COEFFICIENT ESTIMATES OF SOME CLASSES OF RATIONAL FUNCTIONS

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Abstract. Let $\mathcal{A}$ be the class of analytic and univalent functions in the open unit disc $\Delta$ normalized such that $f(0) = 0 = f'(0) - 1$. In this paper, for $\psi \in \mathcal{A}$ of the form $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $0 \leq \alpha \leq 1$, we introduce and investigate interesting subclasses $\mathcal{H}_\alpha(\phi)$, $\mathcal{S}_\sigma(\alpha, \phi)$, $\mathcal{M}_\sigma(\alpha, \phi)$, $\mathcal{I}_\alpha(\alpha, \phi)$ and $\beta_\alpha(\lambda, \phi)$ ($\lambda \geq 0$) of analytic and bi-univalent Ma-Minda star-like and convex functions. Furthermore, we find estimates on the coefficients $|a_1|$ and $|a_2|$ for functions in these classes. Several related classes of functions are also considered.

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1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions $f$ in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Also, by $\wp$ we shall denote the subclass of all functions in $\mathcal{A}$ which are univalent in $\Delta$. Let $\wp$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

which are analytic in $\Delta$ such that

$$p(0) = 1 \quad \text{and} \quad \text{Re}\{p(z)\} > 0 \quad (z \in \Delta).$$

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If the functions $f$ and $g$ are analytic in $\Delta$, then $f$ is said to be subordinate to $g$, written $f(z) \prec g(z)$, provided there is an analytic function $w(z)$ defined on $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$ so that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\Delta$ then we have the following equivalence (see for details, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

Some of the important and well-investigated subclasses of the univalent function class $\varphi$ include (for example) the class $S(\alpha)$ of starlike functions of order $\alpha$ in $\Delta$ and the class $C(\alpha)$ of convex functions of order $\alpha$ in $\Delta$. By definition, we have

$$S(\alpha) = \left\{ f : f \in \varphi \text{ and } \Re \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \Delta, \ 0 \leq \alpha < 1) \right\}$$

and

$$C(\alpha) = \left\{ f : f \in \varphi \text{ and } \Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \Delta, \ 0 \leq \alpha < 1) \right\}.$$  

It readily follows from the definitions ([1] and [2]) that

$$f(z) \in C(\alpha) \iff zf'(z) \in S(\alpha).$$

It is well known that for each $f \in \varphi$, the koebe one-quarter theorem [13] ensures the image of $\Delta$ under $f$ contains a disk of radius $1/4$. Thus every univalent function $f \in \varphi$ has an inverse $f^{-1}$ which satisfies

$$f^{-1}(f(z)) = z \ (|z| < 1)$$

and

$$f(f^{-1}(w)) = w, \ |w| < r_0(f), \ r_0(f) \geq 1/4.$$ 

In fact, the inverse function $g = f^{-1}$ is defined by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + ....$$

A function $f \in A$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\sigma$ denote the class of bi-univalent functions defined in the unit disk $\Delta$ and let $\phi \in P$ and $\phi(\Delta)$ is symmetric with respect to the the real axis, such a function has a Taylor series of the form:

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ... \ (B_1 > 0).$$

In [14], the authors introduced the class $S(\phi)$ of the so-called Ma and Minda starlike functions and the class $C(\phi)$ of Ma and Minda convex functions, unifying several previously studied classes related to those of starlike and convex functions. The class $S(\phi)$ consists of all the functions $f \in A$ satisfying subordination

$$\frac{zf'(z)}{f(z)} \prec \phi(z),$$

whereas $C(\phi)$ is formed with functions $f \in A$ for which the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$ holds. Lewin [15] investigated the class
σ and showed that $|a_2| < 1.51$ for function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \sigma$. Subsequently, Brannan and Clunie [16] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [17], on the other hand, showed that $\max|a_2| = 4/3$ if $f(z) \in \sigma$. Brannan and Taha [18] and Taha [19] introduced certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike and convex functions, they introduced bi-starlike functions and bi-convex functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Recently, many authors investigated bounds for various subclasses of bi-univalent functions (see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. In [34], Mitrinovic essentially investigated certain geometric properties of functions $\psi$ of the form

$$\psi(z) = \frac{z}{f(z)}, \quad f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n. \quad (5)$$

In [35], Reade et al. derived coefficient conditions that guarantee the univalence, starlikeness or convexity of rational functions of the form (5), these results have been improved and generalized in [36]. In this paper, estimates on the initial coefficients for bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type of rational form (5) are obtained. Several related classes are also considered. In order to derive our main results, we require the following lemma.

**Lemma 1.1.** (see [37]) If $p(z) \in P$, then

$$|c_n| \leq 2 \quad (n \in \mathbb{N} = \{1, 2, \ldots\}). \quad (6)$$

2. Coefficients estimates

A function $\psi(z) \in A$ with Re $(\psi'(z)) > 0$ is known to be univalent. This motivates the following class of functions.

**Definition 2.1.** A function $\psi \in \sigma$ given by (5) is said to be in the class $H_\sigma(\phi)$ if the following conditions are satisfied:

$$\psi'(z) \prec \phi(z) \quad (z \in \Delta) \quad \text{and} \quad g'(w) \prec \phi(w) \quad (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2z^2 + \ldots \quad (0 < \gamma \leq 1, \ z \in \Delta)$$

in Definition 2.1 of the bi-univalent function class $H_\sigma(\phi)$ we obtain a new class $H_\sigma(\gamma)$ given by Definition 2.2 below.
**Definition 2.2.** For $0 < \gamma \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $\mathcal{H}_\sigma(\gamma)$ if the following conditions are satisfied:

$$\psi'(z) < \left(\frac{1+z}{1-z}\right)^\gamma \quad (z \in \Delta) \quad \text{and} \quad g'(w) < \left(\frac{1+w}{1-w}\right)^\gamma \quad (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \frac{1 + (1 - 2\nu)z}{1 - z} = 1 + 2(1 - \nu)z + 2(1 - \nu)z^2 + \ldots \quad (0 < \nu \leq 1, \ z \in \Delta)$$

in Definition 2.1 of the bi-univalent function class $\mathcal{H}_\sigma(\phi)$ we obtain, a new class $\mathcal{H}_\sigma(\nu)$ given by Definition 2.3 below.

**Definition 2.3.** For $0 < \nu \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $\mathcal{H}_\sigma(\nu)$ if the following conditions hold true:

$$\psi'(z) < \frac{1 + (1 - 2\nu)z}{1 - z} \quad (z \in \Delta) \quad \text{and} \quad g'(w) < \frac{1 + (1 - 2\nu)w}{1 - w} \quad (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

**Theorem 2.4.** Let $\psi(z) \in \mathcal{H}_\sigma(\phi)$ be of the form (5). Then

$$|a_1| \leq \frac{B_1 \sqrt{B_1}}{\sqrt[3]{3B_1^2 - 4B_2 + 4B_1}} \quad \text{and} \quad |a_2| \leq \frac{1}{3} B_1. \quad (7)$$

**Proof.** Let $\psi(z) \in \mathcal{H}_\sigma(\phi)$ and $g = \psi^{-1}$. Then there exist two functions $u$ and $v$, analytic in $\Delta$, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in \Delta$, such that

$$\psi'(z) = \phi(u(z)) \quad \text{and} \quad g'(w) = \phi(v(w)). \quad (8)$$

Next, define the functions $p_1$ and $p_2$ by

$$p_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \ldots \quad \text{and} \quad p_2(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + b_1 w + b_2 w^2 + \ldots,$$

or, equivalently,

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left[ c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \ldots \right], \quad (9)$$

and

$$v(w) = \frac{p_2(w) - 1}{p_2(w) + 1} = \frac{1}{2} \left[ b_1 w + \left( b_2 - \frac{b_1^2}{2} \right) w^2 + \ldots \right]. \quad (10)$$

Then $p_1$ and $p_2$ analytic in $\Delta$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \to \Delta$, the functions $p_1$ and $p_2$ have a positive real part in $\Delta$, and $|b_i| \leq 2$ and $|c_i| \leq 2$. Clearly, upon substituting from (9) and (10) into (8), if we make use of (4), we find that

$$\psi'(z) = \phi\left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left[ \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right] z^2 + \ldots, \quad (11)$$
and
\[ g'(w) = \phi(\frac{p_2(w) - 1}{p_2(w) + 1}) = 1 + \frac{1}{2} B_1 b_1 w + \left[ \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \right] w^2 + \ldots \tag{12} \]

Since \( \psi \in \sigma \) has the Maclaurin’s series given by
\[ \psi(z) = z - a_1 z^2 + (a_1^2 - a_2) z^3 + \ldots, \tag{13} \]
a computation shows that its inverse \( g = \psi^{-1} \) has the expansion
\[ g(w) = \psi^{-1}(w) = w + a_1 w^2 + (a_1^2 + a_2) w^3 + \ldots. \tag{14} \]

Using (13) and (14) in (11) and (12) respectively, we get
\[ -2a_1 = \frac{1}{2} B_1 c_1 \tag{15} \]
\[ 3(a_1^2 - a_2) = \frac{1}{2} B_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{4} B_2 c_1^2, \tag{16} \]
\[ 2a_1 = \frac{1}{2} B_1 b_1 \tag{17} \]
and
\[ 3(a_1^2 + a_2) = \frac{1}{2} B_1 (b_2 - \frac{b_1^2}{2}) + \frac{1}{4} B_2 b_1^2. \tag{18} \]

From (15) and (17), we have
\[ c_1 = -b_1. \tag{19} \]

Adding (16) and (18) and then using (15) and (19), we get
\[ a_2^2 = \frac{B_1^3 (c_2 + b_2)}{4(3B_1^2 - 4B_2 + 4B_1)}, \]
and now, by applying Lemma 1.1 for the coefficients \( b_2 \) and \( c_2 \), the last equation gives the bound of \( |a_1| \) from (7). By subtracting (18) from (16), further computations using (19) lead to
\[ a_2 = \frac{1}{12} B_1 (b_2 - c_2). \]

The bound of \( |a_2| \), as asserted in (7), is now a consequence of Lemma 1.1 and this completes our proof. \( \square \)

Using the parameter setting of Definition 2.2 in Theorem 2.4 we get the following corollary.

**Corollary 2.5.** For \( 0 < \gamma \leq 1 \), let the function \( \psi \in \mathcal{H}_\sigma(\gamma) \) be of the form (5). Then
\[ |a_1| \leq \frac{\sqrt{2 \gamma}}{\sqrt{\gamma} + 2} \quad \text{and} \quad |a_2| \leq \frac{2}{3} \gamma. \]

Using the parameter setting of Definition 2.3 in Theorem 2.4 we get the following corollary.
Corollary 2.6. For $0 < \nu \leq 1$, let the function $\psi \in \mathcal{H}_\sigma(\nu)$ be given by (5). Then

$$|a_1| \leq \sqrt{\frac{2}{3}(1-\nu)} \quad \text{and} \quad |a_2| \leq \frac{2}{3}(1-\nu).$$

Definition 2.7. A function $\psi \in \sigma$ is given by (5) is said to be in the class $S_\sigma(\alpha, \phi)$ if the following subordinations hold:

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2\psi''(z)}{\psi(z)} \prec \phi(z) \ (z \in \Delta) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2g''(w)}{g(w)} \prec \phi(w) \ (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \ldots \ (0 < \gamma \leq 1, \ z \in \Delta)$$

in Definition 2.7 of the bi-univalent function class $S_\sigma(\alpha, \phi)$, we obtain a new class $S_\sigma(\alpha, \gamma)$ given by Definition 2.8 below.

Definition 2.8. For $0 \leq \alpha \leq 1$ and $0 < \gamma \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $S_\sigma(\alpha, \gamma)$ if the following subordinations hold:

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2\psi''(z)}{\psi(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma \ (z \in \Delta),$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2g''(w)}{g(w)} \prec \left(\frac{1+w}{1-w}\right)^\gamma \ (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = 1 + (1 - 2\nu)z = 1 + 2(1-\nu)z + 2(1-\nu)z^2 + \ldots \ (0 < \nu \leq 1, \ z \in \Delta)$$

in Definition 2.7 of the bi-univalent function class $S_\sigma(\alpha, \phi)$ we obtain a new class $S_\sigma(\alpha, \nu)$ given by Definition 2.9 below.

Definition 2.9. For $0 \leq \alpha \leq 1$ and $0 < \nu \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $S_\sigma(\alpha, \nu)$ if the following subordinations hold:

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2\psi''(z)}{\psi(z)} \prec \frac{1 + (1 - 2\nu)z}{1 - z} \ (z \in \Delta),$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2g''(w)}{g(w)} \prec \frac{1 + (1 - 2\nu)w}{1 - w} \ (w \in \Delta),$$

where $g(w) = \psi^{-1}(w)$.

Note that $S(\phi) = S_\sigma(0, \phi)$. For functions in the class $S_\sigma(\alpha, \phi)$, the following coefficient estimates are obtained,
Theorem 2.10. Let $\psi(z) \in S_\sigma(\alpha, \phi)$ be of the form (4). Then

$$|a_1| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{B_1^2(1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2}},$$

and

$$|a_2| \leq \frac{B_1}{1 + 3\alpha}.$$  \hfill (21)

Proof. Let $\psi \in S_\sigma(\alpha, \phi)$, there are two Schwarz functions $u$ and $v$ defined by (9) and (10) respectively, such that

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2 \psi''(z)}{\psi(z)} = \phi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = \phi(v(w)), \quad (g = \psi^{-1}).$$

Since

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2 \psi''(z)}{\psi(z)} = 1 - (1 + 2\alpha) a_1 z + [(1 + 4\alpha) a_1^2 - 2(1 + 3\alpha) a_2] z^2 + ...$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = 1 + (1 + 2\alpha) a_1 w + [(1 + 4\alpha) a_1^2 + 2(1 + 3\alpha) a_2] w^2 + ...,$$

then (11), (12) and (22) yields

$$-(1 + 2\alpha) a_1 = \frac{1}{2} B_1 c_1$$ \hfill (23)

and

$$(1 + 4\alpha) a_1^2 - 2(1 + 3\alpha) a_2 = \frac{1}{2} B_1 (c_2 - c_1^2) + \frac{1}{4} B_2 c_1^2,$$ \hfill (24)

$$(1 + 2\alpha) a_1 = \frac{1}{2} B_1 b_1$$ \hfill (25)

and

$$(1 + 4\alpha) a_1^2 + 2(1 + 3\alpha) a_2 = \frac{1}{2} B_1 (b_2 - \frac{b_1^2}{2}) + \frac{1}{4} B_2 b_1^2.$$ \hfill (26)

From (23) and (25), we get

$$c_1 = -b_1,$$ \hfill (27)

and after some further calculations using (24)-(27) we find

$$a_1^2 = \frac{B_1^3 (c_2 + b_2)}{4[B_1^2(1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2]},$$

and

$$a_2 = \frac{B_1 (b_2 - c_2)}{4(1 + 3\alpha)}.$$

Applying Lemma 1.1, the estimates in (20) and (21) follow. \hfill \Box

For $\alpha = 0$, Theorem 2.10 readily yields the following coefficient estimates for Ma-Minda bi-starlike functions.
Corollary 2.11. Let \( \psi \) given by (5) be in the class \( S(\phi) \). Then
\[
|a_1| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}}, \quad \text{and} \quad |a_2| \leq B_1.
\]

Using the parameter setting of Definition 2.8 in Theorem 2.10, we get the following corollary.

Corollary 2.12. For \( 0 \leq \alpha \leq 1 \) and \( 0 < \gamma \leq 1 \), let the function \( \psi \in S_\sigma(\alpha, \gamma) \) be of the form (5). Then
\[
|a_1| \leq \frac{2\gamma}{\sqrt{(1 + 2\alpha)^2 + \gamma [1 + 4\alpha - 4\alpha^2]}} \quad \text{and} \quad |a_2| \leq \frac{2\gamma}{1 + 3\alpha}.
\]

Using the parameter setting of Definition 2.9 in Theorem 2.10 we get the following corollary.

Corollary 2.13. For \( 0 \leq \alpha \leq 1 \) and \( 0 < \nu \leq 1 \), let the function \( \psi \in S_\sigma(\alpha, \nu) \) be of the form (5). Then
\[
|a_1| \leq \sqrt{\frac{2(1 - \nu)}{1 + 4\alpha}} \quad \text{and} \quad |a_2| \leq \frac{2(1 - \nu)}{1 + 3\alpha}.
\]

Definition 2.14. A function \( \psi \in \sigma \) given by (5) belongs to the class \( M_\sigma(\alpha, \phi) \) \((0 \leq \alpha \leq 1)\), if the following subordinations hold:
\[
(1 - \alpha) \frac{z\psi'(z)}{\psi(z)} + \alpha(1 + z\psi''(z)/\psi'(z)) \prec \phi(z) \quad (z \in \Delta),
\]
and
\[
(1 - \alpha) \frac{wg'(w)}{g(w)} + \alpha(1 + \frac{wg''(w)}{g'(w)}) \prec \psi(w) \quad (w \in \Delta),
\]
given by (5).

If we set
\[
\phi(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \ldots \quad (0 < \gamma \leq 1, \ z \in \Delta)
\]
in Definition 2.14 of the bi-univalent function class \( M_\sigma(\alpha, \phi) \), we obtain a new class \( M_\sigma(\alpha, \gamma) \) given by Definition 2.15 below.

Definition 2.15. For \( 0 \leq \alpha \leq 1 \) and \( 0 < \gamma \leq 1 \), a function \( \psi \in \sigma \) given by (5) is said to be in the class \( M_\sigma(\alpha, \gamma) \) if the following subordinations hold:
\[
(1 - \alpha) \frac{z\psi'(z)}{\psi(z)} + \alpha(1 + z\psi''(z)/\psi'(z)) \prec \left( \frac{1 + z}{1 - z} \right)^\gamma \quad (z \in \Delta),
\]
and
\[
(1 - \alpha) \frac{wg'(w)}{g(w)} + \alpha(1 + \frac{wg''(w)}{g'(w)}) \prec \left( \frac{1 + w}{1 - w} \right)^\gamma \quad (w \in \Delta),
\]
given by (5).
Corollary 2.16. If we set
\[ \phi(z) = \frac{1 + (1 - 2\nu)z}{1 - z} = 1 + 2(1 - \nu)z + 2(1 - \nu)z^2 + \ldots \quad (0 < \nu \leq 1, \ z \in \Delta) \]
in Definition 2.14 of the bi-univalent function class \( M_\sigma(\alpha, \phi) \) we obtain a new class \( M_\sigma(\alpha, \nu) \) given by Definition 2.17 below.

Definition 2.17. For \( 0 \leq \alpha \leq 1 \) and \( 0 < \nu \leq 1 \), a function \( \psi \in \sigma \) given by (5) is said to be in the class \( M_\sigma(\alpha, \nu) \) if the following subordinations hold:
\[ (1 - \alpha) \frac{z\psi'(z)}{\psi(z)} + \alpha(1 + \frac{z\psi''(z)}{\psi'(z)}) \prec 1 + (1 - 2\nu)z \quad (z \in \Delta), \]
and
\[ (1 - \alpha) \frac{w\psi'(w)}{\psi(w)} + \alpha(1 + \frac{w\psi''(w)}{\psi'(w)}) \prec 1 + (1 - 2\nu)w \quad (w \in \Delta), \]
where \( g(w) := \psi^{-1}(w) \).

A function in the class \( M_\sigma(\alpha, \phi) \) is called bi-Mocanu-convex function of Ma-Minda type. This class unifies the classes \( S(\alpha) \) and \( C(\alpha) \). For functions in the class \( M_\sigma(\alpha, \phi) \), the following coefficients estimates hold.

Theorem 2.18. Let \( \psi(z) \in M_\sigma(\alpha, \phi) \) be of the form (5). Then
\[ |a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1 + \alpha)B_1^2 + (1 + \alpha)(B_1 - B_2)}}, \quad (28) \]
and
\[ |a_2| \leq \frac{B_1}{2(1 + 2\alpha)}. \quad (29) \]

Proof. If \( \psi \in M_\sigma(\alpha, \phi) \), then there exist two Schwarz functions \( u \) and \( v \) defined by (9) and (10) respectively, such that
\[ (1 - \alpha) \frac{z\psi'(z)}{\psi(z)} + \alpha(1 + \frac{z\psi''(z)}{\psi'(z)}) = \phi(u(z)), \quad (30) \]
and
\[ (1 - \alpha) \frac{w\psi'(w)}{g(w)} + \alpha(1 + \frac{w\psi''(w)}{g'(w)}) = \phi(v(w)). \quad (31) \]

Since
\[ (1 - \alpha) \frac{z\psi'(z)}{\psi(z)} + \alpha(1 + \frac{z\psi''(z)}{\psi'(z)}) = 1 - (1 + \alpha)a_1z + [(1 + \alpha)a_1^2 - 2(1 + 2\alpha)a_2] z^2 + \ldots \]
and
\[ (1 - \alpha) \frac{w\psi'(w)}{g(w)} + \alpha(1 + \frac{w\psi''(w)}{g'(w)}) = 1 + (1 + \alpha)a_1w + [(1 + \alpha)a_1^2 + 2(1 + 2\alpha)a_2] w^2 + \ldots, \]
from (11), (12), (30) and (31), it follows that
\[ -(1 + \alpha)a_1 = \frac{1}{2}B_1c_1, \quad (32) \]
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\[ (1 + \alpha)a_1^2 - 2(1 + 2\alpha)a_2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2, \quad (33) \]

\[ (1 + \alpha)a_1 = \frac{1}{2}B_1b_1, \quad (34) \]

and

\[ (1 + \alpha)a_1^2 + 2(1 + 2\alpha)a_2 = \frac{1}{2}B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{4}B_2b_1^2, \quad (35) \]

Eqs. (32) and (34) yields

\[ c_1 = -b_1, \quad (36) \]

and after some further calculations using (33)-(35) we find

\[ a_1^2 = \frac{B_1^3(c_2 + b_2)}{4(1 + \alpha)[B_2^2 + (1 + \alpha)(B_1 - B_2)]}, \]

and

\[ a_2 = \frac{B_1(b_2 - c_2)}{8(1 + 2\alpha)}, \]

Applying Lemma 1.1, the estimates in (28) and (29) follow.

For \( \alpha = 0 \), Theorem 2.18 gives the coefficient estimates for Ma-Minda bi-starlike functions, while for \( \alpha = 1 \), it gives the following estimates for Ma-Minda bi-convex functions.

**Corollary 2.19.** Let \( \psi \) given by (5) be in the class \( C(\phi) \). Then

\[ |a_1| \leq \frac{B_1\sqrt{B_1}}{2|B_1^2 + 2(B_1 - B_2)|}, \quad and \quad |a_2| \leq \frac{B_1}{6}. \]

Using the parameter setting of Definition 2.15 in Theorem 2.18 we get the following corollary.

**Corollary 2.20.** For \( 0 \leq \alpha \leq 1 \) and \( 0 < \gamma \leq 1 \), let the function \( \psi \in M_\sigma(\alpha, \gamma) \) be of the form (7). Then

\[ |a_1| \leq \frac{2\gamma}{\sqrt{(1 + \alpha)[(1 + \alpha) + \gamma(1 - \alpha)]}} \quad and \quad |a_2| \leq \frac{\gamma}{1 + 2\alpha}. \]

Using the parameter setting of Definition 2.17 in Theorem 2.18 we get the following corollary.

**Corollary 2.21.** For \( 0 \leq \alpha \leq 1 \) and \( 0 < \nu \leq 1 \), let the function \( \psi \in M_\sigma(\alpha, \nu) \) be of the form (8). Then

\[ |a_1| \leq \sqrt{\frac{2(1 - \nu)}{1 + \alpha}} \quad and \quad |a_2| \leq \frac{(1 - \nu)}{1 + 2\alpha}. \]
Definition 2.22. A function $\psi \in \sigma$ given by (5) is said to be in the class $\mathcal{I}_\alpha(\alpha, \phi)$ ($0 \leq \alpha \leq 1$), if the following subordinations hold:

$$
\left( \frac{z\psi'(z)}{\psi'(z)} \right)^\alpha \left( 1 + \frac{z\psi''(z)}{\psi'(z)} \right)^{1-\alpha} < \phi(z) \quad (z \in \Delta),
$$

and

$$
\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} < \phi(w) \quad (w \in \Delta),
$$

g(w) := \psi^{-1}(w).$$

This class also reduces to classes of Ma-Minda bi-starlike and bi-convex functions. For functions in this class, the following coefficient estimates are obtained.

Theorem 2.23. Let $\psi(z) \in \mathcal{I}_\alpha(\alpha, \phi)$ be of the form (5). Then

$$|a_1| \leq \frac{2B_1\sqrt{B_1}}{\sqrt{\left[2(\alpha^2 - 3\alpha + 4)B_1^2 + 4(\alpha - 2)^2(B_1 - B_2)\right]}},$$

and

$$|a_2| \leq \frac{B_1}{2|3 - 2\alpha|}.$$

Proof. Let $\psi \in \mathcal{I}_\alpha(\alpha, \phi)$, then there exist are two Schwarz functions $u$ and $v$ defined by (9) and (10) respectively, such that

$$
\left( \frac{z\psi'(z)}{\psi'(z)} \right)^\alpha \left( 1 + \frac{z\psi''(z)}{\psi'(z)} \right)^{1-\alpha} = \phi(u(z)) \quad (39)
$$

and

$$
\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = \phi(v(w)). \quad (40)
$$

Since

$$
\left( \frac{z\psi'(z)}{\psi'(z)} \right)^\alpha \left( 1 + \frac{z\psi''(z)}{\psi'(z)} \right)^{1-\alpha} = 1 - (2 - \alpha) a_1 z
$$

$$
+ \left[ \frac{\alpha^2 - 3\alpha + 4}{2} a_1^2 - 2(3 - 2\alpha) a_2 \right] z^2 + \ldots .
$$

Also

$$
\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = 1 + (2 - \alpha) a_1 w
$$

$$
+ \left[ \frac{\alpha^2 - 3\alpha + 4}{2} a_1^2 + 2(3 - 2\alpha) a_2 \right] w^2 + \ldots ,
$$

from (11), (12), (39) and (40), it follows that

$$
-(2 - \alpha) a_1 = \frac{1}{2} B_1 c_1, \quad (41)
$$

$$
\frac{\alpha^2 - 3\alpha + 4}{2} a_1^2 - 2(3 - 2\alpha) a_2 = \frac{1}{2} B_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{4} B_2 c_1^2, \quad (42)
$$

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\[(2 - \alpha)a_1 = \frac{1}{2}B_1b_1\]  \hspace{1cm} (43)

and

\[\frac{\alpha^2 - 3\alpha + 4}{2}a_1^2 + 2(3 - 2\alpha)a_2 = \frac{1}{2}B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{4}B_2b_1^2.\]  \hspace{1cm} (44)

Eqs. (41) and (43) obviously yield

\[c_1 = -b_1.\]  \hspace{1cm} (45)

Eqs. (42)-(44) and (45) lead to

\[a_2 = \frac{B_1(b_2 - c_2)}{8(3 - 2\alpha)},\]  \hspace{1cm} (46)

which, in view of Lemma 1.1, yields the estimate (48). \(\square\)

**Definition 2.24.** A function \(\psi \in \sigma\) given by (5) is said to be in the class \(\beta_\alpha(\lambda, \phi), \lambda \geq 0\), if the following subordinations hold:

\[(1 - \lambda) \psi(z) = z + \lambda \psi'(z) < \phi(z) (z \in \Delta),\]

and

\[(1 - \lambda) \frac{\psi'(w)}{w} = \psi^{-1}(w).\]

**Theorem 2.25.** Let \(\psi(z) \in \beta_\alpha(\lambda, \phi), \lambda \geq 0\) be of the form (5). Then

\[|a_1| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(1 + 2\lambda) B_1^2 + (1 + \lambda)^2(B_1 - B_2)}},\]  \hspace{1cm} (46)

and

\[|a_2| \leq \frac{B_1}{1 + 2\lambda}.\]  \hspace{1cm} (47)

**Proof.** Let \(\psi \in \beta_\alpha(\lambda, \phi),\) then there exist two Schwarz functions \(u\) and \(v\) defined by (9) and (10) respectively, such that

\[(1 - \lambda) \frac{\psi(z)}{z} = \phi(u(z))\]  \hspace{1cm} (48)

and

\[(1 - \lambda) \frac{g(w)}{w} = \phi(v(w)).\]  \hspace{1cm} (49)

Since

\[(1 - \lambda) \psi(z) = 1 - (1 + \lambda) a_1 z + \left[(1 + 2\lambda) (a_1^2 - a_2)\right] z^2 + ...,\]
and
\[(1 - \lambda) \frac{g'(w)}{w} + \lambda g'(w) = 1 + (1 + \lambda) a_1 w + [(1 + 2\lambda) (a_1^2 + a_2)] w^2 + \ldots,\]

from (11), (12), (48) and (49), it follows that
\[-(1 + \lambda) a_1 = \frac{1}{2} B_1 c_1,\]
\[(1 + 2\lambda) (a_1^2 - a_2) = \frac{1}{2} B_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{4} B_2 c_1^2,\]
\[(1 + \lambda) a_1 = \frac{1}{2} B_1 b_1\]
and
\[(1 + 2\lambda) (a_1^2 + a_2) = \frac{1}{2} B_1 (b_2 - \frac{b_1^2}{2}) + \frac{1}{4} B_2 b_1^2.\]

Now (50) and (52) clearly yield
\[c_1 = -b_1.\]

Eqs. (51), (53) and (54) lead to
\[a_2^2 = \frac{B_1^2 (c_2 + b_2)}{4 \left(1 + \lambda \right) B_1^2 + \left(1 + \lambda \right)^2 (B_1 - B_2)} ,\]

By applying Lemma 1.1 we get the desired estimate of \(|a_1|\) as asserted in (46).

Proceeding similarly as in the earlier proof, using (51)-(54), it follows that
\[a_2 = \frac{B_1 (b_2 - c_2)}{4(1 + 2\lambda)},\]
which, in view of Lemma 1.1 yields the estimate (47). \qed

Competing Interests

The authors declare that they have no competing interests.

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