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Two new results related to the Steffensen integral inequality

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Abstract: This article makes a contribution to the ongoing development of the Steffensen integral inequality by presenting two new results. The first result generalizes the classical Steffensen integral inequality by introducing an additional function that combines key aspects of the Steffensen and Chebyshev integral inequalities. The second result presents a concave integral inequality derived using integration techniques. Numerical examples are provided to demonstrate the validity and application of the results.

Keywords: Steffensen integral inequality, Chebyshev integral inequality, concave integral inequality, integration by parts

MSC: 26D15.

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1. Introduction

Integral inequalities are a fundamental component of mathematical analysis. They are powerful tools for solving problems where an exact analytical solution is either impossible or unnecessary. They play a particularly crucial role in establishing convergence results, estimating errors and solving optimization problems. Well-known examples include the Cauchy-Schwarz, Chebyshev, Jensen, Grönwall, Hardy, Hilbert, Hölder, Minkowski, and Steffensen integral inequalities. Comprehensive discussions of these classical results can be found in the references [1–5].

Of these, the Steffensen integral inequality, introduced in [6], is particularly important. It provides lower and upper bounds for the integral of the product of two functions, one of which is monotonic. The resulting bounds have a distinctive structure because they depend on the integral of one of the functions over the considered interval. This inequality has numerous applications in fields such as numerical analysis and approximation theory. In recent years, it has inspired many generalizations, refinements and adaptations in various mathematical contexts. Notable contributions can be found in [7–17]. Furthermore, the monograph presented in [18] offers a comprehensive and up-to-date account of current research developments on this topic, accompanied by detailed proofs and illustrative examples.

In this article, we make a contribution to the ongoing development of the Steffensen integral inequality by presenting two distinct results. The first introduces an additional function into the classical Steffensen integral inequality framework, yielding a generalized version. The proof incorporates key features of the Steffensen and Chebyshev integral inequalities. The second result establishes a new concave integral inequality derived from the Steffensen integral inequality via integration techniques. Together, these two results demonstrate how the Steffensen integral inequality can be extended and adapted to produce new findings in mathematical analysis. Several numerical examples are also provided to illustrate the validity and applicability of the results.

The remainder of this paper is organized as follows: §2 presents preliminary material, including precise statements of the Steffensen and Chebyshev integral inequalities. §3 is devoted to the first main result, and §4 to the second. §5 contains concluding remarks.

2. Preliminaries

Because of their central role in our study, the theorems below formally state the Steffensen and Chebyshev integral inequalities.

Theorem 1 (Steffensen integral inequality). *Let $a, b \in \mathbb{R}$ with $b > a$, $f : [a, b] \to [0, +\infty)$ be a decreasing integrable function and $g : [a, b] \to [0, 1]$ be an integrable function. Let us set*

$$k = \int_a^b g(x)dx.$$

Then the following inequalities (or double inequality) hold:

$$\int_a^{a+k} f(x)dx \geq \int_a^b f(x)g(x)dx \geq \int_{b-k}^b f(x)dx.$$

We recall that the essential on the Steffensen integral inequality can be found in [18].

Theorem 2 (Chebyshev integral inequality). *Let $a, b \in \mathbb{R}$ with $b > a$ and $f, g : [a, b] \to \mathbb{R}$ be integrable functions that are both increasing or both decreasing. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right).$$

The Chebyshev integral inequality is a key framework for the study of monotonic functions in integral form. Despite its importance, the relationship between this inequality and other classical inequalities, notably the Steffensen integral inequality, remains largely unexplored. This study aims to address this gap to some extent.

3. First result

The theorem below provides a new integral inequality that combines the essential features of the Chebyshev and Steffensen integral inequalities.

Theorem 3. *Let $a, b \in \mathbb{R}$ with $b > a$, $f, g : [a, b] \to [0, +\infty)$ be two decreasing integrable functions and $h : [a, b] \to [0, 1]$ be an integrable function. Let us set*

$$k = \int_a^b h(x)dx.$$

Then the following inequalities hold:

$$\begin{aligned}\left(\int_a^{a+k} f(x)g(x)dx\right)\left(\int_a^b h(x)dx\right) &\geq \left(\int_a^b f(x)h(x)dx\right)\left(\int_a^b g(x)h(x)dx\right)\\ &\geq \left(\int_{b-k}^b f(x)dx\right)\left(\int_{b-k}^b g(x)dx\right).\end{aligned}$$

Proof. Since f and g are decreasing and integrable on $[a, b]$, we can apply the Chebyshev integral inequality to these functions on the interval $[a, a+k]$. This gives

$$\frac{1}{(a+k)-a}\int_a^{a+k} f(x)g(x)dx \geq \left(\frac{1}{(a+k)-a}\int_a^{a+k} f(x)dx\right)\left(\frac{1}{(a+k)-a}\int_a^{a+k} g(x)dx\right),$$

which is equivalent to

$$\int_a^{a+k} f(x)g(x)dx \geq \frac{1}{k}\left(\int_a^{a+k} f(x)dx\right)\left(\int_a^{a+k} g(x)dx\right),$$

so that, by the non-negativity and definition of k,

$$\left(\int_a^{a+k} f(x)g(x)dx\right)\left(\int_a^b h(x)dx\right) \geq \left(\int_a^{a+k} f(x)dx\right)\left(\int_a^{a+k} g(x)dx\right). \tag{1}$$

The next step in our development is to apply the Steffensen integral inequality to f and g. This and the non-negativity of f, g and h give

$$\left(\int_a^{a+k} f(x)dx\right)\left(\int_a^{a+k} g(x)dx\right) \geq \left(\int_a^b f(x)h(x)dx\right)\left(\int_a^b g(x)h(x)dx\right) \geq \left(\int_{b-k}^b f(x)dx\right)\left(\int_{b-k}^b g(x)dx\right). \tag{2}$$

Combining Eqs. (1) and (2), we get

$$\left(\int_a^{a+k} f(x)g(x)dx\right)\left(\int_a^b h(x)dx\right) \geq \left(\int_a^b f(x)h(x)dx\right)\left(\int_a^b g(x)h(x)dx\right) \geq \left(\int_{b-k}^b f(x)dx\right)\left(\int_{b-k}^b g(x)dx\right).$$

This is the stated inequality, concluding the proof. □

If we set $g(x) = 1$, then Theorem 3 coincides with the Steffensen integral inequality. Therefore, Theorem 3 may be viewed as a natural functional generalization.

For a numerical example, let us set $a = 0$, $b = 1$, $f(x) = 1/(1+x^2)$, $g(x) = e^{-x}$ and $h(x) = x$. Then the conditions of Theorem 3 are satisfied, with

$$k = \int_a^b h(x)dx = \int_0^1 xdx = \frac{1}{2},$$

and we have

$$\left(\int_a^{a+k} f(x)g(x)dx\right)\left(\int_a^b h(x)dx\right) = \left(\int_0^{1/2} \frac{e^{-x}}{1+x^2}dx\right)\left(\int_0^1 xdx\right) \approx 0.1841,$$
$$\left(\int_a^b f(x)h(x)dx\right)\left(\int_a^b g(x)h(x)dx\right) = \left(\int_0^1 \frac{x}{1+x^2}dx\right)\left(\int_0^1 xe^{-x}dx\right) \approx 0.0915$$

and

$$\left(\int_{b-k}^b f(x)dx\right)\left(\int_{b-k}^b g(x)dx\right) = \left(\int_{1/2}^1 \frac{1}{1+x^2}dx\right)\left(\int_{1/2}^1 e^{-x}dx\right) \approx 0.0767.$$

It is evident that $0.1841 > 0.0915 > 0.0767$, which confirms the inequality stated in Theorem 3.

For another numerical example, let us set $a = 0$, $b = 1$, $f(x) = 1/(1+x)$, $g(x) = e^{-x^2}$ and $h(x) = e^{-x}$. Then the conditions of Theorem 3 are satisfied, with

$$k = \int_a^b h(x)dx = \int_0^1 e^{-x}dx = 1 - e^{-1},$$

and we have

$$\left(\int_a^{a+k} f(x)g(x)dx\right)\left(\int_a^b h(x)dx\right) = \left(\int_0^{1-e^{-1}} \frac{e^{-x^2}}{1+x}dx\right)\left(\int_0^1 e^{-x}dx\right) \approx 0.2771,$$
$$\left(\int_a^b f(x)h(x)dx\right)\left(\int_a^b g(x)h(x)dx\right) = \left(\int_0^1 \frac{e^{-x}}{1+x}dx\right)\left(\int_0^1 e^{-x^2}e^{-x}dx\right) \approx 0.2349,$$

and

$$\left(\int_{b-k}^b f(x)dx\right)\left(\int_{b-k}^b g(x)dx\right) = \left(\int_{e^{-1}}^1 \frac{1}{1+x}dx\right)\left(\int_{e^{-1}}^1 e^{-x^2}dx\right) \approx 0.1500.$$

It is evident that $0.2771 > 0.2349 > 0.1500$, which confirms the inequality stated in Theorem 3.

Numerous other examples can be constructed to further illustrate the result.

4. Second result

The theorem below provides a new concave integral inequality that uses the features of the Steffensen integral inequality.

Theorem 4. *Let $a, b \in \mathbb{R}$ with $b > a$, $f : [a, b] \to [0, +\infty)$ be a twice differentiable concave function and $g : [a, b] \to [0, 1]$ be a differentiable integrable function. Then the following inequalities hold:*

$$
\begin{aligned}
f\left(b - \int_a^b g(x)dx\right) - f(b)(1 - g(b)) - f(a)g(a) &\geq \int_a^b f(x)g'(x)dx \\
&\geq f(b)g(b) + f(a)(1 - g(a)) - f\left(a + \int_a^b g(x)dx\right).
\end{aligned}
$$

Proof. First, we transpose the problem of working with f' instead of g' into the main integral. This is possible thanks to an integration by parts, which gives

$$
\begin{aligned}
\int_a^b f(x)g'(x)dx &= [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx \\
&= f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx. \qquad (3)
\end{aligned}
$$

The next step in our development is to apply the Steffensen integral inequality to f' which is decreasing since f is concave. Setting $k = \int_a^b g(x)dx$, this gives

$$
\begin{aligned}
f\left(a + \int_a^b g(x)dx\right) - f(a) = \int_a^{a+k} f'(x)dx &\geq \int_a^b f'(x)g(x)dx \\
&\geq \int_{b-k}^b f'(x)dx = f(b) - f\left(b - \int_a^b g(x)dx\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
f\left(b - \int_a^b g(x)dx\right) - f(b) &\geq -\int_a^b f'(x)g(x)dx \\
&\geq -f\left(a + \int_a^b g(x)dx\right) + f(a). \qquad (4)
\end{aligned}
$$

Combining Eqs. (3) and (4), we get

$$
\begin{aligned}
f(b)g(b) - f(a)g(a) + f\left(b - \int_a^b g(x)dx\right) - f(b) &\geq \int_a^b f(x)g'(x)dx \\
&\geq f(b)g(b) - f(a)g(a) - f\left(a + \int_a^b g(x)dx\right) + f(a),
\end{aligned}
$$

which can be rearranged as follows:

$$
\begin{aligned}
f\left(b - \int_a^b g(x)dx\right) - f(b)(1 - g(b)) - f(a)g(a) &\geq \int_a^b f(x)g'(x)dx \\
&\geq f(b)g(b) + f(a)(1 - g(a)) - f\left(a + \int_a^b g(x)dx\right).
\end{aligned}
$$

This is the stated inequality, concluding the proof. □

For a numerical example, let us set $a = 0$, $b = 1$, $f(x) = \log(1+x)$ and $g(x) = e^{-x}$. Then the conditions of Theorem 4 are satisfied, and we have

$$f\left(b - \int_a^b g(x)dx\right) - f(b)(1-g(b)) - f(a)g(a) = \log\left(2 - \int_0^1 e^{-x}dx\right) - \log(2)(1-e^{-1}) \approx -0.1248,$$

$$\int_a^b f(x)g'(x)dx = -\int_0^1 \log(1+x)e^{-x}dx \approx -0.2084,$$

and

$$f(b)g(b) + f(a)(1-g(a)) - f\left(a + \int_a^b g(x)dx\right) = \log(2)e^{-1} - \log\left(2 + \int_0^1 e^{-x}dx\right) \approx -0.7127.$$

It is evident that $-0.1248 > -0.2084 > -0.7127$, which confirms the inequality stated in Theorem 4.

For another numerical example, let us set $a = 0$, $b = 1$, $f(x) = \sqrt{x}$ and $g(x) = 1/(1+x^2)$. Then the conditions of Theorem 4 are satisfied, and we have

$$f\left(b - \int_a^b g(x)dx\right) - f(b)(1-g(b)) - f(a)g(a) = \sqrt{1 - \int_0^1 \frac{1}{1+x^2}dx} - \sqrt{1}\left(1 - \frac{1}{1+1^2}\right) \approx -0.0367,$$

$$\int_a^b f(x)g'(x)dx = -\int_0^1 \frac{2x\sqrt{x}}{(1+x^2)^2}dx \approx -0.3669,$$

and

$$f(b)g(b) + f(a)(1-g(a)) - f\left(a + \int_a^b g(x)dx\right) = \sqrt{1}\frac{1}{1+1^2} - \sqrt{\int_0^1 \frac{1}{1+x^2}dx} \approx -0.3862.$$

It is evident that $-0.0367 > -0.3669 > -0.3862$, which confirms the inequality stated in Theorem 4. Numerous other examples can be constructed to further illustrate the result.

5. Conclusion

In conclusion, this article has built upon the classical Steffensen integral inequality by establishing a generalized form involving an additional function, and by deriving a new concave integral inequality using integration techniques. These results demonstrate the versatility of the Steffensen integral inequality framework and its potential for generating further analytical developments. Future research could explore multidimensional extensions, fractional versions or applications of these inequalities in areas such as numerical analysis and optimization.

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