x+1}\right) = \frac{-2}{(2x+1)^2} < 0.$$

Next,

$$k = \int_0^1 g(x)d\mu(x) = \int_0^1 x(2x+1)dx = \frac{7}{6} \in [0, W].$$

Define $\Phi(x) := \int_0^x w(t)dt = x^2 + x$. Then $x_-, x_+ \in [0,1]$ are determined by

$$\Phi(x_-) = k = \frac{7}{6}, \qquad \Phi(x_+) = W - k = \frac{5}{6}.$$

Solving the quadratic equations $x^2 + x = \frac{7}{6}$ and $x^2 + x = \frac{5}{6}$ yields

$$x_- = \frac{-1 + \sqrt{1 + \frac{28}{6}}}{2} \approx 0.69023807, \qquad x_+ = \frac{-1 + \sqrt{1 + \frac{20}{6}}}{2} \approx 0.54083300.$$

Then,

$$\int_0^1 f(x)\frac{g'(x)}{w(x)}d\mu(x) = \int_0^1 f(x)\frac{1}{w(x)}w(x)dx = \int_0^1 (1-x^2)dx = \frac{2}{3}.$$

The bounds in (7) are

$$\text{Upper bound} = f(x_+) = 1 - x_+^2 \approx 0.70749967,$$

and (since $f(1) = 0$, $g(1) = 1$, $f(0) = 1$, $g(0) = 0$)

$$\text{Lower bound} = f(1)g(1) + f(0)\big(1 - g(0)\big) - f(x_-) = 1 - (1 - x_-^2) = x_-^2 \approx 0.47642860.$$

Hence

$$0.70749967 \geq \frac{2}{3} \geq 0.47642860.$$

Note that Theorem 1 is formulated for the Lebesgue measure dx and does not involve the μ-dependent endpoints $x_\pm$ defined via $\mu([0, x_-]) = k$ and $\mu([x_+, 1]) = k$.

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