LAKHDAR T. RACHDI, SAMIA SGHAIER

Abstract. We define fractional transforms $R_\mu$ and $H_\mu$, $\mu > 0$ on the space $\mathbb{R} \times \mathbb{R}^n$. First, we study these transforms on regular function spaces and we establish that these operators are topological isomorphisms and we give the inverse operators as integro differential operators. Next, we study the $L^p$-boundedness of these operators. Namely, we give necessary and sufficient condition on the parameter $\mu$ for which the transforms $R_\mu$ and $H_\mu$ are bounded on the weighted spaces $L^p([0, +\infty[\times\mathbb{R}^n, r^{2\alpha}dr \otimes dx)$ and we give their norms.

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1. Introduction

Let $D_j$, $1 \leq j \leq n$, and $\Xi_\mu$, $\mu > 0$, be the singular partial differential operators defined by

$$
\begin{align*}
D_j &= \frac{\partial}{\partial x_j} \\
\Xi_\mu &= (\frac{\partial}{\partial r})^2 + \frac{2\mu}{r} \frac{\partial}{\partial r} + \sum_{j=1}^{n} (\frac{\partial}{\partial x_j})^2; (r, x) \in [0, +\infty[\times\mathbb{R}^n, \mu > 0.
\end{align*}
$$

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Corresponding Author
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Ξ₀ is a Bessel-Laplace operator. When \( \mu = \frac{n-1}{2}; n \in \mathbb{N}^* \), \( \Xi_{n-1/2} \) is the Laplacian operator on \( \mathbb{R}^n \times \mathbb{R}^n \) when acting on the functions \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C} \), that are radial with respect to the first variable.

For every \((\lambda_0, \lambda) \in \mathbb{C} \times \mathbb{C}^n\), the system

\[
\begin{cases}
D_j u(r, x) = -i\lambda_j u(r, x), 1 \leq j \leq n \\
\Xi_{\mu} u(r, x) = -(\lambda_0^2 + \lambda^2) u(r, x) \\
u(0, 0) = 1, \frac{\partial}{\partial r} u(0, x) = 0, \forall x \in \mathbb{R}^n
\end{cases}
\]

admits a unique solution given by

\[
\psi_{\lambda_0, \lambda}(r, x) = j_{\mu-1/2}(r \lambda_0) e^{-i\langle \lambda | x \rangle},
\]

where \( \lambda^2 = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 \), \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \)

\( \langle \lambda | x \rangle = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \)

\( j_{\mu-1/2} \) is the modified Bessel function given by

\[
j_{\mu-1/2}(s) = 2^{\mu-1/2} \Gamma(\mu + \frac{1}{2}) \frac{J_{\mu-1/2}(s)}{s^{\mu-1/2}}
\]

\[
= \frac{\Gamma(\mu + \frac{1}{2})}{\sqrt{2\pi}} \Gamma(\mu + 1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{s^2}{2} \right)^{2k}
\]

\[
= \frac{2 \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi}} \Gamma(\mu + 1) \int_0^1 (1 - t^2)^{\mu-1} \cos(\lambda | x |) dt,
\]

and \( J_{\mu-1/2} \) is the Bessel function of first kind and index \( \mu - \frac{1}{2} \) ([11] [12] [13] [14]).

The eigenfunction \( \psi_{\lambda_0, \lambda} \) allows us to define the Fourier transform \( \tilde{F}_{\mu-1/2} \) connected with the operators \( D_j \), \( 1 \leq j \leq n \) and \( \Xi_{\mu} \) by

\[
\tilde{F}_{\mu-1/2}(f)(\lambda_0 \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \psi_{\lambda_0, \lambda}(r, x) d\nu_{\mu}(r, x)
\]

\[
= \int_0^\infty \int_{\mathbb{R}^n} f(r, x) j_{\mu-1/2}(r \lambda_0) e^{-i\langle \lambda | x \rangle} d\nu_{\mu}(r, x),
\]

where \( f \) is any integrable function on \([0, +\infty)[\times \mathbb{R}^n\) with respect to the measure

\[
d\nu_{\mu}(r, x) = \frac{e^{2\mu dr}}{2^{\mu-1/2} \Gamma(\mu + 1/2)} \otimes \frac{dx}{(2\pi)^{n/2}}.
\]

Many harmonic analysis results related to the Fourier transform \( \tilde{F}_{\mu-1/2} \) are established ([15] [16] [17] [18] [19]).

Also, many uncertainty principles have been checked for this transform ([20] [21] [22] [23]).
On the other hand, the eigenfunction $\psi_{0,\lambda}$ admits the Poisson integral representation

$$
\psi_{0,\lambda}(r, x) = \frac{2 \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} r^{1-2\mu} \int_0^r (r^2 - t^2)^{-\mu-1} \cos(\lambda_0 t) e^{-i(\lambda_1 x)} dt
$$

(4)

Using the relation (4), we define the fractional transform $\mathcal{R}_\mu$ on $C_c(\mathbb{R} \times \mathbb{R}^n)$ (the space of continuous functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable) by

$$
\mathcal{R}_\mu(f)(r, x) = \frac{2 \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} r^{1-2\mu} \int_0^r (r^2 - t^2)^{-\mu-1} f(t, x) dt; \quad (r, x) \in \mathbb{R} \times \mathbb{R}^n.
$$

(5)

This involves in particular, that

$$
\psi_{0,\lambda}(r, x) = \mathcal{R}_\mu(\cos(\lambda_0 r) e^{-i(\lambda_1 \cdot)}) (r, x),
$$

(6)

which gives the mutual connection between the functions $\psi_{0,\lambda}$ and $\cos(\lambda_0 r) e^{-i(\lambda_1 \cdot)}$.

On the other hand, we shall prove in the next section that for every integrable function $f$ on $[0, +\infty] \times \mathbb{R}^n$ with respect to the measure $d\nu_\mu(r, x)$ and for every bounded function $g$ on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, we have the duality relation

$$
\int_0^\infty \int_{\mathbb{R}^n} f(r, x) \mathcal{R}_\mu(g)(r, x) d\nu_\mu(r, x) = \int_0^\infty \int_{\mathbb{R}^n} g(r, x) \mathcal{H}_\mu(f)(r, x) dm(r, x)
$$

where $dm$ is the Lebesgue measure on $[0, +\infty] \times \mathbb{R}^n$,

$$
dm(r, x) = \frac{\sqrt{2\pi}}{\pi} dr \otimes \frac{dx}{(2\pi)^{\frac{n}{2}}}. \tag{8}
$$

$\mathcal{H}_\mu$ is the fractional transform defined by

$$
\mathcal{H}_\mu(f)(r, x) = \frac{1}{2^\mu \Gamma(\mu)} \int_r^\infty (t^2 - r^2)^{-\mu-1} f(t, x) 2t dt.
$$

The relations (2), (6) and (7) show that for all integrable functions $f, g$ on $[0, +\infty] \times \mathbb{R}^n$ with respect to the measure $d\nu_\mu(r, x)$, we have

$$
\tilde{\mathcal{F}}_{-\frac{1}{2}}(f) = \Lambda_0 \mathcal{H}_\mu(f) \tag{9}
$$

and

$$
\mathcal{H}_\mu(f * g) = \mathcal{H}_\mu(f) * \mathcal{H}_\mu(g), \tag{10}
$$
where $\Lambda$ is the usual Fourier transform defined by
\[
\Lambda(f)(\lambda_0, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \cos(\lambda_0 r) e^{-i\langle \lambda | x \rangle} dm(r, x),
\]
\([\lambda_0, \lambda] \in \mathbb{R} \times \mathbb{R}^n\).

$\ast$ is the convolution product associated with the Fourier transform $\widetilde{F}_\mu$, $\ast_o$ is the usual convolution product defined by
\[
f \ast_o g(r, x) = \int_0^\infty \int_{\mathbb{R}^n} f(s, y) \sigma_{r,x}(g)(s, -y) dm(s, y)
\]
and $\sigma_{r,x}$ is the usual translation operator given by
\[
\sigma_{r,x}(f)(s, y) = \frac{1}{2} \left( f(r + s, x + y) + f(|r - s|, x + y) \right).
\]
(11)

Our purpose in this work is to study the fractional transforms $R_\mu$ and $H_\mu$ in two ways.

In the second section, we will prove that the operator $R_\mu$ is a topological isomorphism from $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ (the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable) onto itself and we give the inverse operator $R^{-1}_\mu$ as integro-differential operator.

Next, we show that the fractional transform $H_\mu$ can be extended to $\mu \in \mathbb{R}$ and that for every $\mu \in \mathbb{R}$, $H_\mu$ is a topological isomorphism from the Schwartz’s space $\mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n)$ (the subspace of $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ consisting of rapidly decreasing functions together with all their derivatives) onto itself whose inverse operator is $H^{-1}_\mu = H^{-\mu}$.

The precedent results imply in particular that $R_\mu$ and $H_\mu$ are transmutation operators of $D_j$, $1 \leq j \leq n$, and $\Xi_\mu$ to $D_j$, $1 \leq j \leq n$ and $\Delta$, where
\[
\Delta = \left( \frac{\partial}{\partial r} \right)^2 + \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} \right)^2.
\]

That is, for every $f \in \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$
\[
D_j R_\mu(f) = R_\mu D_j(f), \quad 1 \leq j \leq n
\]
\[
\Xi_\mu R_\mu(f) = R_\mu \Delta(f),
\]
and for every $f \in \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n)$
\[
D_j H_\mu(f) = H_\mu D_j(f), \quad 1 \leq j \leq n
\]
\[
\Delta H_\mu(f) = H_\mu \Xi_\mu(f).
\]

The third section contains the main results of this paper. In fact, we study the $L^p$–boundedness of the operators $R_\mu$ and $H_\mu$ on the weighted spaces $L^p([0, +\infty[ \times \mathbb{R}^n, r^{2\mu} dr \otimes dx), p \in [1, +\infty[$. We recall in this context, that studying the $L^p$–boundedness of integral transforms connected with differential systems is an interesting subject because knowing the range of parameters $\mu$, $p$ for which an operator is bounded on Lebesgue space gives quantitative information about
the rate of growth of the transformed functions \((15, 16, 17)\).

In this work, we give necessary and sufficient conditions on the parameters \(\mu, a, p\) for which the operator \(R_\mu\) (respectively \(H_\mu\)) satisfies

\[ ||R_\mu(f)||_p,a \leq C_{p,a,\mu} ||f||_p,a, \quad (12) \]

respectively

\[ ||H_\mu(f)||_p,a \leq D_{p,a,\mu} ||r^{2\alpha}f||_p,a. \quad (13) \]

Moreover, we give the best (the smallest) constants \(C_{p,a,\mu}\) and \(D_{p,a,\mu}\) that satisfy the relations (12) and (13).

2. Fractional transforms

2.1. The fractional transform \(R_\mu\). The space \(E_c(\mathbb{R} \times \mathbb{R}^n)\) is equipped with the topology generated by the family of semi-norms

\[ P_{m,k}(f) = \sup_{||r,x|| \leq m} |D^\alpha(f)(r,x)|, \quad (m,k) \in \mathbb{N}^2. \]

and the distance

\[ d(f,g) = \sum_{m,k=0}^{+\infty} \left( \frac{1}{2} \right)^{m+k} \frac{P_{m,k}(f-g)}{1 + P_{m,k}(f-g)}. \]

Lemma 2.1. i. For every \(\mu > 0\), the transform \(R_\mu\) is continuous from \(E_c(\mathbb{R} \times \mathbb{R}^n)\) into itself.

ii. The operator \(\frac{\partial}{\partial r^2} = \frac{1}{r} \frac{\partial}{\partial r}\) is continuous from \(E_c(\mathbb{R} \times \mathbb{R}^n)\) into itself.

Proof. i. For every \(f \in E_c(\mathbb{R} \times \mathbb{R}^n)\), we have

\[ R_\mu(f)(r,x) = \frac{2\Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\mu)} \int_0^1 (1 - t^2)^{\mu-1} f(tr,x)dt, \]

this shows that the function \(R_\mu(f)\) belongs to the space \(E_c(\mathbb{R} \times \mathbb{R}^n)\). Moreover, for every \((\alpha_0, \alpha) \in \mathbb{N} \times \mathbb{N}^n\)

\[ D^{(\alpha_0,\alpha)}(R_\mu(f))(r,x) = \frac{2\Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\mu)} \int_0^1 (1 - t^2)^{\mu-1} t^{\alpha_0} D^{(\alpha_0,\alpha)}(f)(tr,x)dt, \]

thus, for every \((m,k) \in \mathbb{N}^2, P_{m,k}(R_\mu(f)) \leq P_{m,k}(f)\).

ii. For every \(f \in E_c(\mathbb{R} \times \mathbb{R}^n)\)

\[ \frac{\partial}{\partial r^2}(f)(r,x) = \int_0^1 \frac{\partial^2 f}{\partial x^2}(rt,x)dt. \]

Hence, the function \(\frac{\partial}{\partial r^2}(f)\) belongs to the space \(E_c(\mathbb{R} \times \mathbb{R}^n)\) and for every \((\alpha_0, \alpha) \in \mathbb{N} \times \mathbb{N}^n\)

\[ D^{(\alpha_0,\alpha)}\left(\frac{\partial}{\partial r^2}f\right)(r,x) = \int_0^1 t^{\alpha_0} D^{(\alpha_0+2,\alpha)}(f)(rt,x)dt, \]
so, for every \((m, k) \in \mathbb{N}^2\), \(P_{m, k}(\frac{\partial}{\partial r^2} (f)) \leq P_{m, k+2}(f)\). \(\square\)

In the following, we shall prove that \(\mathcal{R}_\mu\) is a topological isomorphism from \(\mathscr{E}_e(\mathbb{R} \times \mathbb{R}^n)\) onto itself and we give the inverse operator. For this we need following notations:\n\r

\(r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\) is the space defined by \(r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n) = \{ f : \mathbb{R} \setminus \{0\} \times \mathbb{R}^n \rightarrow \mathbb{C}, f\) is even with respect to the first variable and \(f(r, x) = \tilde{f}^{2a} g(r, x), \ g \in \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\}\n\r

\(r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\) is equipped by the family of semi-norms:
\r

\[ E(r, x) = 2^{\alpha} (r^2 + t^2)^{\mu-1} t^{2a} g(t, x)dt, \ r > 0. \]
\r

**Proposition 2.2.** i. For every \(a > -\frac{1}{2}\), the operator \(\Box\) defined by
\r

\[ \Box (f)(r, x) = \frac{\partial}{\partial r} \left( \frac{f(r, x)}{r} \right) \]
\r

is continuous from \(r^{2(a+1)} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\) into \(r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\).

ii. The transform \(\tilde{\mathcal{R}}_\mu\) is continuous from \(r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\) into \(r^{2(a+\mu)} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\).

**Proof.** i. Let \(f \in r^{2(a+1)} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n); f(r, x) = r^{2a+2} g(r, x), g \in \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\)
\r

\[ \Box f(r, x) = r^{2a} \left( (2a + 1) g(r, x) + r \frac{\partial g}{\partial r}(r, x) \right). \]
\r

Since, the map \(g \rightarrow (2a + 1) g + r \frac{\partial g}{\partial r}\) is continuous from \(\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\) into itself, then, the function \(\Box (f)\) belongs to \(r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\).

Moreover, for every \((m, k) \in \mathbb{N}^2\)
\r

\[ \tilde{P}_{m, k, a}(\Box (f)) = P_{m, k}(2a + 1) g + r \frac{\partial g}{\partial r} \]
\r

\[ \leq CP_{m', k'}(g) = CP_{m', k', a+1}(f), \]
\r

where \(C\) is a constant.

ii. For every \(f \in r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n), f = r^{2a} g, g \in \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\) and \(a > -\frac{1}{2}\), the function
\r

\[ \tilde{\mathcal{R}}_\mu (f)(r, x) = \frac{2r}{2^{\mu} \Gamma(\mu)} \int_0^r (r^2 - t^2)^{\mu-1} t^{2a} g(t, x)dt \]
\r

\[ = \frac{2^{2a+2}}{2^{\mu} \Gamma(\mu)} \int_0^1 (1 - t^2)^{\mu-1} t^{2a} g(tr, x)dt \]
\r

belongs to the space \(r^{2(a+\mu)} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)\), and for every \((m, k) \in \mathbb{N}^2\)
\r

\[ \tilde{P}_{m, k, a+\mu}(\tilde{\mathcal{R}}_\mu (f)) = P_{m, k}(\frac{2}{2^{\mu} \Gamma(\mu)} \int_0^1 (1 - t^2)^{a-1} t^{2a} g(tr, x)dt) \]
Proposition 2.4.  

\[ \text{i. This completes the proof. } \]

\[ \square \]

Proof.  

\[ \text{i. However, for every } \mu \in \mathbb{R}, \nu > 0 \text{ and } f \in \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n), \, a > -\frac{1}{2}, \text{ we have} \]

\[ \mathcal{R}_\mu \circ \mathcal{R}_\nu(f) = \mathcal{R}_{\mu + \nu}(f). \]

Proof.  

\[ \text{For all } \mu, \nu > 0 \text{ and } f \in \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n), \, a > -\frac{1}{2}, \text{ we have} \]

\[ \mathcal{R}_\mu \circ \mathcal{R}_\nu(f)(r, x) = \frac{2^r}{2^{\mu+\nu} \Gamma(\mu) \Gamma(\nu)} \int_0^r (r^2 - t^2)^{\mu-1} 2t \left( \int_0^t (t^2 - s^2)^{\nu-1} f(s, x)ds \right) dt. \]

Applying Fubini’s theorem we get

\[ \mathcal{R}_\mu \circ \mathcal{R}_\nu(f)(r, x) = \frac{2^r}{2^{\mu+\nu} \Gamma(\mu) \Gamma(\nu)} \int_0^r f(s, x) \left( \int_s^r (r^2 - t^2)^{\mu-1} (t^2 - s^2)^{\nu-1} 2tdt \right) ds, \]

however, \[ \int_0^r (r^2 - t^2)^{\mu-1} (t^2 - s^2)^{\nu-1} 2tdt = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)} (r^2 - s^2)^{\mu+\nu-1}. \]

This completes the proof.  

\[ \square \]

Proposition 2.4.  

\[ \text{i. For every } \mu > 1 \text{ and } f \in \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n), \, a > -\frac{1}{2}, \text{ we have} \]

\[ \square \mathcal{R}_\mu(f) = \mathcal{R}_{\mu-1}(f). \]

In particular, for every \( \mu > 0, \, k \in \mathbb{N} \)

\[ \square^k \mathcal{R}_{\mu+k}(f) = \mathcal{R}_\mu(f). \]  \hfill (14)

ii. For every \( f \in \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n), \, a > -\frac{1}{2} \text{ and } \mu > 0 \)

\[ \mathcal{R}_\mu(\square f) = \square \mathcal{R}_\mu(f). \]  \hfill (15)

In particular, for every \( f \in \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n), \, a > -\frac{1}{2}, \, k \in \mathbb{N} \)

\[ \mathcal{R}_\mu(\square^k f) = \square^k \mathcal{R}_\mu(f). \]  \hfill (16)

Proof.  

i. Let \( f \in \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n), \)

\[ \square \mathcal{R}_\mu(f)(r, x) = \frac{\partial}{\partial r} \left( \frac{2}{2^{\mu} \Gamma(\mu)} \int_0^r (r^2 - t^2)^{\mu-1} f(t, x)dt \right) \]

\[ = \frac{2.2r(\mu - 1)}{2^{\mu} \Gamma(\mu)} \int_0^r (r^2 - t^2)^{\mu-2} f(t, x)dt \]

\[ = \mathcal{R}_{\mu-1}(f)(r, x), \]
and by induction, we deduce that for all \( \mu > 0, \ k \in \mathbb{N} \)
\[ \square^k \mathcal{R}_\mu f = \mathcal{R}_\mu f. \]

i. Let \( f \in r^{2(a+1)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \), by Proposition 2.2, the function \( \square(f) \) belongs to the space \( r^{2\mu} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \) and we have
\[ \mathcal{R}_\mu f(r, x) = \frac{r}{2^\mu \Gamma(\mu + 1)} \int_0^r \frac{\partial}{\partial t} ((r^2 - t^2)^\mu \frac{f(t, x)}{t}) dt. \]
Integrating by parts, we get
\[ \mathcal{R}_\mu f(r, x) = \frac{r}{2^\mu \Gamma(\mu + 1)} \int_0^r (r^2 - t^2)^\mu \square f(t, x) dt, \]
so,
\[ \square \mathcal{R}_\mu f(r, x) = \frac{2r}{2^\mu \Gamma(\mu)} \int_0^r (r^2 - t^2)^{\mu-1} \square f(t, x) dt \]
\[ = \mathcal{R}_\mu \square(f)(r, x). \]

Now, suppose that for every \( f \in r^{2(a+k)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \), \( \square^k \mathcal{R}_\mu f = \mathcal{R}_\mu (\square^k f) \), let \( g \in r^{2(a+k+1)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \).
Then, the function \( \square g \) belongs to \( r^{2(a+k)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \), and by hypothesis
\[ \square^k \mathcal{R}_\mu (\square g)(r, x) = \mathcal{R}_\mu (\square^{k+1} g), \]
on the other hand, by relation \([15]\) and the fact that \( \square g \in r^{2(a+k)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \subset r^{2(a+1)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \), we have
\[ \square^k \mathcal{R}_\mu (\square g)(r, x) = \square^{k+1} \mathcal{R}_\mu (g). \]
The proof is complete by induction. \( \square \)

**Theorem 2.5.** For every \( k \in \mathbb{N} \setminus \{0\} \), the operator \( \mathcal{R}_k \) is an isomorphism from \( r^{2\mu} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \) onto \( r^{2(a+k)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \); \( a > -\frac{1}{2} \). The inverse operator is given by
\[ \mathcal{R}_k^{-1} = \square^k. \]

**Proof.** Let \( f \in r^{2a} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \). From Proposition 2.2, the function \( \mathcal{R}_k f \) belongs to \( r^{2(a+k)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \) and by relation \([14]\), we have
\[ \square^k \mathcal{R}_k (f) = \square \square^{k-1} \mathcal{R}_{k+1}(f) \]
\[ = \mathcal{R}_{k+1}(f) \]
\[ = f. \]
Let \( g \in r^{2(a+k)} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \subset r^{2a} \mathcal{E}_c(\mathbb{R} \times \mathbb{R}^n) \), by relation \([16]\)
\[ \mathcal{R}_k (\square^k (g)) = \square^k \mathcal{R}_k (g) \]
\[ = g. \]
This achieves the proof. \( \square \)
Let \( \omega \) be given by
\[ \Theta_r \phi(x) = \int_{|s-x|=r} e^{ixs} \frac{e^{-|s-x|^2}}{(2\pi)^n} \, ds. \]

By the change of variables \( r \rightarrow r \phi \) from Theorem 2.6. For every \( I \in r^{2a+2\mu}E_c(\mathbb{R} \times \mathbb{R}^n) \), we have
\[ \int_{|s|<a} |s|^a \, ds = 0. \]

We deduce that
\[ \overline{\Theta}_{1-\mu}(g)(r, x) = \Theta_{1-\mu}(g)(r, x), \]
and
\[ \overline{\Theta}_{1-\mu}(g)(r, x) = \Theta_{1-\mu}(g)(r, x). \]

Proof. Let \( g \in r^{2(\alpha+\mu)}E_c(\mathbb{R} \times \mathbb{R}^n) \),
\[ g(r, x) = r^{2a+2\mu}h(r, x); \quad h \in E_c(\mathbb{R} \times \mathbb{R}^n), \]
\[ \square \overline{\Theta}_{1-\mu}(g)(r, x) = \frac{\partial}{\partial r} \left( \frac{2}{2\Gamma(1-\mu)} \right) \int_0^r (r^2 - t^2)^{-\mu} t^{2a+2\mu} h(t, x) dt \]
\[ = \frac{\partial}{\partial r} \left( \frac{2}{2\Gamma(1-\mu)} \right) \int_0^1 (1 - t^2)^{-\mu} t^{2a+2\mu} h(t, x) dt \]
\[ = 2(2a+1) \frac{r^{2a+1}}{2\Gamma(1-\mu)} \int_0^1 (1 - t^2)^{-\mu} t^{2a+2\mu+1} \frac{\partial h}{\partial t} (t, x) dt \]
\[ + \frac{2}{2\Gamma(1-\mu)} \int_0^r (r^2 - t^2)^{-\mu} t^{2a+2\mu} h(t, x) dt \]
\[ + \frac{2}{2\Gamma(1-\mu)} \frac{1}{r} \int_0^r (r^2 - t^2)^{-\mu} t^{2a+2\mu+1} \frac{\partial h}{\partial t} (t, x) dt. \]

We deduce that
\[ \overline{\Theta}_{\mu} \left( \square \overline{\Theta}_{1-\mu}(g) \right)(r, x) \]
\[ = \frac{2(2a+1)2r}{2\Gamma(1-\mu)} \int_0^r (r^2 - t^2)^{-\mu} \frac{1}{t} \left( \int_0^t (t^2 - s^2)^{-\mu} s^{2a+2\mu} h(s, x) ds \right) dt + \]
\[ + \frac{2.2r}{2\Gamma(1-\mu)} \int_0^r (r^2 - t^2)^{-\mu} \frac{1}{t} \left( \int_0^t (t^2 - s^2)^{-\mu} s^{2a+2\mu+1} \frac{\partial h}{\partial s} (s, x) ds \right) dt \]
\[ = I_{1, \mu}(r, x) + I_{2, \mu}(r, x). \]

From Fubini’s theorem, we have
\[ I_{1, \mu}(r, x) = \frac{(2a+1)r}{2\Gamma(1-\mu)} \int_0^r h(s, x) \left( \int_s^r (r^2 - t^2)^{\mu-1} (t^2 - s^2)^{-\mu} \frac{2t}{t^2} dt \right) s^{2a+2\mu} ds. \]

Let
\[ J(r, s) = \int_s^r (r^2 - t^2)^{\mu-1} (t^2 - s^2)^{-\mu} \frac{2t}{t^2} dt. \]

By the change of variables \( \omega = \frac{r^2-t^2}{t^2-s^2} \), we get
\[ J(r, s) = \frac{1}{r^2} \int_0^1 \frac{\omega^{\mu-1}(1-\omega)^{-\mu}}{1 - \frac{r^2}{r^2-s^2} \omega} d\omega. \]
\[
\begin{align*}
&= \frac{1}{r^2} \sum_{k=0}^{\infty} \left( \frac{r^2 - s^2}{r^2} \right)^k \int_0^1 \omega^{k+\mu-1}(1 - \omega)^{-\mu} d\omega \\
&= \frac{\Gamma(1 - \mu)}{r^2} \sum_{k=0}^{\infty} \frac{\Gamma(k + \mu)}{k!} \left( \frac{r^2 - s^2}{r^2} \right)^k \\
&= \Gamma(\mu) \Gamma(1 - \mu) r^{2\mu - 2} s^{-\mu}.
\end{align*}
\]

So,
\[
I_{1,\mu}(r, x) = (2a + 1) r^{2\mu - 1} \int_0^r h(s, x) s^{2a} ds
\]

As the same way,
\[
I_{2,\mu}(r, x) = \frac{r}{\Gamma(\mu) \Gamma(1 - \mu)} \int_0^r \frac{\partial h}{\partial s}(s, x) \left( \int_s^r (t^2 - t^2)^{\mu-1}(t^2 - s^2)^{\mu-2} \frac{2t}{t^2} dt \right) s^{2a+2\mu+1} ds
\]

Consequently,
\[
\widetilde{R}_\mu(\Box^{k_1}_{1} - \mu (g))(r, x) = r^{2\mu - 1} \int_0^r \left( (2a + 1) s^{2a} h(s, x) + s^{2a+1} \frac{\partial h}{\partial s}(s, x) \right) ds
\]

On the other hand, from Proposition 2.3 and for every \( f \in r^{2a} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n) \),
\[
\Box^{k_2}_{1} - \mu \widetilde{R}_\mu(f) = \Box^{k_1}_{1}(f) = f.
\]

This completes the proof. \( \square \)

**Lemma 2.7.** Let \( \mu \in \mathbb{R}, \mu > 0 \). For every \( k_1, k_2 \in \mathbb{N} \setminus \{0\} \), \( k_1 - \mu > 0, k_2 - \mu > 0 \) and for every \( f \in r^{2(a+\mu)} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n) \), we have
\[
\Box^{k_1}_{k_1-\mu}(f) = \Box^{k_2}_{k_2-\mu}(f).
\]

**Proof.** Let \( k_1, k_2 \in \mathbb{N} \setminus \{0\} \), \( k_1 - \mu > 0, k_2 - \mu > 0 \), and \( k_1 < k_2 \),
\[
\Box^{k_2}_{k_2-\mu}(f) = \Box^{k_1}_{k_2-\mu}(f) - \Box^{k_1}_{k_1-\mu}(f),
\]

applying relation (14), we get
\[
\Box^{k_2}_{k_2-\mu}(f) = \Box^{k_1}_{k_2-\mu}(f).
\]

\( \square \)
The previous Lemma allows us to define the fractional transform $\tilde{R}_\mu$ for every $\mu \in \mathbb{R}$.

**Definition 2.8.** For every $\mu \in \mathbb{R}$, $\mu \geq 0$, the fractional transform $\tilde{R}_\mu$ is defined on $r^{2(a+\mu)}c_c(\mathbb{R} \times \mathbb{R}^n)$ by

$$\tilde{R}_\mu(f) = \Box^k \tilde{R}_{k-\mu}(f),$$

where $k \in \mathbb{N} \setminus \{0\}$, $k - \mu > 0$.

In particular, for $f \in r^{2(a+\mu)}c_c(\mathbb{R} \times \mathbb{R}^n)$

$$\tilde{R}_\mu(f) = \Box^{E(\mu)+1} \tilde{R}_{E(\mu)+1-\mu}(f),$$

where $E(\mu)$ is the entire party of $\mu$.

**Remark 2.9.** According to definition 2.8 and for every $f \in r^{2a}c_c(\mathbb{R} \times \mathbb{R}^n)$, $a > -\frac{1}{2}$, we have

$$\tilde{R}_0(f) = \Box \tilde{R}_1(f) = f,$$

that is

$$\tilde{R}_0 = \text{Id}_{r^{2a}c_c(\mathbb{R} \times \mathbb{R}^n)}.$$

**Theorem 2.10.** For $\mu > 0$, the fractional transform $\tilde{R}_\mu$ is a topological isomorphism from $r^{2a}c_c(\mathbb{R} \times \mathbb{R}^n)$ onto $r^{2(a+\mu)}c_c(\mathbb{R} \times \mathbb{R}^n)$, $a > -\frac{1}{2}$. The inverse operator is given by

$$\tilde{R}_\mu^{-1} = \tilde{R}_{-\mu}.$$

**Proof.** For $\mu \in \mathbb{N}$, the result follows from Theorem 2.5 and Remark 2.9. Let $\mu \in [0, +\infty] \setminus \mathbb{N}$, for every $f \in r^{2a}c_c(\mathbb{R} \times \mathbb{R}^n)$ and from Proposition 2.3 and Theorem 2.5, we have

$$\tilde{R}_\mu = \Box \tilde{R}_{1} = \Box \tilde{R}_{1}.$$

Conversely, for every $g \in r^{2(a+\mu)}c_c(\mathbb{R} \times \mathbb{R}^n)$,

$$\tilde{R}_\mu \circ \tilde{R}_{-\mu}(g) = \tilde{R}_\mu \Box^{E(\mu)+1} \tilde{R}_{E(\mu)+1-\mu}(g),$$

let $\nu = \mu - E(\mu)$, then $\nu \in [0, 1]$, and

$$\tilde{R}_{\nu} \circ \tilde{R}_{-\nu}(g) = \tilde{R}_{\nu} \Box^{E(\mu)} \tilde{R}_{E(\mu)+1-\nu}(g).$$

Since, $\Box \tilde{R}_{1-\nu}(g)$ belongs to $r^{2(a+E(\mu))}c_c(\mathbb{R} \times \mathbb{R}^n)$, then, Theorem 2.5 involves that

$$\tilde{R}_\mu \circ \tilde{R}_{-\mu}(g) = \tilde{R}_{\nu} \Box \tilde{R}_{1-\nu}(g).$$

The result follows from Theorem 2.6.

Now, we have the following important result.
Theorem 2.11. For every $\mu > 0$, the fractional transform $\mathcal{R}_\mu$ defined by relation (7) is a topological isomorphism from $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ onto itself.

Proof. For every $f \in \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$,

$$\mathcal{R}_\mu(r, x) = \frac{2^\mu \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi}} r^{-2\mu} \mathcal{F}_\mu(f)(r, x).$$

From Theorem 2.10, the transform $\mathcal{F}_\mu$ is a topological isomorphism from $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ onto $r^{2\mu} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$. On the other hand, the map

$$f \mapsto r^{-2\mu} f$$

is a topological isomorphism from $r^{2\mu} \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ onto $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$. Consequently, $\mathcal{R}_\mu$ is a topological isomorphism from $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ onto itself.

Moreover,

$$\mathcal{R}^{-1}_\mu(f)(r, x) = \frac{\sqrt{\pi}}{2^\mu \Gamma(\mu + \frac{1}{2})} \mathcal{F}_\mu^{-1}(r^{2\mu} f)(r, x)$$

$$= \frac{\sqrt{\pi}}{2^\mu \Gamma(\mu + \frac{1}{2})} \mathcal{F}_\mu^{-1}(f)(r, x).$$

$\square$

2.2. The fractional transform $\mathcal{H}_\mu$. We recall that the space $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ is equipped with the topology generated by the family of norms

$$N_m(f) = \max_{(r, x) \in \mathbb{R} \times \mathbb{R}^n} \max_{k + |\alpha| \leq m} (1 + r^2 + |x|^2)^k |D^\alpha(f)(r, x)|,$$

$m \in \mathbb{N}$.

By a standard argument, for every $f \in \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$, the function $\frac{\partial}{\partial r^2}(f)$ belongs to $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ and for every $m \in \mathbb{N}$,

$$N_m\left(\frac{\partial}{\partial r^2}(f)\right) \leq 2^{m+1} N_{m+3}(f).$$

This shows that the operator $\frac{\partial}{\partial r^2}$ is continuous from $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ into itself and consequently the operator $\Xi_\mu$ is also continuous from $\mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ into itself.

On the other hand, for every $f \in \mathcal{E}_e(\mathbb{R} \times \mathbb{R}^n)$ and for every $k \in \mathbb{N}$, we have

$$(1 + \lambda_0^2 + |\lambda|^2)^k \mathcal{F}_\mu^{-\frac{1}{2}}(f)(\lambda_0, \lambda) = \mathcal{F}_\mu^{-\frac{1}{2}}((I - \Xi_\mu)^k(f))(\lambda_0, \lambda).$$

Where $I$ is the identity operator.

Using the relation (17) and the inversion formula for $\mathcal{F}_\mu^{-\frac{1}{2}}$ that is for every $f \in L^1(dv_\mu)$ such that $\mathcal{F}_\mu^{-\frac{1}{2}}(f)$ belongs to $L^1(dv_\mu)$, we have

$$f = \mathcal{F}_\mu^{-\frac{1}{2}} \circ \mathcal{F}_\mu^{-\frac{1}{2}}(f) \text{ a.e.}$$
we deduce that the transform $\tilde{\mathcal{F}}_{\mu - \frac{1}{2}}$ is a topological isomorphism from $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ onto itself and

$$\tilde{\mathcal{F}}_{\mu - \frac{1}{2}}^{-1}(f) = \tilde{\mathcal{F}}_{\mu - \frac{1}{2}}(\tilde{f})$$

where $\tilde{f}(r, x) = f(r, -x)$.

**Lemma 2.12.** For every $f \in L^1(d\nu_{\mu})$ and $\mu > 0$, the function

$$\mathcal{H}_\mu(f)(t, x) = \frac{1}{2^\mu \Gamma(\mu)} \int_0^\infty (r^2 - t^2)^{\mu-1} f(r, x) 2rdr,$$

is defined almost everywhere, belongs to $L^1(dm)$, where $dm$ is the Lebesgue measure given by relation \([8]\), and we have

$$||\mathcal{H}_\mu(f)||_{1,m} \leq ||f||_{1,\nu_{\mu}}.$$  

**Proof.** By Fubini-Tonnelli Theorem's, we have

$$\int_0^\infty \int_{\mathbb{R}^n} |\mathcal{H}_\mu(f)(t, x)| dm(t, x)$$

$$\leq \sqrt{\frac{2}{\pi}} \frac{1}{2^\mu \Gamma(\mu)(2\pi)^{\frac{n}{2}}} \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^\infty (r^2 - t^2)^{\mu-1} |f(r, x)| 2rdr \right) dt dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2^\mu \Gamma(\mu)(2\pi)^{\frac{n}{2}}} \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \left( \int_0^\infty (r^2 - t^2)^{\mu-1} dt \right) 2rdr dx$$

$$= \frac{1}{2^{\mu-\frac{1}{2}} \Gamma(\mu + \frac{1}{2})(2\pi)^{\frac{n}{2}}} \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| r^{2\mu} dr dx$$

$$= ||f||_{1,\nu_{\mu}}.$$

**Proposition 2.13.** i. For every $f \in L^1(d\nu_{\mu})$ and every bounded measurable function $g$ on $[0, +\infty[\times\mathbb{R}^n$, we have the duality relation

$$\int_0^\infty \int_{\mathbb{R}^n} f(r, x) \mathcal{H}_\mu(g)(r, x) d\nu_{\mu}(r, x) = \int_0^\infty \int_{\mathbb{R}^n} \mathcal{H}_\mu(f)(r, x) g(r, x) dm(r, x).$$

ii. For every $f \in L^1(d\nu_{\mu})$

$$\tilde{\mathcal{F}}_{\mu - \frac{1}{2}}(f) = \Lambda \circ \mathcal{H}_\mu(f),$$

where, $\Lambda$ is the usual Fourier transform defined on $L^1(dm)$ by

$$\Lambda(f)(\lambda_0, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \cos(r\lambda_0) e^{-i(\lambda|x|)} dm(r, x).$$

**Proof.** i. It is clear that for every bounded function $g$ on $[0, +\infty[\times\mathbb{R}^n$, the function $\mathcal{H}_\mu(g)$ is also bounded on $[0, +\infty[\times\mathbb{R}^n$. 

Consequently, the integral \( \int_0^\infty \int_{\mathbb{R}^n} f(r, x) R_\mu(g)(r, x) d\nu_\mu(r, x) \) is well defined, and we have
\[
\int_0^\infty \int_{\mathbb{R}^n} f(r, x) R_\mu(g)(r, x) d\nu_\mu(r, x) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \frac{2r}{2^{\mu-\frac{1}{2}} \sqrt{\pi} (2\pi)^{\frac{\mu}{2}} \Gamma(\mu)} \times \left( \int_0^r (r^2 - t^2)^{\mu-1} g(t, x) dt \right) dr dx.
\]

By Fubini’s Theorem,
\[
\int_0^\infty \int_{\mathbb{R}^n} f(r, x) R_\mu(g)(r, x) d\nu_\mu(r, x) = \int_0^\infty \int_{\mathbb{R}^n} g(t, x) \mathcal{H}_\mu(f)(t, x) dm(t, x).
\]

ii. Let \( f \in L^1(d\nu_\mu) \), we have
\[
\mathcal{F}_{\mu-\frac{1}{2}}(f)(\lambda_0, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \Psi_{\lambda_0, \lambda}(r, x) d\nu_\mu(r, x)
\]
and by the relation (6),
\[
\mathcal{F}_{\mu-\frac{1}{2}}(f)(\lambda_0, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) R_\mu(\cos(\lambda_0 r) e^{-i(\lambda_0 x)})(r, x) d\nu_\mu(r, x),
\]
and by the relation of duality, Proposition 2.13, we obtain
\[
\mathcal{F}_{\mu-\frac{1}{2}}(f)(\lambda_0, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} H_\mu(f)(r, x) \cos(\lambda_0 r) e^{-i(\lambda_0 x)} dm(r, x)
= \Lambda \circ H_\mu(f)(\lambda_0, \lambda).
\]

**Corollary 2.14.** For every \( \mu > 0 \), the fractional transform \( H_\mu \) is a topological isomorphism from \( \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n) \) onto itself.

**Proof.** Since the Fourier transforms \( \Lambda \) and \( \mathcal{F}_{\mu-\frac{1}{2}} \) are topological isomorphisms from \( \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n) \) onto itself, the result follows from the relation (18). \( \square \)

Next, we will prove that the fractional transform \( H_\mu \) can be extended to \( \mu \in \mathbb{R} \) and we give the inverse operator \( H_{-\mu}^{-1} \).

**Proposition 2.15.** For every \( \mu, \nu > 0 \) and \( f \in \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n) \),
\[
H_\mu \circ H_\nu(f) = H_{\mu+\nu}(f).
\]
Proof. Let $\mu, \nu > 0$ and $f \in \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n)$
\[
\mathcal{H}_\mu \circ \mathcal{H}_\nu(f)(r, x) = \frac{1}{2^{\mu+\nu} \Gamma(\mu)\Gamma(\nu)} \int_{r}^{\infty} (t^2 - r^2)^{\mu-1} \left( \int_{t}^{\infty} (s^2 - t^2)^{\nu-1} f(s, x) 2s \, ds \right) 2t \, dt.
\]
Applying Fubini’s Theorem we get
\[
\mathcal{H}_\mu \circ \mathcal{H}_\nu(f)(r, x) = \frac{1}{2^{\mu+\nu} \Gamma(\mu)\Gamma(\nu)} \int_{r}^{\infty} f(s, x) \left( \int_{s}^{\infty} (s^2 - t^2)^{\nu-1} (t^2 - r^2)^{\mu-1} 2t \, dt \right) 2s \, ds,
\]
however,
\[
\int_{s}^{\infty} (s^2 - t^2)^{\nu-1} (t^2 - r^2)^{\mu-1} 2t \, dt = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)} (s^2 - r^2)^{\mu+\nu-1},
\]
this completes the proof. \(\square\)

**Proposition 2.16.** i. For every $f \in \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n)$ and $\mu > 0$, we have
\[
\frac{\partial}{\partial t^2} \mathcal{H}_\mu(f)(t, x) = \mathcal{H}_\mu \left( \frac{\partial}{\partial t^2} f \right). \tag{19}
\]

ii. For every $f \in \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n)$ and $\mu > 0$, we have
\[
- \mathcal{H}_{\mu+1} \left( \frac{\partial}{\partial t^2} f \right) = \mathcal{H}_\mu(f). \tag{20}
\]

**Proof.** i. Integrating by parts, we get for every $f \in \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n)$,
\[
\mathcal{H}_\mu(f)(t, x) = - \frac{1}{2^\mu \Gamma(\mu + 1)} \int_{t}^{\infty} (r^2 - t^2)^{\mu-1} \frac{\partial f}{\partial r}(r, x) \, dr.
\]
Hence,
\[
\frac{\partial}{\partial t^2} \mathcal{H}_\mu(f)(t, x) = \frac{1}{2^\mu \Gamma(\mu)} \int_{t}^{\infty} (r^2 - t^2)^{\mu-1} \frac{\partial f}{\partial r^2}(r, x) 2r \, dr
\]
\[
= \mathcal{H}_{\mu+1} \left( \frac{\partial}{\partial t^2} f \right)(t, x).
\]

ii. For every $f \in \mathcal{S}_e(\mathbb{R} \times \mathbb{R}^n)$, $\mu > 0$, and from relation (19),
\[
\frac{\partial}{\partial t^2} \mathcal{H}_{\mu+1}(f) = \mathcal{H}_{\mu+1} \left( \frac{\partial}{\partial t^2} f \right).
\]
So, for every $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,
\[
\mathcal{H}_{\mu+1} \left( \frac{\partial}{\partial t^2} f \right)(t, x) = \frac{\partial}{\partial t^2} \left( \frac{1}{2^{\mu+1} \Gamma(\mu + 1)} \int_{t}^{\infty} (r^2 - t^2)^{\mu} f(r, x) 2r \, dr \right)
\]
\[
= - \mathcal{H}_\mu(f)(t, x).
\] \(\square\)
Corollary 2.17. Let $\mu$ be a real number. For all $k_1$, $k_2 \in \mathbb{N}$, $k_1 + \mu > 0$, $k_2 + \mu > 0$ and for every $f \in \mathcal{S}_c(\mathbb{R} \times \mathbb{R}^n)$, we have
\[
(-1)^{k_1} \mathcal{H}_{\mu+k_1} \left( \frac{\partial}{\partial t^2} \right)^{k_1} f = (-1)^{k_2} \mathcal{H}_{\mu+k_2} \left( \frac{\partial}{\partial t^2} \right)^{k_2} f.
\]
Proof. Let $k_1$, $k_2 \in \mathbb{N}$, $k_1 < k_2$, $k_1 + \mu > 0$ and $k_2 + \mu > 0$. From Proposition 2.16, it follows that for every $f \in \mathcal{S}_c(\mathbb{R} \times \mathbb{R}^n)$,
\[
(-1)^{k_2} \mathcal{H}_{\mu+k_2} \left( \frac{\partial}{\partial t^2} \right)^{k_2} f = (-1)^{k_1} \mathcal{H}_{\mu+k_1} \left( \frac{\partial}{\partial t^2} \right)^{k_1} \mathcal{H}_{\mu+k_1} \left( \frac{\partial}{\partial t^2} \right)^{k_2} f.
\]
Therefore, we have
\[
(-1)^{k_1} \mathcal{H}_{\mu+k_1} \left( \frac{\partial}{\partial t^2} \right)^{k_1} f = (-1)^{k_1} \mathcal{H}_{\mu+k_1} \left( \frac{\partial}{\partial t^2} \right)^{k_1} \mathcal{H}_{\mu+k_2} \left( \frac{\partial}{\partial t^2} \right)^{k_2} f.
\]

Definition 2.18. For every $\mu \in \mathbb{R}$, the fractional transform $\mathcal{H}_\mu$ is defined on $\mathcal{S}_c(\mathbb{R} \times \mathbb{R}^n)$ by
\[
\mathcal{H}_\mu(f) = (-1)^k \mathcal{H}_{\mu+k} \left( \frac{\partial}{\partial t^2} \right)^k f = (-1)^k \left( \frac{\partial}{\partial t^2} \right)^k \mathcal{H}_{\mu+k} f,
\]
where $k \in \mathbb{N}$, $k + \mu > 0$.

From Corollary 2.17, the expression $\mathcal{H}_\mu$ in Definition 2.18 is independent of the choice of $k \in \mathbb{N}$, $k + \mu > 0$.

For every $f \in \mathcal{S}_c(\mathbb{R} \times \mathbb{R}^n)$,
\[
\mathcal{H}_0(f)(t,x) = -\frac{\partial}{\partial t} \mathcal{H}_1(f)(t,x)
\]
\[
= -\frac{1}{t} \frac{\partial}{\partial t} \left( \int_0^\infty f(r,x)rdr \right) = f(t,x). \tag{21}
\]

Proposition 2.19. i. For every $\mu$, $\nu \in \mathbb{R}$ and $f \in \mathcal{S}_c(\mathbb{R} \times \mathbb{R}^n)$,
\[
\mathcal{H}_\mu \circ \mathcal{H}_\nu(f) = \mathcal{H}_{\mu+\nu}(f) \tag{22}
\]
ii. For every $\mu \in \mathbb{R}$, the fractional transform $\mathcal{H}_\mu$ is a topological isomorphism from $\mathcal{S}_c(\mathbb{R} \times \mathbb{R}^n)$ onto itself whose inverse isomorphism is
\[
\mathcal{H}_\mu^{-1} = \mathcal{H}_{-\mu}.
\]

Proof. i. Let $\mu$, $\nu \in \mathbb{R}$, $k_1$, $k_2 \in \mathbb{N}$, $k_1 + \mu > 0$, $k_2 + \mu > 0$ and $f \in \mathcal{S}_c(\mathbb{R} \times \mathbb{R}^n)$, we have
\[
\mathcal{H}_\mu \circ \mathcal{H}_\nu(f) = \mathcal{H}_\mu \left( (-1)^{k_2} \left( \frac{\partial}{\partial t^2} \right)^{k_2} \mathcal{H}_{\nu+k_2}(f) \right)
\]
\[
= (-1)^{k_1+k_2} \mathcal{H}_{\mu+k_1+k_2} \left( \left( \frac{\partial}{\partial t^2} \right)^{k_1} \mathcal{H}_{\nu+k_2} \left( \left( \frac{\partial}{\partial t^2} \right)^{k_2} f \right) \right)
\]
\[
= (-1)^{k_1+k_2} \mathcal{H}_{\mu+k_1} \circ \mathcal{H}_{\nu+k_2} \left( \left( \frac{\partial}{\partial t^2} \right)^{k_1+k_2} f \right).
\]
Now, from Proposition 2.15, we deduce that
\[
\mathcal{H}_\mu \circ \mathcal{H}_\nu(f) = (-1)^{k_1+k_2} \mathcal{H}_{\mu+\nu+k_1+k_2} \left( \left( \frac{\partial}{\partial t} \right)^{k_1+k_2} (f) \right)
\]

because \( \mu + \nu + k_1 + k_2 > 0 \).

\[\text{ii. The result follows from relations (21) and (22).} \]

3. \( L^p \)-boundedness of the fractional transform \( \mathcal{R}_\mu \) and \( \mathcal{H}_\mu \)

This section contains the main results of this work. In fact, we study the boundedness of the operators \( \mathcal{R}_\mu \) and \( \mathcal{H}_\mu \) on the weighted Lebesgue spaces \( L^p([0, +\infty[ \times \mathbb{R}^n, r^{2\alpha} \, dr \, dx] \), \( p \in [1, +\infty[ \) equipped with the norm

\[
||f||_{p,a} = \begin{cases} 
\left( \int_0^{+\infty} \int_{\mathbb{R}^n} |f(r,x)|^p r^{2\alpha} \, dr \, dx \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < +\infty \\
\text{ess sup}_{(r,x) \in [0, +\infty[ \times \mathbb{R}^n} |f(r,x)|, & \text{if } p = +\infty.
\end{cases}
\]

For convenience we refer to this space as \( L^p(d\gamma_a) \) with \( d\gamma_a(r,x) = r^{2\alpha} \, dr \, dx \).

3.1. \( L^p \)-boundedness of the fractional transform \( \mathcal{R}_\mu \).

**Proposition 3.1.** For every \( a \in \mathbb{R} \) and every \( \mu > 0 \), the fractional transform \( \mathcal{R}_\mu \) is bounded from \( L^\infty(d\gamma_a) \) into itself and

\[
||\mathcal{R}_\mu||_{\infty,\gamma_a} = \sup_{||f||_{\infty,a} \leq 1} ||\mathcal{R}_\mu(f)||_{\infty,a} = 1.
\]

**Proof.** Let \( f \) be a bounded measurable function on \([0, +\infty[ \times \mathbb{R}^n\). For every \((r,x) \in [0, +\infty[ \times \mathbb{R}^n\),

\[
|\mathcal{R}_\mu(f)(r,x)| \leq \frac{2\Gamma(\mu + \frac{3}{2})}{\sqrt{\pi} \Gamma(\mu)} \int_0^{1} (1 - t^2)^{\mu-1} |f(tr,x)| \, dt
\]

\[
\leq ||f||_{\infty,a} \frac{2\Gamma(\mu + \frac{3}{2})}{\sqrt{\pi} \Gamma(\mu)} \int_0^{1} (1 - t^2)^{\mu-1} \, dt
\]

\[
= ||f||_{\infty,a}.
\]

This shows that the operator \( \mathcal{R}_\mu \) is bounded from \( L^\infty(d\gamma_a) \) into itself and that

\[
||\mathcal{R}_\mu||_{\infty,\gamma_a} \leq 1.
\]

However, \( \mathcal{R}_\mu(1) = 1 \), this shows that

\[
||\mathcal{R}_\mu||_{\infty,\gamma_a} = 1.
\]

**Theorem 3.2.** The operator \( \mathcal{R}_\mu; \mu > 0 \) is bounded from \( L^1(d\gamma_a) \) into itself if and only if \( a < 0 \) and in this case

\[
||\mathcal{R}_\mu||_{1,\gamma_a} = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(-a)}{\sqrt{\pi} \Gamma(\mu - a)}.
\]
Proof. Let \( a \in \mathbb{R}, \ a < 0 \). By Fubini-Tonnelli Theorem's and for every \( f \in L^1(d\gamma_a) \),
\[
\int_0^\infty \int_{\mathbb{R}^n} |\mathcal{R}_\mu(f)(r,x)|d\gamma_a(r,x) \\
\leq \frac{2\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^1 (1-t^2)^{\mu-1} |f(tr,x)|dt \right) d\gamma_a(r,x) \\
= \frac{2\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} \int_0^1 (1-t^2)^{-1} \left( \int_0^\infty \int_{\mathbb{R}^n} |f(tr,x)|d\gamma_a(r,x) \right) dt \\
= ||f||_{1,a} \frac{2\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu-a)} \frac{\Gamma(-a)}{\Gamma(\mu-a)} \\
= \frac{\Gamma(\mu + \frac{1}{2})\Gamma(-a)}{\sqrt{\pi} \Gamma(\mu-a)} ||f||_{1,a}.
\]
Consequently for \( a < 0 \), the transform \( \mathcal{R}_\mu \) is a bounded operator from \( L^1(d\gamma_a) \) into itself and
\[
||\mathcal{R}_\mu||_{1,\gamma_a} \leq \frac{\Gamma(\mu + \frac{1}{2})\Gamma(-a)}{\sqrt{\pi} \Gamma(\mu-a)}.
\]
On the other hand, for every nonnegative \( f \in L^1(d\gamma_a) \), we have
\[
||\mathcal{R}_\mu(f)||_{1,a} = \frac{\Gamma(\mu + \frac{1}{2})\Gamma(-a)}{\sqrt{\pi} \Gamma(\mu-a)} ||f||_{1,a}
\]
We conclude that
\[
||\mathcal{R}_\mu||_{1,\gamma_a} = \frac{\Gamma(\mu + \frac{1}{2})\Gamma(-a)}{\sqrt{\pi} \Gamma(\mu-a)}.
\]
For converse, let \( a \in \mathbb{R}, \ a > 0 \) and let \( f \in L^1(d\gamma_a) \) be a nonnegative function such that \( ||f||_{1,a} = 1 \). We have
\[
||\mathcal{R}_\mu(f)||_{1,\gamma_a} = \frac{\Gamma(\mu + \frac{1}{2})\Gamma(-a)}{\sqrt{\pi} \Gamma(\mu-a)} = +\infty.
\]
This completes the proof. \( \square \)

**Theorem 3.3.** Let \( p \in [1, +\infty[ \). The operator \( \mathcal{R}_\mu, \ \mu > 0, \) is bounded from \( L^p(d\gamma_a) \) into itself if and only if \( 2a + 1 < p \) and in this case
\[
||\mathcal{R}_\mu||_{p,\gamma_a} = \frac{\Gamma(\mu + \frac{1}{2})(\frac{p-2a-1}{2p})}{\sqrt{\pi} \Gamma(\mu + \frac{p-(2a+1)}{2p})}.
\]

**Proof.** Let \( p \in [1, +\infty[ , \ 2a + 1 < p \). From Minkowski's inequality \([IS]\) and for every \( f \in L^p(d\gamma_a) \),
\[
||\mathcal{R}_\mu(f)||_{p,a} \leq \frac{2\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} \int_0^1 (1-t^2)^{\mu-1} \left( \int_0^\infty \int_{\mathbb{R}^n} |f(tr,x)|^p d\gamma_a(r,x) \right)^{\frac{1}{p}} dt \\
= \frac{2\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} \||f||_{p,a} \int_0^1 (1-t^2)^{\mu-1} t^{-\frac{2a+1}{p}} dt
\]
Let \( \eta > 0 \) and let \( f_0(r,x) = r^{\eta-(2a+1)/p} \Pi_{j=1}^n 1_{[0,1]}(r) \Pi_{j=1}^n 1_{[0,1]}(x_j) \), then \( f_0 \) belongs to \( L^p(d\gamma_a) \) and
\[
||f_0||_{p,a} = \left( \frac{1}{\eta} \right)^{\frac{1}{p}}.
\]
On the other hand,
\[
|\mathcal{R}_\mu(f_0)(r,x)| \geq \frac{2\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} r^{1-2\mu} \left( \int_0^r (r^2 - t^2)^{\mu-1} t^{\eta-(2a+1)/p} dt \right) 1_{[0,1]}(r) \Pi_{j=1}^n 1_{[0,1]}(x_j)
\]
\[
= \frac{2\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} f_0(r,x) \int_0^1 (1-t^2)^{\mu-1} t^{\eta-(2a+1)/p} dt
\]
\[
= \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\frac{1}{2} + \frac{\eta-(2a+1)}{2p})}{\sqrt{\pi} \Gamma(\mu + \frac{1}{2} + \frac{\eta-(2a+1)}{2p})} f_0(r,x).
\]
Integrating over \( [0, +\infty) \times \mathbb{R}^n \) with respect to the measure \( d\gamma_a \), we deduce that for every \( \eta > 0 \),
\[
||\mathcal{R}_\mu||_{p,\gamma_a} \geq \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\frac{1}{2} + \frac{\eta-(2a+1)}{2p})}{\sqrt{\pi} \Gamma(\mu + \frac{1}{2} + \frac{\eta-(2a+1)}{2p})}.
\]
This involves that
\[
||\mathcal{R}_\mu||_{p,\gamma_a} \geq \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi} \Gamma(\mu + \frac{p-(2a+1)}{2p})}.
\]

The relations (23) and (24) imply that for every \( a \), \( 2a + 1 < p \)
\[
||\mathcal{R}_\mu||_{p,\gamma_a} = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi} \Gamma(\mu + \frac{p-(2a+1)}{2p})}.
\]

Now, we prove that, for \( 2a + 1 > p \), \( \mathcal{R}_\mu \) does not map \( L^p(d\gamma_a) \) into itself. To prove this we have following two cases:

**Case 1.** Suppose that \( 2a + 1 = p \) and let
\[
g_0(r,x) = \frac{1}{r(1 - \ln(r))} 1_{[0,1]}(r) \Pi_{j=1}^n 1_{[0,1]}(x_j),
\]
then, $g_0$ belongs to $L^p(d\gamma_a)$ and we have

$$
\|g_0\|^p_{p,a} = \int_0^1 \frac{dr}{r(1 - \ln(r))^p} = \int_{-\infty}^0 \frac{ds}{(1 - s)^p} = \frac{1}{p - 1}.
$$

However, for every $(r, x) \in ]0, 1[ \times ]0, 1[^n$,

$$
\mathcal{R}_\mu(g_0)(r, x) = \frac{2 \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} r^{1-2\mu} \int_0^r (r^2 - t^2)^{\mu-1} \frac{dt}{t(1 - \ln(t))} = +\infty,
$$
in particular $\mathcal{R}_\mu(g_0)$ does not belong to $L^p(d\gamma_a)$.

**Case 2.** Suppose that $2a + 1 > p$ and let $\eta \in \mathbb{R}; -\frac{2a+1}{p} < \eta < -1$ and let

$$
h_0(r, x) = r^\eta 1_{]0,1[}(r) \prod_{j=1}^n 1_{]0,1[}(x_j).
$$

Then the function $h_0$ lies in $L^p(d\gamma_a)$ and

$$
\|h_0\|^p_{p,a} = \frac{1}{p\eta + 2a + 1}.
$$

But, for every $(r, x) \in ]0, 1[ \times ]0, 1[^n$,

$$
\mathcal{R}_\mu(h_0)(r, x) = \frac{2 \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)} r^{\eta} \int_0^1 (1 - t^2)^{\mu-1} t^{\eta} dt = +\infty.
$$

Hence, for $2a + 1 > p$, $\mathcal{R}_\mu$ does not map $L^p(d\gamma_a)$ into itself and this completes the proof of theorem. \hfill \Box

Combining Proposition (3.1), Theorem (3.2) and Theorem (3.3), we claim the following interesting result.

**Theorem 3.4.** For every $p \in [1, +\infty]$, the fractional operator $\mathcal{R}_\mu$ is bounded on $L^p(d\gamma_a)$ if and only if $2a + 1 < p$ and in this case

$$
\|\mathcal{R}_\mu\|^p_{p,\gamma_a} = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\frac{p-(2a+1)}{2p})}{\sqrt{\pi} \Gamma(\mu + \frac{p-(2a+1)}{2p})}.
$$

**Remark 3.5.** The case $a = \mu$ in Theorem (3.4) is important because the measure $d\nu_\mu$ defined by the relation (3) is connected with the operators $D_j$, $1 \leq j \leq n$ and $\Xi$ and the Fourier-Hankel transform $\mathcal{F}_{\mu - \frac{1}{2}}$ given by relation (2) and in this occurrence, $\mathcal{R}_\mu$ is bounded from $L^p(d\nu_\mu)$ into itself if and only if $2\mu + 1 < p$ and we have

$$
\|\mathcal{R}_\mu\|^p_{p,\nu_\mu} = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\frac{p-(2\mu+1)}{2p})}{\sqrt{\pi} \Gamma(\mu + \frac{p-(2\mu+1)}{2p})}.
$$
3.2. $L^p$-boundedness of the fractional transform $\mathcal{H}_\mu$. We denote by $r^{-2\mu} L^p(d\gamma_a)$ the space defined by $r^{-2\mu} L^p(d\gamma_a) = \{ f : ]0, +\infty[ \times \mathbb{R}^n \to \mathbb{C}, f \text{ is measurable and the function } (r, x) \mapsto r^{2\mu} f(r, x) \text{ belongs to } L^p(d\gamma_a) \}$.

$r^{-2\mu} L^p(d\gamma_a)$ is equipped with the norm $N_{p, a}(f) = \| r^{2\mu} f \|_{p,a}$.

**Theorem 3.6.** The operator $\mathcal{H}_\mu$, $\mu > 0$ is bounded from $r^{-2\mu} L^1(d\gamma_a)$ into $L^1(d\gamma_a)$ if and only if $2a + 1 > 0$ and in this case $N_{1, \gamma_a}(\mathcal{H}_\mu) = \sup_{\| r^{2\mu} f \|_{1,a} \leq 1} \| \mathcal{H}_\mu(f) \|_{1,a} = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu \Gamma(\mu + \frac{2a+1}{2})}$.

**Proof.** Suppose that $a > -\frac{1}{2}$ and let $f \in r^{-2\mu} L^1(d\gamma_a)$. We have

$$|\mathcal{H}_\mu(f)(r, x)| \leq \frac{r^{2\mu}}{2^\mu \Gamma(\mu)} \int_0^\infty (t^2 - 1)^{\mu-1} |f(rt, x)| 2tdt.$$

Applying Fubini-Tonnelli Theorem’s, we get

$$\int_0^\infty \int_{\mathbb{R}^n} |\mathcal{H}_\mu(f)(r, x)| d\gamma_a(r, x)$$

$$\leq \frac{1}{2^\mu \Gamma(\mu)} \int_1^\infty (t^2 - 1)^{\mu-1} \left( \int_0^\infty \int_{\mathbb{R}^n} r^{2\mu+2a} |f(tr, x)| dr dx \right) 2tdt$$

$$= \| r^{2\mu} f \|_{1,a} \frac{1}{2^\mu \Gamma(\mu)} \int_1^\infty (t^2 - 1)^{\mu-1} t^{-2\mu-2a-1} 2tdt.$$

By the change of variable $s = \frac{1}{t^2}$, we have

$$\frac{1}{2^\mu \Gamma(\mu)} \int_1^\infty (t^2 - 1)^{\mu-1} t^{-2\mu-2a-1} 2tdt = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu \Gamma(\mu + \frac{2a+1}{2})}.$$

This shows that for every $f \in r^{-2\mu} L^1(d\gamma_a)$, the function $\mathcal{H}_\mu(f)$ belongs to $L^1(d\gamma_a)$ and

$$\| \mathcal{H}_\mu(f) \|_{1,a} \leq \frac{\Gamma(\frac{2a+1}{2})}{2^\mu \Gamma(\mu + \frac{2a+1}{2})} \| r^{2\mu} f \|_{1,a}.$$

On the other hand, for every nonnegative function $f \in r^{-2\mu} L^1(d\gamma_a)$, we have

$$\| \mathcal{H}_\mu(f) \|_{1,a} = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu \Gamma(\mu + \frac{2a+1}{2})} \| r^{2\mu} f \|_{1,a}. \tag{25}$$

Hence, for $a > -\frac{1}{2}$, the fractional transform $\mathcal{H}_\mu$ is continuous from $r^{-2\mu} L^1(d\gamma_a)$ into $L^1(d\gamma_a)$ and

$$N_{1, \gamma_a}(\mathcal{H}_\mu) = \frac{\Gamma(\frac{2a+1}{2})}{2^\mu \Gamma(\mu + \frac{2a+1}{2})}.$$
For Converse, let \( a \leq - \frac{1}{2} \) and let \( f \in r^{-2\mu}L^1(d\gamma_a) \), \( f \) nonnegative function such that 
\[ ||r^{2\mu}f||_{1,a} = 1. \]
From relation [25]
\[ ||\mathcal{H}_\mu(f)||_{1,a} = +\infty, \]
which proves that for \( a \leq - \frac{1}{2} \), the operator \( \mathcal{H}_\mu \) does not map the space \( r^{-2\mu}L^1(d\gamma_a) \) into \( L^1(d\gamma_a) \).

**Theorem 3.7.** For every \( p \in [1, +\infty[ \), the fractional transform \( \mathcal{H}_\mu \) is bounded from \( r^{-2\mu}L^p(d\gamma_a) \) into \( L^p(d\gamma_a) \) if and only if \( 2a + 1 > 0 \) and in this case
\[ N_{p,\gamma_a}(\mathcal{H}_\mu) = \sup_{||r^{2\mu}f||_{p,a} \leq 1} ||\mathcal{H}_\mu(f)||_{p,a} = \frac{\Gamma\left(\frac{2a+1}{2p}\right)}{2^\mu \Gamma\left(\mu + \frac{2a+1}{2p}\right)}. \]

**Proof.** Let \( a > - \frac{1}{2} \) and \( f \in r^{-2\mu}L^p(d\gamma_a) \). By Minkowski’s inequality, we have
\[ ||\mathcal{H}_\mu(f)||_{p,a} \leq \frac{1}{2^\mu \Gamma(\mu)} \int_1^\infty (t^2 - 1)^{\mu-1} \left( \int_0^\infty \left( \int \left( r^{2\mu} |f(tr,x)| \right)^p r^{2a} dr dx \right)^{\frac{1}{p}} dt \right) \]
\[ = ||r^{2\mu}f||_{p,a} \frac{1}{2^\mu \Gamma(\mu)} \int_1^\infty (t^2 - 1)^{\mu-1} t^{-\frac{2a+1}{p}} dt \]
\[ = \frac{\Gamma\left(\frac{2a+1}{2p}\right)}{2^\mu \Gamma\left(\mu + \frac{2a+1}{2p}\right)} ||r^{2\mu}f||_{p,a}. \]
Consequently, for \( a > - \frac{1}{2} \), \( \mathcal{H}_\mu \) is a bounded operator from \( r^{-2\mu}L^p(d\gamma_a) \) into \( L^p(d\gamma_a) \) and
\[ N_{p,\gamma_a}(\mathcal{H}_\mu) \leq \frac{\Gamma\left(\frac{2a+1}{2p}\right)}{2^\mu \Gamma\left(\mu + \frac{2a+1}{2p}\right)}. \] (26)

Let \( \eta \in \mathbb{R}, \eta > 0 \), and let
\[ f_0(r, x) = r^{-2\mu-\frac{2a+n+1}{p}} 1_{[1, +\infty]}(r) \prod_{j=1}^n 1_{[0,1]}(x_j). \]
The function \( f_0 \) belongs to \( r^{-2\mu}L^p(d\gamma_a) \) and
\[ ||r^{2\mu}f_0||_{p,a} = \left( \frac{1}{\eta} \right)^\frac{1}{p}. \]
Moreover,
\[ ||\mathcal{H}_\mu(f_0)(r, x)||_{p,a} \]
\[ = \mathcal{H}_\mu(f_0)(r, x) \]
\[ \geq \frac{1}{2^\mu \Gamma(\mu)} \left( \int_r^\infty (t^2 - r^2)^{\mu-1} t^{-2\mu-\frac{2a+1+n}{p}} 2dt \right) 1_{[1, +\infty]}(r) \prod_{j=1}^n 1_{[0,1]}(x_j) \]
\[ = \frac{\Gamma\left(\frac{2a+n+1}{2p}\right)}{2^\mu \Gamma\left(\mu + \frac{2a+n+1}{2p}\right)} \mu^{2\mu} f_0(r, x). \]
Thus,

$$||H\mu(f_0)||_{p,a} \geq \frac{\Gamma\left(\frac{2a+1+\eta}{2p}\right)}{2^{\mu} \Gamma(\mu + \frac{2a+1+\eta}{2p})} ||r^{2\mu}f_0||_{p,a}$$

and then, for every $\eta > 0$,

$$N_{p,\gamma}(H\mu) \geq \frac{\Gamma\left(\frac{2a+1+\eta}{2p}\right)}{2^{\mu} \Gamma(\mu + \frac{2a+1+\eta}{2p})}.$$ 

This implies that

$$N_{p,\gamma}(H\mu) \geq \frac{\Gamma\left(\frac{2a+1}{2p}\right)}{2^{\mu} \Gamma(\mu + \frac{2a+1}{2p})}. \quad (27)$$

Combining the relations (26) and (27), we deduce that for $a > -\frac{1}{2}$, the fractional transform $H\mu$ is a bounded operator from $r^{-2\mu}L^p(\gamma_0 \, d\gamma)$ into $L^p(\gamma_0 \, d\gamma)$ and that

$$N_{p,\gamma}(H\mu) = \frac{\Gamma\left(\frac{2a+1}{2p}\right)}{2^{\mu} \Gamma(\mu + \frac{2a+1}{2p})}.$$ 

Now we prove that, for $a \geq \frac{1}{2}$, the operator $H\mu$ does not map the space $r^{-2\mu}L^p(\gamma_{-\frac{1}{2}})$ into $L^p(\gamma_{-\frac{1}{2}})$. We have two cases:

**Case 1.** Suppose that $2a + 1 = 0$ and let

$$g_0(r, x) = \frac{1}{r^{2\mu}(1 + \ln(r))^{\mu}} 1_{[1, +\infty]}(r) \prod_{j=1}^{n} 1_{[0, 1]}(x_j).$$

The function $g_0$ belongs to $r^{-2\mu}L^p(\gamma_{-\frac{1}{2}})$ and

$$||r^{2\mu}g_0||_{p, -\frac{1}{2}} = \left( \int_{1}^{\infty} \frac{dr}{r(1 + \ln(r))^{\mu}} \right)^\frac{1}{2} = \left( \int_{0}^{\infty} \frac{du}{(1 + u)^{\mu}} \right)^\frac{1}{2} = \left( \frac{1}{p-1} \right)^{\frac{1}{2}}.$$ 

But for every $(r, x) \in [1, +\infty] \times [0, 1]^n$,

$$H\mu(g_0)(r, x) = \int_{0}^{\infty} \frac{2t}{t^{2\mu}(1 + \ln(r))} \, dt = +\infty.$$ 

This shows that for $a = -\frac{1}{2}$, the operator $H\mu$ does not map the space $r^{-2\mu}L^p(\gamma_{-\frac{1}{2}})$ into $L^p(\gamma_{-\frac{1}{2}})$.

**Case 2.** Finally, suppose that $a < -\frac{1}{2}$ and let $\eta \in \mathbb{R}$; $\frac{1}{2} < \eta < -a$. Let

$$h_0(r, x) = r^{-2\mu - \frac{2a+2a}{p}} 1_{[1, +\infty]}(r) \prod_{j=1}^{n} 1_{[0, 1]}(x_j).$$
The function $h_0$ belongs to $r^{-\mu}L^p(d\gamma_a)$, and

$$||r^{2\mu}h_0||_{p,a} = \left( \int_1^{\infty} r^{-2\eta} dr \right)^{\frac{1}{p}} = \left( \frac{1}{2\eta - 1} \right)^{\frac{1}{p}}.$$ 

However, for every $(r, x) \in [1, +\infty] \times [0, 1]^n$,

$$|r_{p,a} h_0| = \frac{1}{2\mu} \frac{\Gamma(\mu)}{\Gamma(\mu + 2\eta + 1)} \int_r^{\infty} (t^2 - r^2)^{\mu - 1} t^{-2\mu - 2\eta + 2} dt = +\infty,$$

because $a + \eta < 0$.

Hence, for $a < -\frac{1}{2}$, the operator $\mathcal{H}_\mu$ does not map the space $r^{-\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_a)$.

The proof of theorem is complete. □

**Remark 3.8.** For every $a \in \mathbb{R}$, the fractional transform $\mathcal{H}_\mu$ does not map the space $r^{-\mu}L^\infty(d\gamma_a)$ into itself.

In fact, the function $f(r, x) = r^{2\mu} 1_{[1, +\infty]}(r)$ belongs to $r^{-\mu}L^\infty(d\gamma_a)$, but for every $(r, x) \in [0, +\infty] \times \mathbb{R}^n$

$$\mathcal{H}_\mu(f)(r, x) = \frac{1}{2\mu} \frac{\Gamma(\mu)}{\Gamma(\mu + 2\eta + 1)} \int_r^{\infty} (t^2 - r^2)^{\mu - 1} t^{2\mu + 2\eta + 2} dt = +\infty.$$

We conclude that for every $p \in [1, +\infty[$, the transform $\mathcal{H}_\mu$, $\mu > 0$, is bounded from $r^{-\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_a)$ if and only if $2a + 1 > 0$ and

$$N_{p,\gamma_a}(\mathcal{H}_\mu) = \sup_{||r^{2\mu}f||_{p,\gamma_a} \leq 1} ||\mathcal{H}_\mu(f)||_{p,\gamma_a} \leq \frac{\Gamma(2a + 1)}{2\mu \Gamma(\mu + 2a + 1)}.$$ 

In particular, for $a = \mu > 0$, the fractional transform $\mathcal{H}_\mu$ is bounded from $r^{-\mu}L^p(d\gamma_a)$ into $L^p(d\gamma_\mu)$ and for every $f \in r^{-\mu}L^p(d\gamma_\mu)$,

$$||\mathcal{H}_\mu(f)||_{p,\gamma_\mu} \leq \frac{\Gamma(2a + 1)}{2\mu \Gamma(\mu + 2a + 1)} ||r^{2\mu}f||_{p,\gamma_\mu}.$$

**Competing Interests**

The author declares that he has no competing interests.

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**Lakhdar T. Rachdi**  
Université de Tunis El manar, Faculté des Sciences de Tunis, UR11ES23 Analyse géométrique et harmonique, 2092 Tunis, Tunisia.  
e-mail: lakhdartannech.rachdi@fst.rnu.tn

**Samia Sghaier**  
Université de Tunis El manar, Faculté des Sciences de Tunis, UR11ES23 Analyse géométrique et harmonique, 2092 Tunis, Tunisia.  
e-mail: samiasghaier21@gmail.com